KHANMAMEDOV Ag.Kh.

ON THE THEORY OF INVERSE SCATTERING PROBLEMS FOR A SYSTEM OF DIFFERENCE EQUATIONS

Abstract

In paper transformation operators for difference equations are constructed necessary and sufficient conditions of solvability of the inverse scattering problem are established.

Consider the system of the difference equations

$$\begin{cases} a_{1,n}y_{2,n} + a_{2,n}y_{2,n-1} = \lambda y_{1,n} \\ a_{1,n}y_{1,n} + a_{2,n}y_{1,n+1} - \lambda y_{2,n} \end{cases} \qquad n = 0,\pm1,\pm2,... \tag{1}$$

with the coefficients $a_{i,n}$ $(i = 1,2)$ satisfying the conditions

$$\sum_{n=\infty}^{\infty}|n|\left|a_{i,n} - A_i\right| < \infty, \quad i = 1,2\,. \tag{2}$$

where $A_i > 0$, $i = 1,2$.

In this work the transformation operator for the system of equations (1) with a condition on infinity is studied, basic equations by Gelphand-Levitan are obtained, necessary and sufficient conditions of solvability of the inverse scattering problem, are established.

Note that for $A_1 = 1$, $A_0 = -1$, in more restricted class of coefficients the inverse problem is studied in [1-3].

Let Γ be a complex λ-plane with section on segments $\left[-|A_1 + A_2|, -|A_1 - A_2|\right]$, and $\left[|A_1 - A_2|, |A_1 + A_2|\right]$, and $\partial\Gamma$ be its boundary.

In the plane Γ we'll consider the function

$$z = z(\lambda) = \frac{\lambda^2 - A_1^2 - A_2^2}{2A_1A_2} + \sqrt{\left(\frac{\lambda^2 - A_1^2 - A_2^2}{2A_1A_2}\right)^2 - 1}$$

choosing such a branch of the root that $z(\infty)$.

Theorem 1. *For the conditions (2) the system of equations (1) has special solutions* $f_n^{\pm}(\lambda) - \left(f_{1,n}^{\pm}(\lambda), f_{2,n}^{\pm}(\lambda)\right)$ *representable in the form of*

$$f_{i,n}^{\pm}(\lambda) = \alpha_{i,n}^{\pm}\left(\frac{A_1z^{\pm1} + A_2}{\lambda z^{\pm1}}\right)^{2-i}\left(1 + \sum_{m=\pm1}^{\pm\infty}K_i^{\pm}(n,m)z^{\pm m}\right)z^{\pm n}, \quad i = 1,2\,, \tag{3}$$

where the quantities $\alpha_{i,n}^{\pm}$, $K_i^{\pm}(n,m)$ *have the properties*

$$\frac{a_{1,n}}{A_1} = \frac{\alpha_{2,n}^{+}}{\alpha_{1,n}^{+}} = \frac{\alpha_{1,n}^{-}}{\alpha_{2,n}^{-}}, \quad \frac{a_{2,n}}{A_2} = \frac{\alpha_{1,n+1}^{+}}{\alpha_{2,n}^{+}} = \frac{\alpha_{1,n}^{-}}{\alpha_{1,n+1}^{-}},$$

$$\frac{a_{1,n}^2}{A_1^2} = 1 + \frac{A_{3-i}}{A_i}\left\{K_i^{+}(n,1) - K_{3-i}^{+}(n+i-2,1)\right\} =$$

$$= 1 + \frac{A_{3-i}}{A_i}\left\{K_{3-i}^{-}(n+i-1,1) - K_i^{-}(n+1,-1)\right\},$$

$$K_i^{\pm}(n,m)=O\left(\sum_{l=n+\left[\frac{m}{2}\right]}^{\pm\infty}\left\{\left|a_{1,e}-A_1\right|+\left|a_{2,l}-A_2\right|\right\}\right),\ n+m\to\pm\infty. \tag{4}$$

For proving theorem substituting the expression (3) in (1), the equations for the kernels $K_i^{\pm}(n,m)$ are obtained. Solving these equalities by method of successive approximations we'll prove the property (4).

