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ON A CLASS OF FIRST ORDER NORMAL DIFFERENTIAL OPERATORS

Abstract

In this study, a first order differential operator in Hilbert space of vector-functions from the finite interval into a Hilbert space has been considered and the relationships between formally normality property and the coefficients of this operator have been investigated. Furthermore, all normal extensions of a suitably chosen minimal operator have been described in terms of boundary values.

Introduction

A densely defined, closed operator N in a Hilbert space H is called formally normal if $D(N)\subset D(N^*)$ and $\|Nf\|=\|N^*f\|$ for each $f\in D(N)$. If a formally normal operator N satisfies the condition $D(N)=D(N^*)$, then it is called normal operator [1]. In work [14], it has been shown that a densely defined , closed operator N is normal iff $NN^*=N^*N$.

The abstract theory of normal extensions of unbounded formally normal operators in H has been extensively learned in works [1], [3]-[5], [9]-[10], [15]. The application of normal extension theory of formally normal operators to the differential operators has a great importance as in Von Neumann's Theory. Because, many of physical and technical processes require non-self adjoint operators to be applied in Hilbert spaces. From this viewpoint, this theory has been less learned (for example: [2], [7],[12]).

Let us denote a seperable Hilbert space by H and let us denote $L_2(0,1)$ by H_0. Let $H_1=L_2(H,(0,1))$ denote the Hilbert space of vector-functions from the interval [0,1] into H.

1. In this section we will investigate the Relationships Between Formally Normality Property and The Coefficients:

Let us consider the differential-operator expression given by

$$l(u)=A(t)u'+B(t)u,\quad 0\le t\le 1, \qquad (1.1)$$

where the coefficients $A(t)$, $B(t)$ are linear, bounded and self-adjoint operators in H for each $t\in[0,1]$ and A(t) is strongly differentiable on [0,1], B(t) is strongly continue in [0,1].

The differential operator expression

$$l^+(v)=-(A(t)v)'+B(t)v,\quad 0\le t\le 1 \qquad (1.2)$$

is called formally adjoint of the differential expression (1.1).

Let us define two operators as follows:

$$L_0'u=l(u), L_0^{+'}v=l^+(v), u,v\in C_0^\infty(H,(0,1)).$$

The closure of L_0' ($L_0^{+'}$) in H_1 is called minimal operator generated by the differential-operator expression (1.1)((1.2)) and it is denoted by L_0 (L_0^+). The operator L(L^+), adjoint of L_0^+(L_0) in H_1, is called maximal operator generated by $l(u)$($l^+(v)$). Thus we have the following results:

$$L = (L_0^+)^*, L^+ = L_0^* \text{ and } L_0 \subset L, L_0^+ \subset L^+.$$

In this section Suppose A(t), B(t) are two-times and one-time differentiable respectvely on $[0,1]$ *such that* $C(t) = A(t)A''(t) + A''(t)A(t) - 2A(t)B'(t) - 2B'(t)A(t)$ *is uniformly continuous on* $[0,1]$, *and* $A(t) \geq E$ for each $t \in [0,1]$.

Theorem1.1. *Minimal operator* L_0 *is formally normal in* $L_2(H,(0,1))$ *if and only if* $C(t) \equiv 0,\ 0 < t < 1$.

Proof. In first note that $D(L_0) = \overset{0}{W}{}_2^1(H,(0,1))$. Let us minimal operator L_0 is formally normal. Then we have $\|Au' + Bu\|_{H_1} = \|(Au)' - Bu\|_{H_1}$ for each $u(t) \in D(L_0)$. From this relation we obtain that $\|A'u\|_{H_1}{}^2 + \langle A'u, Au'\rangle_{H_1} + \langle Au', A'u\rangle_{H_1} - \langle Bu, A'u + 2Au\rangle_{H_1} - \langle A'u + 2Au', Bu\rangle_{H_1} = 0$ for each $u(t) \in D(L_0)$. Now replacing $u(t)$ by $\varphi(t)f, \varphi(t) \in \overset{0}{W}{}_2^1(0,1), f \in H$ $(see$:[13] *for* $W_p^l(.)$ *and* $\overset{0}{W}{}_p^l(.))$, in the last that expression and then integrating term-by-term we get

$$\int_0^1 \sigma(t)\varphi^2(t)dt = 0,\ \sigma(t) = \langle (A''A + AA'' - 2AB' - 2B'A)f, f\rangle_H. \qquad (1.3)$$

Let us now prove that if (1.3) is valid for each function $\varphi(t) \in \overset{0}{W}{}_2^1(0,1)$, then $\sigma(t) \equiv 0,\ 0 < t < 1$.

