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HORIZONTAL LIFTS OF TENSOR FIELDS OF TYPE (1,1)

Abstract

The purpose of this paper is to study the behaviour along cross-sections of horizontal lifts of tensor fields of type $(1,1)$ from manifold to its tensor bundle.

1. Introduction.

Let M_n be a differentiable manifold of class C^∞ and finite dimension n, and let $T_q^p(M_n)$, $p+q>0$ be the bundle over M_n of tensor of type (p,q): $T_q^p(M_n)=\bigcup\limits_{P\in M_n} T_q^p(P)$, where $T_q^p(P)$ denotes the vector (tensor) spaces of tensor of type (p,q) at $P\in M_n$. The purpose of this paper is to study the behaviour along cross-sections of the horizontal lifts of tensor fields of type $(1,1)$ from a manifold M_n to its tensor bundle $T_q^p(M_n)$.

We list below notations used in this paper.

1. $\pi : T_q^p(M_n) \mapsto M_n$ is the projection $T_q^p(M_n)$ onto M_n.
2. The indices $i,j,k,...$ run from 1 to n, the indices $\bar{i},\bar{j},\bar{k},...$ from $n+1$ to $n+n^{p+q}$ and the indices $I=(i,\bar{i})$, $J=(j,\bar{j})$, $K=(k,\bar{k}),...$ from 1 to $n+n^{p+q}$.
3. $\mathfrak{I}(M)$ is the ring of real-valued C^∞ functions on M_n. $\mathfrak{T}_q^p(M_n)$ is the vector (tensor) space of C^∞ tensor field of type (p,q) over the real number R (of infinite dimensions). We may also regard $\mathfrak{T}_q^p(M_n)$ as a module over $\mathfrak{I}(M)$.
4. Vector fields in M_n are denoted by $V,W,...$. Tensor field of type $(1,1)$ is denoted by φ.

2. Horizontal lifts of vector fields on a cross-section.

Denoting by x^j the local coordinates of $P=\pi(\widetilde{P})\left(\widetilde{P}\in T_q^p(M_n)\right)$ in a neighborhood $U\subset M_n$ and if we make $\left(x^j, t_{j_1\ldots j_q}^{i_1\ldots i_p}\right)=\left(x^j, x^{\bar{j}}\right)$ correspond to the point $\widetilde{P}\in\pi^{-1}(U)$, we can introduce a system of local coordinates $\left(x^j,x^{\bar{j}}\right)$ in a neighborhood $\pi^{-1}(U)\subset T_q^p(M_n)$, where $t_{j_1\ldots j_q}^{i_1\ldots i_p}\overset{def}{=}x^{\bar{j}}$ are components of $t\in T_q^p(P)$ with respect to the natural frame ∂_i.

If $\alpha\in\mathfrak{T}_p^q(M_n)$, it is regarded, in a natural way (by contraction), as a function in $T_q^p(M_n)$, which we denote by $i\alpha$. If α has the local expression $\alpha=\alpha_{i_1\ldots i_p}^{j_1\ldots j_q}\partial_{j_1}\otimes\ldots\otimes\partial_{j_q}\otimes dx^{i_1}\otimes\ldots\otimes dx^{i_p}$ in a coordinate neighborhood $U(x^i)\subset M_n$, then $i\alpha$ has the local expression

$$i\alpha=\alpha(t)=\alpha_{i_1\ldots i_p}^{j_1\ldots j_q}t_{j_1\ldots j_q}^{i_1\ldots i_p}$$

with respect to the coordinates $\left(x^j, x^{\bar{j}}\right)$ in $\pi^{-1}(U)$.

Suppose that $A \in \mathfrak{T}_q^p(M_n)$. We define the vertical lift $A \in \mathfrak{T}_0^1\left(T_q^p(M_n)\right)$ of A to $T_p^q(M_n)$ (see [1]) by

$$^V A(i\alpha) = \alpha(A) \circ \pi = ^V(\alpha(A)), \tag{2.1}$$

where $^V(\alpha(A))$ is the vertical lift of the function $\alpha(A) \in \mathfrak{I}(M_n)$. We note that, the vertical lift $^V(\alpha(A))$ of the arbitrary function $f \in \mathfrak{I}(M_n)$ is constant along any fibre $\pi^{-1}(P)$.