From (3) it follows that for $\lambda\in\partial\Gamma$, $\lambda^2\neq(A_1\pm A_2)^2$, the solutions $f_n^{\pm}$ are connected among themselves by relations

$$f_n^-(\lambda)=a(\lambda)\overline{f_n^+(\lambda)}+b(\lambda)f_n^+(\lambda), \tag{5}$$

$$f_n^+(\lambda)=a(\lambda)\overline{f_n^-(\lambda)}-b(\lambda)f_n^+(\lambda), \tag{6}$$

where

$$\alpha(\lambda)=\frac{\lambda W\left[f_n^+(\lambda),f_n^-(\lambda)\right]}{A_1A_2\left(z^{-1}-z\right)},\qquad b(\lambda)=\frac{\lambda W\left[\overline{f_n^+(\lambda)},f_n^-(\lambda)\right]}{A_1A_2\left(z^{-1}-z\right)} \tag{7}$$

here $W\left[f_n^+,f_n^-\right]=a_{1,n}\left(f_{1,n}^+f_{2,n}^- - f_{2,n}^+f_{1,n}^-\right)$ is a wronscian solution of $f_n^{\pm}$.

According to (7) the function $a(\lambda)$ allows an analytical extension in the plane Γ and

$$a(\lambda)\to\alpha_{1,n}^+,\alpha_{1,n}^->0. \tag{8}$$

By a standard method as in [1-3] it is proved that the function $a(\lambda)$ has a finite number of prime real zeros $\pm\lambda_k$, $\lambda_k>0$ $\quad(k=1...N)$. In these zeros the relations

$$f_n^-(\pm\lambda_k)=c_kf_n^+(\pm\lambda_k),$$
$$A_1A_2\left[\left(z^{-1}-z\right)\frac{da(\lambda)}{d\lambda}\right]\Bigg|_{\lambda=\pm_k}=c_k\left(m_k^+\right)^{-2}=c_k^{-1}\left(m_k^-\right)^2, \tag{9}$$

where $r^{\pm}(\lambda)$ are valid.

Let introduce the coefficients $r^+(\lambda)$ assuming

$$r^+(\lambda)=\frac{b(\lambda)}{a(\lambda)},\quad r^-(\lambda)=-\frac{\overline{b(\lambda)}}{a(\lambda)}.$$

Taking into consideration (5)-(6) we'll find that

$$1-\left|r^{\pm}(\lambda)\right|^2=\left|a(\lambda)\right|^{-2}.$$

A set of quantities $\left\{r^-(\lambda),\pm\lambda_k,m_k^-,k=1,...,N\right\}$ and $\left\{r^+(\lambda),\pm\lambda_k,m_k^+,k=1,...,N\right\}$ is called left and right scattering data of the system of equations (1) respectively. The inverse problem of the scattering theory for this system of equations is the restoration of the coefficients $a_{i,n}(i=1,2)$ on the left or right scattering data.

In solution of the inverse problem Gelphand-Levitan type basic equations play a significant role.

Multiply from the left both sides of equalities (5) and (6) by

$$\frac{1}{2\pi i}\begin{pmatrix} \frac{A_1 + A_2 z}{\lambda} \\ 1 \end{pmatrix}\left\{A_1 A_2\left(z^{-1} - z\right)a(\lambda)\right\}^{-1} z^{-n+m},$$

$$\frac{1}{2\pi i}\begin{pmatrix} \frac{A_1 z + A_2}{\lambda z} \\ 1 \end{pmatrix}\left\{A_1 A_2\left(z^{-1} - z\right)a(\lambda)\right\}^{-1} z^{n+m}, \quad \text{and}$$

respectively, and integrate on $\partial\Gamma$. Using the theorem on residues and the relations (3), (8), (9) we'll obtain that the kernels $K_i^{\pm}(n,m)$ satisfy the equations

$$F_i^{\pm}(2n+m) + K_i^{\pm}(n,m) + \sum_{l=\pm 1}^{\pm\infty} F_i^{\pm}(2n+m+l)K_i^{\pm}(n,l) = 0, \tag{10}$$

$$\left(\alpha_{i,n}^{\pm}\right)^{-2} = 1 + F_i^{\pm}(2n) + \sum_{l=\pm 1}^{\pm\infty} F_i^{\pm}(2n+l)K_i^{\pm}(n,l), \tag{11}$$

where the quantities $F_i^{\pm}(n)$ are completely determined by the scattering data

$$F_i^{\pm}(n) = 2\sum_{k=1}^{N}\left(m_k^{\pm}\right)^2\left[\left(\frac{A_1 z^{\pm 1} + A_2}{\lambda z^{\pm 1}}\right)^{4-2i} z^{\pm n}\right]\Bigg|_{\lambda=\lambda_k} +$$

$$+\frac{1}{2\pi i}\int_{\partial\Gamma}\frac{\lambda r^{\pm}(\lambda)}{A_1 A_2\left(z^{-1} - z\right)}\left(\frac{A_1 z^{\pm 1} + A_2}{\lambda z^{\pm 1}}\right)^{4-2i} z^{\pm n} d\lambda .$$

Basic equations (10), (12) allows to solve the inverse problem. At first $K_i^{\mp}(n,m)$ are found from the equations (10). Then the coefficients $a_{i,n}$ $(i=1,2)$ are determined by the formula (4).