If the function $\sigma(t) \in C[0,1]$ satisfies the condition $\sigma(t) \geq 0 (\leq 0),\ 0 < t < 1$, then from (1.3), $\sigma(t) \equiv 0,\ 0 < t < 1$, is obvious.

Now assume that σ is negative on some intervals $\Delta_n = (\alpha_n, \beta_n) \subset [0,1],\ n = 1,2,\ldots$, and non-negative on the remainder part of $[0,1] \setminus \bigcup_n \Delta_n$. For simplicity, suppose that n=1, i.e. $\sigma(t) < 0,\ t \in \Delta = (\alpha, \beta)$ and $\sigma(t) \geq 0,\ t \in [0,1] \setminus \Delta$. (The general case can be proved in an analogous manner)

Let us define the function $\varphi_{\alpha\beta}$ as

$$\varphi_{\alpha\beta} = \begin{cases} 0, & 0 \leq t \leq \alpha \\ (t-\alpha)(t-\beta), & \alpha \leq t \leq \beta. \\ 0, & \beta \leq t \leq 1 \end{cases}$$

Then $\varphi_{\alpha\beta} \in \overset{0}{W}{}_2^1(0,1)$ is clear and from (1.3)

$$\int_0^1 \sigma(t)\varphi_{\alpha\beta}^2(t)dt = \int_\Delta \sigma(t)\varphi_{\alpha\beta}^2(t)dt = 0.$$

Notice that $\sigma'(t)\varphi_{\alpha\beta}^2(t) \in C[\alpha,\beta]$, $\sigma(t)\varphi_{\alpha\beta}^2(t) < 0,\ \varphi_{\alpha\beta} \neq 0, t \in \Delta$. So we get $\sigma'(t) < 0,\ t \in \Delta = (\alpha, \beta)$.

Let us consider the function $\varphi_{0\alpha}\ (\varphi_{\beta 1})$ defined by

$$\varphi_{0\alpha}(t) = \begin{cases} t(t-1), & 0 \leq t \leq \alpha \\ 0, & \alpha \leq t \leq 1 \end{cases} \left(\varphi_{\beta 1}(t) = \begin{cases} 0, & 0 \leq t \leq \beta \\ (t-\beta)(t-1), & \beta \leq t \leq 1 \end{cases} \right)$$

Replacing $\varphi(t)$ by $\varphi_{0\alpha}(t)$ and $\varphi_{\beta 1}(t)$ in oreder in (1.3), we obtain

$$\sigma(t)=0,\ t\in(0,\alpha)\ and\ \sigma(t)=0,\ t\in(\beta,1).$$

Combining these results we see that $\sigma(t)=0,\ t\in(0,\alpha)\cup(\alpha,\beta)\cup(\beta,1)$.

Since σ is contiuous on $[0,1]$, it shows that $\sigma(t)\equiv 0,\ 0<t<1$, i.e.

$$C(t)\equiv 0,\ 0<t<1.$$

If σ is positive on Δ and non-positive on $[0,1]\setminus\Delta$, then the proof can be given in an analogous manner.

To prove the sufficiency, let us choose a function $u(t)\in D(L_0)$. Then

$$l(u)=A(t)u'+B(t)u'\in H_1$$

and

$$l^+(u)=-A(t)u'+(B(t)-A'(t))u=-(A(t)u'+B(t)u)-A'(t)u+2B(t)u.$$

Since

$$\int_0^1\|A'(t)u\|_H^2\,dt\le\left(\max_{t\in[0,1]}\|A'(t)\|_H^2\right)\cdot\|u\|_{H_1}^2$$

and

$$\int_0^1\|B(t)u\|_H^2\,dt\le\left(\max_{t\in[0,1]}\|B(t)\|_H^2\right)\cdot\|u\|_{H_1}^2,$$

it shows that $l^+(u)\in H_1$, i.e., $u(t)\in D(L^+)$ and thus $D(L_0)\subset D(L^*)$.

Finally, let us show that the second condition of formally normality is provided.

Assume that $u(t)\in D(L_0)\ and\ u(t)=\varphi(t)f, \varphi(t)\in \overset{0}{W}{}_2^1(0,1), f\in H$. It can easily be shown that

$$\|L_0u\|_{H_1}^2-\|L^+u\|_{H_1}^2=\int_0^1\langle(A''A+AA''-2AB'-2B'A)f,f\rangle_H\varphi^2(t)dt=0,$$ i.e., the operator L_0 satisfies the second condition of formally normality. This completes the proof.