If $^V A = ^V A^k \partial_k + ^V A^{\bar{k}} \partial_{\bar{k}}$, $x^{\bar{k}} \overset{def}{=} t_{k_1 \ldots k_q}^{i_1 \ldots i_p}$, then we have from (2.1)

$$^V A^k t_{j_1 \ldots j_q}^{i_1 \ldots i_p} \partial_k \alpha_{i_1 \ldots i_p}^{j_1 \ldots j_q} + ^V A^{\bar{k}} \alpha_{l_1 \ldots l_p}^{k_1 \ldots k_q} = \alpha_{l_1 \ldots l_p}^{k_1 \ldots k_q} A_{k_1 \ldots k_q}^{l_1 \ldots l_p}.$$

But $\alpha_{l_1 \ldots l_p}^{k_1 \ldots k_q}$ and $\partial_k \alpha_{i_1 \ldots i_p}^{j_1 \ldots j_q}$ can take any preassigned values as each point. Thus, we have from the equation above

$$^V A^k t_{j_1 \ldots j_q}^{i_1 \ldots i_p} = 0, \quad ^V A^{\bar{k}} = A_{k_1 \ldots k_q}^{l_1 \ldots l_p}.$$

Hence

$$^V A^k = 0,$$

at all points of $T_q^p(M_n)$ except possibly those at all the components $x^{\bar{j}} = t_{j_1 \ldots j_q}^{i_1 \ldots i_p}$ are zero: that is, at points of the base space. Thus we see that the components $^V A^k$ are zero a point such that $x^{\bar{j}} \neq 0$, that is, in $T_q^p(M_n) - M_n$. However, $T_q^p(M_n) - M_n$ is dense in $T_q^p(M_n)$ and the components of $^V A$ are continuous at every point of $T_q^p(M_n)$. Hence, we have $^V A^k = 0$ at all points of $T_q^p(M_n)$. Thus, the vertical lift $^V A$ of A to $T_q^p(M_n)$ has components

$$^V A = \begin{pmatrix} ^V A^j \\ ^V A^{\bar{j}} \end{pmatrix} = \begin{pmatrix} 0 \\ A_{j_1 \ldots j_q}^{i_1 \ldots i_p} \end{pmatrix} \tag{2.2}$$

with respect to the coordinates $\left(x^j, x^{\bar{j}}\right)$ in $T_q^p(M_n)$.

Suppose that ∇ is an affine connection (with zero torsion) on M_n. Let ∇_V be the covariant differntiation with respect to $V \in \mathfrak{T}_0^1(M_n)$. We define the horizontal lift $^H V = \overline{\nabla}_X$ of V to $T_q^p(M_n)$ [1] by

$$^H V(i\alpha) = i(\nabla_V \alpha), \alpha \in \mathfrak{T}_p^q(M_n). \tag{2.3}$$

If $^H V = ^H V \partial_k + ^H V^{\bar{k}} \partial_{\bar{k}}$, then we have from (2.3):

$$^H V^k t_{j_1 \ldots j_q}^{i_1 \ldots i_p} \partial_k \alpha_{i_1 \ldots i_p}^{j_1 \ldots j_q} + ^H V^{\bar{k}} \alpha_{l_1 \ldots l_p}^{k_1 \ldots k_q} = t_{j_1 \ldots j_q}^{i_1 \ldots i_p} \Bigg(V^k \partial_k \alpha_{i_1 \ldots i_p}^{j_1 \ldots j_q} +$$

$$+ \sum_{\mu=1}^{q} V^m \Gamma_{ms}^{j_\mu} \alpha_{i_1 \ldots i_p}^{j_1 \ldots s \ldots j_q} - \sum_{\lambda=1}^{p} V^m \Gamma_{mi_\lambda}^{s} \alpha_{i_1 \ldots s \ldots i_p}^{j_1 \ldots j_q} \Bigg) = V^k t_{j_1 \ldots j_q}^{i_1 \ldots i_p} \partial_k \alpha_{i_1 \ldots i_p}^{j_1 \ldots j_q} +$$

$$+V^{m}\left(\sum_{\mu=1}^{q}\Gamma_{mk_{\mu}}^{s}t_{k_1\ldots s\ldots k_q}^{l_1\ldots l_p}-\sum_{\lambda=1}^{p}\Gamma_{ms}^{l_\lambda}t_{k_1\ldots k_q}^{l_1\ldots s\ldots l_p}\right)\alpha_{l_1\ldots l_p}^{k_1\ldots k_q}, \tag{2.4}$$

where Γ_{ij}^{k} are components of ∇ with respect to the local coordinates in $U\subset M_n$.