From above mentioned reasoning the next basic theorem yields.

For the set of quantities $\left\{r^{+}(\lambda),\ \lambda \in \partial\Gamma;\ \pm\lambda_k, \lambda_k \notin \partial\Gamma;\ m_k^{+}, m_k^{+} > 0,\ k = 1 \ldots N\right\}$ to be the scattering data of some systems of equations (1) with the coefficients $a_{i,n}$ $(i=1,2)$ satisfying the conditions (2), it is necessary and sufficient that the next conditions be fulfilled:

1) The function $z^{+}(\lambda)$ is continuous at $\lambda \in \partial K$ and

$$r^{+}(\lambda - i0) = \overline{r^{+}(\lambda + i0)} = r^{+}(-\lambda + i0),\ (A_1 - A_2)^2 \le \lambda^2 \le (A_1 + A_2)^2,$$
$$-1 \le r^{+}(A_1 \pm A_2) < 1;$$

2) The quantities

$$R_i^{+}(n) = \frac{1}{2\pi i}\int_{\partial\Gamma}\frac{\lambda r^{+}(\lambda)}{A_1 A_2\left(z - z^{-1}\right)}\left(\frac{A_1 z + A_2}{\lambda z}\right)^{4-2i} z^{n} d\lambda$$

for all $N > \infty$ satisfy the conditions

$$\sum_{m=N}^{\infty} |n|\left|R_i^{+}(n+2) - R_j^{+}(n)\right| < \infty, \quad i,j = 1,2;$$

3) The function $\left(z - z^{-1}\right)a(\lambda)$ where

$$a(\lambda)=\exp\left\{-\frac{i}{4\pi A_1A_2}\int_{\partial\Gamma}\frac{\ln\left(1-\left|r^+(\mu)\right|^2\right)}{z(\mu)-z^{-1}(\mu)}\frac{z(\mu)+z(\lambda)}{z(\mu)-z(\lambda)}\mu d\mu\right\}\prod_{k=1}^{N} sign\, z(\lambda_k)\frac{z(\lambda)-z(\lambda_k)}{z(\lambda_k)z(\lambda)-1}$$

is continuous in the plane Γ up to its boundaries $\partial\Gamma$;

4) The quantities

$$R_i^-=-\frac{1}{2\pi i}\int_{\partial\Gamma}\frac{\lambda\overline{z^+(\lambda)}}{A_1A_2\left(z-z^{-1}\right)}\frac{\overline{a(\lambda)}}{a(\lambda)}\left(\frac{A_1+A_2z}{\lambda}\right)^{4-i}z^{-n}d\lambda$$

for all $N<\infty$ satisfy the conditions

$$\sum_{n=-\infty}^{\infty}|n|\left|R_i^-(n+2)-R_j^-(n)\right|<\infty,\quad i,j=1,2.$$

References

[1]. Азимова Г.М., Гусейнов И.М. *Прямая и обратная задачи теории рассеяния для системы разностных уравнений первого порядка* // Изв. АН Аз.ССР, с. физ.-тех. и мат. наук, 1987, №3, с.3-8.

[2]. Мамедов Х.Р. *Обратная задача теории рассеяния для системы разностных уравнений первого на всей оси* // Прикладные задачи функционального анализа. Баку, АГУ, 1987, с.57-65.

[3]. Khanmamedov A.Kh. *Scattering problem for difference equations* // Transactions of AS Azerbaijan, ser. of phys.-tech. and math. sciences, 1998, v.18, №3-4, p.86-92.

Khanmamedov Ag.Kh.
Baku State University named after M. Rasulzadeh.
23, Z.I. Khalilov str., 370148, Baku, Azerbaijan.

Received April 17, 1998; Revised October 27, 2000.
Translated by Mirzoyeva K.S.