Corollary 1.1. *Suppose A=constant then the minimal operator L_0 generated by the differential-operator expression $l(u)=Au'+B(t)u$ is formally normal in H_1 if and only if*

$$AB'(t)+B'(t)A\equiv 0,\ 0<t<1.$$

If $A=E$, then $B(t)\equiv constant$, $0<t<1$ and if $A(t)=constant$, $B(t)\equiv constant$, then L_0 is evidently formally normal where E is the identity operator on $L_2(H,(0,1))$.

If $\dim H=1$, $a(t)\in C^2(0,1), b(t)\in C^1(0,1)$ and $a(t)\ne 0, t\in[0,1]$, then the necessary and sufficient condition for the normality property of the minimal operator l_0 generated by $l(u)=a(t)u'+b(t)u$ is the following:

$$a(t)=-2\int_0^1 b(\tau)d\tau+C_1t+C_2,$$ where e_1 and e_2 any complex numbers.

Let us show for which necessary conditions the minimal operator L_0 has a normal extension.

Theorem 1.2. *If L_0 has a normal extension, then the following conditions are satisfied:*

$$C(t) \equiv 0,\ 0 < t < 1,$$
$$\{u : K_1(t)u(t) = 0\} \cap \{u : K_2(t)u(t) = 0\} \neq \{0\}.$$

Where,

$$K_1(t)u(t) = -A^2u'' + (BA - AB - A'A - AA')u' + (E + B^2 - A'B - AB')u,$$
$$K_2(t)u(t) = -A^2u'' + (AB - BA - 2AA')u' + (E + B^2 - BA' + AB' - AA'')u$$

Proof. Let $\widetilde{L}$ be a normal extension of $L_0 \subset \widetilde{L} \subset L$ *and* $L_0^+ \subset \widetilde{L}^* \subset L^+$. It is seen from this that

$$D(L_0) \subset D(\widetilde{L}) = D(L^*) \subset D(L^+), i.e.,\ D(L_0) \subset D(L^+).$$

On the other hand,

$$\|L_0u\|_{H_1} = \|\widetilde{L}u\|_{H_1} = \|\widetilde{L}^*u\|_{H_1} = \|L^+u\|_{H_1},\ \text{since}\ u(t) \in D(L_0).$$

Thus, the minimal operator L_0 is formally normal.

Consequently, by Theorem (1.1),

$$C(t) \equiv 0,\ 0 < t < 1.$$

Let us now show the second condition. The simple calculations show that

$$\text{M} = Ker(E + L^*L) =$$
$$\{u(t) : -A^2u'' + (BA - AB - A'A - AA')u' + (E + B^2 - A'B - AB')u = 0\}$$

and

$$\text{M}^1 = Ker(E + LL^*) =$$
$$\{u(t) : -A^2u'' + (AB - BA - 2AA')u' + (E + B^2 - BA' + AB' - AA'')u = 0\}.$$

From the assumption that the minimal operator L_0 has a normal extension $\text{M} \cap \text{M}^1 \neq \{0\}$ (see:[1]). This completes the proof.

Especially, if *A(t)=constant*, *B(t)≡constant*, *AB=BA* and $o \in \rho(A)$ (where $\rho(A)$ is the resolvet set of *A*), then the minimal operator L_0 generated by the expression $l(u) = Au'(t) + Bu(t)$ is formally normal and $\text{M} = \text{M}^1 = \{u(t) : -A^2u'' + (E + B)u = 0\} \neq \{0\}$.

Corollary 1.2. *If A(t)=Constant, then A[B(1)-B(0)]=[B(0)-B(1)]A.*

Proof. Considering that L_0 is formally normal,

$C(t) = AA'' + A''A + 2AB' + 2B'A = 2AB' + 2B'A = 0$. Since *A(t)=constant*, from $AB' + B'A = 0$ we obtain:

$$\int_0^1 \langle B'(t)f, f^A\rangle_H dt + \int_0^1 \langle Af, B'(t)f\rangle_H dt = \langle B(t)f^A, f\rangle_H\Big|_0^1 + \langle Af, B(t)f\rangle_H\Big|_0^1 = 0$$

for each $f \in H$. In other words,

$$\langle (A[B(1) - B(0)]) + ([B(1) - B(0)]A)f, f\rangle_H = 0 \ \text{for each}\ f \in H.$$