Thus, discussing in the same way as in the case of the vertical lift, from (2.4) we see that, ${}^{H}V$ has components

$${}^{H}V^{k}=V^{k},\ {}^{H}V^{\bar{k}}=V^{m}\left(\sum_{\mu=1}^{q}\Gamma_{mk_{\mu}}^{s}t_{k_1\ldots s\ldots k_q}^{l_1\ldots l_p}-\sum_{\lambda=1}^{p}\Gamma_{ms}^{l_\lambda}t_{k_1\ldots k_q}^{l_1\ldots s\ldots l_p}\right) \tag{2.5}$$

with respect to the coordinates $\left(x^{k},x^{\bar{k}}\right)$ in $T_q^p(M_n)$.

If we put $p=1,\ q=0$ $(p=0,\ q=1)$, then ${}^{H}V^{K}$ are the components of the horizontal lift of V from a manifold M_n to its tangent (cotangent) bundle [2], [4, p.87] ([3], [4, p.276]).

Suppose that there is given a tensor field $\xi\in\mathcal{T}_q^p(M_n)$. Then the correspondence $X\mapsto\xi_X$, ξ_X being the value of ξ at $X\in M_n$, determines a mapping $\sigma_\xi:M_n\mapsto T_q^p(M_n)$, such that $\pi\circ\sigma_\xi=id_{M_n}$, and the n dimensional submanifold $\sigma_\xi(M_n)$ of $T_q^p(M_n)$ is called the cross-section determined by ξ. If the tensor field ξ has the local components $\xi_{k_1\ldots k_q}^{l_1\ldots l_p}\left(x^{k}\right)$, the cross-section $\sigma_\xi(M_n)$ is locally expressed by

$$\begin{cases}x^{k}=x^{k}\\ x^{\bar{k}}=\xi_{k_1\ldots k_q}^{l_1\ldots l_p}\left(x^{k}\right)\end{cases} \tag{2.6}$$

with respect to the coordinates $\left(x^{k},x^{\bar{k}}\right)$ in $T_q^p(M_n)$. Differentiating (2.6) by x^{j}, we see that the n tangent vector fields B_j to $\sigma_\xi(M_n)$ have components

$$\left(B_j^{K}\right)=\left(\frac{\partial x^{K}}{\partial x^{j}}\right)=\begin{pmatrix}\delta_j^{k}\\ \partial_j\xi_{k_1\ldots k_q}^{l_1\ldots l_p}\end{pmatrix}, \tag{2.7}$$

with respect to the natural frame $\{\partial_k,\partial_{\bar{k}}\}$ in $T_q^p(M_n)$.

On the other hand, the fibre is locally expressed by

$$\begin{cases}x^{k}=const,\\ t_{k_1\ldots k_q}^{l_1\ldots l_p}=t_{k_1\ldots k_q}^{l_1\ldots l_p},\end{cases}$$

$t_{k_1\ldots k_q}^{l_1\ldots l_p}$ being consider as parameters. Thus, on differentiating with respect to $x^{\bar{j}}=t_{j_1\ldots j_q}^{i_1\ldots i_p}$, we see that the n^{p+q} tangent vector fields $C_{\bar{j}}$ to the fibre have components

$$\left(C_{\bar{j}}^{K}\right)=\left(\frac{\partial x^{K}}{\partial x^{\bar{j}}}\right)=\begin{pmatrix}0\\ \delta_{i_1}^{l_1}\ldots\delta_{i_p}^{l_p}\delta_{k_1}^{j_1}\ldots\delta_{k_q}^{j_q}\end{pmatrix} \tag{2.8}$$

with respect to the natural frame $\{\partial_k,\partial_{\bar{k}}\}$ in $T_q^p(M_n)$.

We consider in $\pi^{-1}(U)\subset T_q^p(M_n)$, $n+n^{p+q}$ local vector fields B_j and $C_{\bar{j}}$ along $\sigma_\xi(M_n)$. They form a local family of frames $\{B_j,C_{\bar{j}}\}$ along $\sigma_\xi(M_n)$, which is called the adapted (B,C)-frame of $\sigma_\xi(M_n)$ in $\pi^{-1}(U)$. From ${}^H V = {}^H V^k \partial_k + {}^H V^{\bar{k}} \partial_{\bar{k}}$ and ${}^H V = {}^H \widetilde{V}^j B_j + {}^H \widetilde{V}^{\bar{j}} C_{\bar{j}}$, we easily obtain ${}^H V^k = {}^H \widetilde{V}^j B_j^k + {}^H \widetilde{V}^{\bar{j}} C_{\bar{j}}^k$, ${}^H V^{\bar{k}} = {}^H \widetilde{V}^j B_j^{\bar{k}} + {}^H \widetilde{V}^{\bar{j}} C_{\bar{j}}^{\bar{k}}$. Now, taking account of (2.5) on the cross-section ξ, and also (2.7) and (2.8), we have ${}^H \widetilde{V}^j = {}^H V^j = V^j$, ${}^H \widetilde{V}^{\bar{j}} = -\nabla_V \xi_{j_1 \ldots j_q}^{i_1 \ldots i_p}$.

Thus, ${}^H V$ has along $\sigma_\xi(M_n)$ components of the form

$$ {}^H V = \begin{pmatrix} {}^H \widetilde{V}^j \\ {}^H \widetilde{V}^{\bar{j}} \end{pmatrix} \tag{2.9} $$

with respect tot he adapted (B,C)-frame.

3. Horizontal lifts of affinor fields on a pure cross-section.

Let $\varphi \in \mathfrak{T}_1^1(M_n)$. We define a tensor field ${}^H \varphi \in \mathfrak{T}_1^1(T_q^p(M_n))$ along the cross-section $\sigma_\xi(M_n)$ by

$$ \begin{cases} {}^H \varphi({}^H V) = {}^H(\varphi(V)), \forall V \in \mathfrak{T}_0^1(M_n), \\ {}^H \varphi({}^V A) = {}^V(\varphi(A)), \forall A \in \mathfrak{T}_q^p(M_n), \end{cases} \tag{3.1} $$

where $\varphi(A) = C(\varphi \otimes A) \in \mathfrak{T}_q^p(M_n)$ and call ${}^H \varphi$ the horizontal lift of $\varphi \in \mathfrak{T}_1^1(M_n)$ to $T_q^p(M_n)$ along $\sigma_\xi(M_n)$.

From (2.2), (2.7), (2.8) and ${}^V A = {}^V \widetilde{A}^j B_j + {}^V \widetilde{A}^{\bar{j}} C_{\bar{j}}$, we easily obtain ${}^V \widetilde{A}^j = 0$, ${}^V \widetilde{A}^{\bar{j}} = {}^V A^{\bar{j}} = A_{j_1 \ldots j_q}^{i_1 \ldots i_p}$. Thus the vertical lift ${}^V A$ also has components of the form (2.2) with respect to the adapted (B,C) - frame of $\sigma_\xi(M_n)$.

Let ${}^H \widetilde{\varphi}_L^K$ be components of ${}^H \varphi$ with respect to the adapted (B,C) - frame of the cross-section $\sigma_\xi(M_n)$. Then, from (3.1) we have

$$ \begin{cases} {}^H \widetilde{\varphi}_L^K \, {}^H \widetilde{V}^L = {}^H (\widetilde{\varphi}(V))^K, & (i) \\ {}^H \widetilde{\varphi}_L^K \, {}^V \widetilde{A}^L = {}^V (\widetilde{\varphi}(A))^K, & (ii) \end{cases} \tag{3.2} $$

where ${}^V(\varphi(A)) = \begin{pmatrix} 0 \\ {}^V(\varphi \widetilde{(} A))^{\bar{K}} \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi_m^{i_1} A_{k_1 \ldots k_q}^{m i_2 \ldots i_p} \end{pmatrix}$.

Since the horizontal lift ${}^H V$ is projectable, and so is ${}^H \varphi$ by virtue of (3.2) (i). Then ${}^H \varphi$ has components (see [5])

$$ {}^H \widetilde{\varphi}_l^k = \varphi_l^k, \quad {}^H \widetilde{\varphi}_{\bar{l}}^k = 0 \tag{3.3} $$

with respect to the adapted (B,C) - frame, which is also projectable. Therefore, in the case $K = k$ we get from (i) of (3.2) the identity $\varphi_l^k = \varphi_l^k$.