Hence,

$$A[B(1) - B(0)] = [B(0) - B(1)]A.$$

2. In this section we shall consider the differntial-operator expression given by

$$l(u) = Au' + Bu \quad , \tag{2.1}$$

where $A = A^* \geq \gamma_1 > 0,\ B = B^* \geq \gamma_2 > 0,\ AB = BA$ *and* $A, B \in B(H)$ -space of linear boundeal operators in H. In this case we will describe, in terms of boundary values, all normal extensions of the minimal operator L_0 generated in H_1 by the expression (2.1). To do this, we need some additional informations.

In the following definition, it is assumed T is a closed, symmetric operator in the Hilbert space H such that T has the same deficiency indices.

Definition 2.1.([6]). *Let* H *be a Hilbert space and* γ_1, γ_2 *be two linear mappings from* $D(T^*)$ into H. *The triplet* $(\mathrm{H}, \gamma_1, \gamma_2)$ *is called a space of boundary values of the operator T if:*

1) $\langle T^* f, g \rangle - \langle f, T^* g \rangle_H = \langle \gamma_1 f, \gamma_2 g \rangle_H - \langle \gamma_2 f, \gamma_1 g \rangle_H$ *for all f,g in* $D(T^*)$;

2) *For any* F_1, F_2 *in* H, *there exists* $f \in D(T^*)$ *such that* $\gamma_1 f = F_1$ *and* $\gamma_2 f = F_2$.

If T is any symmetric operator in the Hilbert space H such that T has the same deficiency indices, say $n(n \leq \infty)$, then there exists a space of boundary values $(\mathrm{H}, \gamma_1, \gamma_2)$ of T such that $dim\mathrm{H} = n$ ([11]).

Let us now suppose that $(\mathrm{H}, \gamma_1, \gamma_2)$ is any space of boundary values of the operator T. Then following theorem is true:

Theorem 2.1.([6]) *For the any unitary operator W in* H *the restriction of* T^* *on the manifold of vectors* $f \in D(T^*)$ *satisfying the condition*

$$(W - E)\gamma_1(f) + i(W + E)\gamma_2(f) = 0 \tag{2.2}$$

or

$$(W - E)\gamma_1(f) - i(W + E)\gamma_2(f) = 0 \tag{2.3}$$

is defined a self-adjoint extension of the operator T. Conversely, any self-adjoint extension of T is the restriction of T^* *to the manifold of vectors* $f \in D(T^*)$ *satisfying condition (2.2) or (2.3) such that the unitary operator W is defined uniquely by the extension.*

The following theorem can be proved with the help of this facts:

Theorem 2.2. *Let W be an unitary operator in H with the property (AB)W=W(AB). Then the restriction of the operator L to the manifold of vector-functions* $u(t) \in D(L)$ *satisfying the boundary condition*

$$u(1) = A^{-\frac{1}{2}} W A^{\frac{1}{2}} u(0) \tag{2.4}$$

is defined an normal extension in H_1 *of the operator* L_0. *Conversely, any normal extension of* L_0 *is the restriction of* L^* *to the manifold of vector-functions* $u(t) \in D(L^*)$ *satisfying condition (2.4) such that the unitary operator W in condition (2.4) is defined uniquely by the extension.*

Proof. Let $\widetilde{L}$ be a normal extension in H_1 of the minimal operator L_0. The operator

$$\tilde{L}_J = \frac{\widetilde{L} - \widetilde{L}^*}{2i}$$

is a self-adjoint extension in H_1 of the closed symmetric and minimal operator L_0^J generated by the formally symmetric operator $l^I(u) = \dfrac{l(u) - l^+(u)}{2i} = -iAu'$. It can be proved that the triplet $(\mathrm{H}, \gamma_1, \gamma_2)$, $\mathrm{H}=H$, $\gamma_1(u) = \dfrac{A^{1/2}u(0) + A^{1/2}u(1)}{\sqrt{2}}$, $\gamma_2(u) = \dfrac{A^{1/2}u(0) - A^{1/2}u(1)}{\sqrt{2}}$ determines a space of boundary values of the minimal operator L_0. Then by the Theorem 2.1, there exists an unitary operator W in H such that the boundary condition $(W - E)\gamma_1(u) + i(W + E)\gamma_2(u) = 0,\ u \in D(\widetilde{L}_I)$, is valid. As for this condition,

$$u(1) = A^{-\frac{1}{2}} W A^{\frac{1}{2}} u(0).$$

Also, the operator $L_R = \dfrac{\widetilde{L} + \widetilde{L}^*}{2}$ is self-adjoint extension in H_1 of the minimal operator L_0^R generated by the formally symmetric operator

$$l^R(u) = Bu(t),\ u(t) \in D(\widetilde{L}_R) = D(\widetilde{L}_J) = D(\tilde{L}).$$