When $K = k$, (ii) of (3.2) can be rewritten, by virtue of (2.2) and (3.3), as $0 = 0$.

When $K = \bar{k}$, (ii) of (3.2) reduces to

$$ {}^H \widetilde{\varphi}_l^{\bar{k}} \, {}^V \widetilde{A}^l + {}^H \widetilde{\varphi}_{\bar{l}}^{\bar{k}} \, {}^V \widetilde{A}^{\bar{l}} = {}^V \left(\varphi \widetilde{(} A)\right)^{\bar{k}} $$

or

$$^{H}\widetilde{\varphi}_{\bar{I}}^{\bar{K}} A_{r_1\ldots r_q}^{s_1\ldots s_p} = \varphi_m^{l_1} A_{k_1\ldots k_q}^{m l_2\ldots l_p} = \varphi_{s_1}^{l_1}\delta_{s_2}^{l_2}\ldots\delta_{s_p}^{l_p}\delta_{k_1}^{r_1}\ldots\delta_{k_q}^{r_q} A_{r_1\ldots r_q}^{s_1\ldots s_p},$$

for all $A \in \mathfrak{T}_q^p(M_n)$, which implies

$$^{H}\widetilde{\varphi}_{\bar{I}}^{\bar{K}} = \varphi_{s_1}^{l_1}\delta_{s_2}^{l_2}\ldots\delta_{s_p}^{l_p}\delta_{k_1}^{r_1}\ldots\delta_{k_q}^{r_q}, \quad p \geq 1, \tag{3.4}$$

where $x^{\bar{I}} = t_{r_1\ldots r_q}^{s_1\ldots s_p}$, $x^{\bar{K}} = t_{k_1\ldots k_q}^{l_1\ldots l_p}$.

Similarly, we have

$$^{H}\widetilde{\varphi}_{\bar{I}}^{\bar{K}} = \delta_{s_1}^{l_1}\ldots\delta_{s_p}^{l_p}\varphi_{k_1}^{r_1}\delta_{k_2}^{r_2}\ldots\delta_{k_q}^{r_q}, \quad q \geq 1. \tag{3.4'}$$

When $K = \bar{k}$, (i) of (3.2) reduces to

$$^{H}\widetilde{\varphi}_{l}^{\bar{K}}\ {}^{H}\widetilde{V}^{l} + {}^{H}\widetilde{\varphi}_{\bar{I}}^{\bar{K}}\ {}^{H}\widetilde{V}^{\bar{I}} = {}^{H}\left(\varphi\widetilde{(V)}\right)^{\bar{K}}. \tag{3.5}$$

We will investigate components $^{H}\widetilde{\varphi}_{l}^{\bar{K}}$.

Let $\xi \in \mathfrak{T}_q^p(M_n)$. We consider the Vishnevskii operator

$$\left(\Phi_{\varphi}\xi\right)_{lk_1\ldots k_q}^{l_1\ldots l_p} = \varphi_l^m \nabla_m \xi_{k_1\ldots k_q}^{l_1\ldots l_p} - \begin{cases} \varphi_m^{l_1}\nabla_l \xi_{k_1\ldots k_q}^{m l_2\ldots l_p}, & p \geq 1, \\ \varphi_{k_1}^{m}\nabla_l \xi_{m k_2\ldots k_q}^{l_1\ldots l_p}, & q \geq 1 \end{cases} \tag{3.6}$$

Remark 1. Let φ be an integrable φ-structure in M_n and $\nabla\varphi = 0$. If $\xi \in \mathfrak{T}_q^p(M_n)$ be a pure tensor field with respect to φ-structure, i.e.

$$\varphi_m^{l_1}\xi_{k_1\ldots k_q}^{m l_2\ldots l_p} = \ldots = \varphi_m^{l_p}\xi_{k_1\ldots k_q}^{l_1\ldots m} = \varphi_{k_1}^{m}\xi_{m k_2\ldots k_q}^{l_1\ldots l_p} = \ldots = \varphi_{k_q}^{m}\xi_{k_1\ldots m}^{l_1\ldots l_p} = \overset{*}{\xi}{}_{k_1\ldots k_q}^{l_1\ldots l_p},,$$

then equation $\left(\Phi_{\varphi}\xi\right)_{lk_1\ldots k_q}^{l_1\ldots l_p} = 0$ is the condition for ξ to be holomorphic (see [6, p.184]).

From (3.6), we have

$$V^l\left(\Phi_{\varphi}\xi\right)_{lk_1\ldots k_q}^{l_1\ldots l_p} + \varphi_l^{l_1}\nabla_V \xi_{k_1\ldots k_q}^{l l_2\ldots l_p} = \nabla_{\varphi(V)}\xi_{k_1\ldots k_q}^{l_1\ldots l_p}, \quad p \geq 1 \tag{3.7}$$

Using (2.9), from (3.7) we have

$$V^l\left(\Phi_{\varphi}\xi\right)_{lk_1\ldots k_q}^{l_1\ldots l_p} + \varphi_l^{l_1}\nabla_V \xi_{k_1\ldots k_q}^{l l_2\ldots l_p} = V^l\left(\Phi_{\varphi}\xi\right)_{lk_1\ldots k_q}^{l_1\ldots l_p} + \varphi_{s_1}^{l_1}\delta_{s_2}^{l_2}\ldots\delta_{s_p}^{l_p}\delta_{k_1}^{r_1}\ldots\delta_{k_q}^{r_q}$$

$$\nabla_V \xi_{r_1\ldots r_q}^{s_1\ldots s_p} = \left(\Phi_{\varphi}\xi\right)_{lk_1\ldots k_q}^{l_1\ldots l_p}\ {}^{H}\widetilde{V}^{l} - \varphi_{s_1}^{l_1}\delta_{s_2}^{l_2}\ldots\delta_{s_p}^{l_p}\delta_{k_1}^{r_1}\ldots\delta_{k_q}^{r_q}\ {}^{H}\widetilde{V}^{\bar{I}} = -{}^{H}\left(\varphi\widetilde{(V)}\right)^{\bar{K}}$$

or

$$\left(\Phi_{\varphi}\xi\right)_{lk_1\ldots k_q}^{l_1\ldots l_p}\ {}^{H}\widetilde{V}^{l} - \varphi_{s_1}^{l_1}\delta_{s_2}^{l_2}\ldots\delta_{s_p}^{l_p}\delta_{k_1}^{r_1}\ldots\delta_{k_q}^{r_q}\ {}^{H}\widetilde{V}^{\bar{I}} = -{}^{H}\left(\varphi\widetilde{(V)}\right)^{\bar{K}}. \tag{3.8}$$

Comparing (3.5) and (3.8), and making use of (3.4), we get

$$\left({}^{H}\widetilde{\varphi}_{l}^{\bar{K}} + \left(\Phi_{\varphi}\xi\right)_{lk_1\ldots k_q}^{l_1\ldots l_p}\right)V^l = 0,$$

for any $V \in \mathfrak{T}_0^1(M_n)$, from which

$$^{H}\widetilde{\varphi}_{l}^{\bar{K}} = -\left(\Phi_{\varphi}\xi\right)_{lk_1\ldots k_q}^{l_1\ldots l_p}, \quad p \geq 1.$$

Similarly, we obtain

$$^{H}\widetilde{\varphi}_{l}^{\bar{K}} = -\left(\Phi_{\varphi}\xi\right)_{lk_1\ldots k_q}^{l_1\ldots l_p}, \quad q \geq 1.$$

Thus the horizontal lift $^{H}\varphi$ of φ has along the cross-section $\sigma_{\xi}(M_n)$ components

$$\begin{cases} {}^H\tilde{\varphi}_l^k = \varphi_l^k, \quad {}^H\tilde{\varphi}_{\bar{l}}^k = 0, \quad {}^H\tilde{\varphi}_l^{\bar{k}} = -\left(\Phi_\varphi \xi\right)_{lk_1\ldots k_q}^{l_1\ldots l_p} \\ {}^H\tilde{\varphi}_{\bar{l}}^{\bar{k}} = \begin{cases} \varphi_{s_1}^{l_1}\delta_{s_2}^{l_2}\ldots\delta_{s_p}^{l_p}\delta_{k_1}^{r_1}\ldots\delta_{k_q}^{r_q}, & p \geq 1, \\ \delta_{s_1}^{l_1}\ldots\delta_{s_p}^{l_p}\delta_{k_1}^{r_1}\delta_{k_2}^{r_2}\ldots\delta_{k_q}^{r_q}, & q \geq 1, \end{cases} \end{cases} \tag{3.9}$$

with respect to the adapted (B,C)-frame of $\sigma_\xi(M_n)$, where $\Phi_\varphi\xi$ is the Vishnevskii operator.