From the second condition of normality, we obtain that

$$\langle Au', Bu\rangle_{H_1} + \langle Bu, Au'\rangle_{H_1} = \langle (ABu)', u\rangle_{H_1} + \langle u, (ABu)'\rangle_{H_1} =$$

$$\langle u, ABu\rangle'_{H_1} = \langle u(1), ABu(1)\rangle_{H_1} - \langle u(0), ABu(0)\rangle_{H_1} =$$

$$\left\|(AB)^{1/2}u(1)\right\|_H^1 - \left\|(AB)^{1/2}u(0)\right\|_H^2 = 0,\ \mathrm{U}(0) \in \mathrm{d}(\tilde{\mathrm{L}})$$

This shows that there exist an isometric operator V in H such that

$$(AB)^{1/2}u(1) = V(AB)^{1/2}u(0),\ u(t) \in D(\widetilde{L})$$

or

$$u(1) = (AB)^{-1/2}V(AB)^{1/2}u(0).$$

From this said, it is clear that the selfadjoint extension $\widetilde{L}_J$ is generated by two conditions:

$$(W - E)\gamma_1(u) + i(W + E)\gamma_2(u) = 0,$$

$$(W_1 - E)\gamma_1(u) + i(W_1 + E)\gamma_2(u) = 0,\ W_1 = (AB)^{-1/2}V(AB)^{1/2},\ u \in D(\widetilde{L}).$$

Since, for any self-adjoint $\widetilde{L}_J$ the unitary operator W is unique (see: Theorem 2.1), this implies that W=W_1 , i.e.,

$$W = (AB)^{-1/2}V(AB)^{1/2}$$

From this, we have $V = (AB)^{1/2}W(AB)^{-1/2}$, that this says actually $(AB)W$=$W(AB)$.

Conversely, let W be an unitary operator in H with the property $(AB)W=W(AB)$ and let us denoted by $L(W)$ the restriction of the maximal operator L to the manifold of vector-functions satisfying condition (2.4). Then $L_0 \subset L(W) \subset L$ is clear.

Let us now prove that the operator $L(W)$ is a normal operator in H_1. Consider the extension of the minimal-operator L_0^+ generated by the differential-operator expression $l^+(v) = -Av' + Bv$ satisfying the boundary condition

$$v(0) = A^{-1/2} W^* A^{1/2} v(1). \tag{2.5}$$

In deed, this considered operator is adjoint to

$L(W)$, i.e., it is $L^*(W)$.

It is easily seen if W is unitary operator, then the boundary condition (2.4) and (2.5) determine the same manifold of vector-functions. Thus, $D(L(W)) = D(L^*(W))$.

On the other hand, for each $u(t) \in D(L(W))$,

$$\|L(W)u\|_{H_1}^2 - \|L^*(W)u\|_{H_1}^2 = 2\left[\langle ABu(1), u(1)\rangle_H - \langle ABu(0), u(0)\rangle_H\right] =$$

$$= 2\left[\left\langle AB(A^{1/2} W^* A^{-1} W A^{1/2} - E)u(0), u(0)\right\rangle_H\right].$$

From the condition (2.4), for all $u(t)$ in $D(L(W)) = D(L^*(W))$, $u(0) = A^{1/2} W^* A^{1/2} u(1) = A^{1/2} W^* A^{-1/2} A^{-1/2} W^{1/2} u(0)$. Using this fact in the previous relation, we see that

$$\|L(W)u\|_{H_1} = \|L^*(W)u\|_{H_1},\ u(t) \in D(L(W)).$$

Hence, the operator $L(W)$ is a normal extension of L_0 and this completes the proof.

Example: Let us lim H=1. In this ease all normale extension $\tilde{l}$ of the minimal operator $l_0$generated by differential expression l(u)=au'+bu, a>0, b>0 in the space $L_2(0,1)$ in the form of boundary values writes:

$$u(1) = e^{iQ} u(0), Q \in (0, 2\pi].$$

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