Remark 2. The formula (3.6) is valid if and only if $\Phi_\varphi\xi$ is the Vishnevskii operator, i.e. ${}^H\varphi$ in the form (3.9) is unique solution of (3.1). Therefore, if $\overset{*}{\varphi}$ is element of $\mathcal{T}_1^1(T_q^p(M_n))$, such that $\overset{*}{\varphi}({}^H V) = {}^H\varphi({}^H V) = {}^H(\varphi(V))$, $\overset{*}{\varphi}({}^V A) = {}^H\varphi({}^V A) = {}^V(\varphi(A))$, then $\overset{*}{\varphi} = {}^H\varphi$.

Theorem 3.1. *The horizontal lift* ${}^H : End\, M_n \mapsto End T_q^p(M_n)$ *along the cross-section* $\sigma_\xi(M_n)$ *is a monomorphism.*

Proof. The pecularity

$$\left(\Phi_{a\varphi_1+b\varphi_2}\xi\right)_{lk_1\ldots k_q}^{l_1\ldots l_p} = a\left(\Phi_{\varphi_1}\xi\right)_{lk_1\ldots k_q}^{l_1\ldots l_p} + b\left(\Phi_{\varphi_2}\xi\right)_{lk_1\ldots k_q}^{l_1\ldots l_p}, \ \forall a,b \in R$$

of the Vishnevskii operator and from (3.9), we find that, ${}^H : End\, M_n \mapsto End T_q^p(M_n)$ is a linear. From (3.1), we write

$$\begin{aligned} {}^H(\varphi\circ\psi)({}^H V) &= {}^H((\varphi\circ\psi)(V)) = {}^H(\varphi(\psi(V))) \\ &= {}^H\varphi({}^H\psi(V)) = {}^H\varphi({}^H\psi({}^H V)) = ({}^H\varphi\circ{}^H\psi)({}^H V), \\ {}^H(\varphi\circ\psi)({}^V A) &= {}^V((\varphi\circ\psi)(A)) = {}^V(\varphi(\psi(A))) \\ &= {}^H\varphi({}^V(\psi(A))) = {}^H\varphi({}^H\psi({}^V A)) = ({}^H\varphi\circ{}^H\psi)({}^V A). \end{aligned}$$

With respect to the Remark 2, we find

$${}^H(\varphi\circ\psi) = {}^H\varphi\circ{}^H\psi, \tag{3.10}$$

i.e. ^{H} is a homeomorphism. However, ${}^H\varphi = 0$ if and only if $\varphi = 0$, i.e. ^{H} is a monomorphism.

Let $\mathcal{A}_m$ be an associative commutative unital algebra of finite dimension m over the field R of real numbers. An algebraic Π-structure on M_n is a collection $\Pi = \left\{\underset{\alpha}{\varphi}\right\}$, $\alpha = 1,\ldots,m$ of tensor fields of type $(1,1)$ such that $\underset{\alpha}{\varphi}\circ\underset{\beta}{\varphi} = C_{\alpha\beta}^\gamma\underset{\gamma}{\varphi}$, where $C_{\alpha\beta}^\gamma$ are the structure constants of the algebra $\mathcal{A}_m$. From (3.10), we obtain ${}^H\underset{\alpha}{\varphi}\circ{}^H\underset{\beta}{\varphi} = {}^H\left(\underset{\alpha}{\varphi}\circ\underset{\beta}{\varphi}\right) = {}^H\left(C_{\alpha\beta}^\gamma\underset{\gamma}{\varphi}\right) = C_{\alpha\beta}^\gamma\,{}^H\underset{\gamma}{\varphi}$. Thus we have

Theorem 3.2. *If* $\Pi = \left\{\underset{\alpha}{\varphi}\right\}$, $\alpha = 1,\ldots,m$ *defines an algebraic* Π*-structure on* M_n, *so does* ${}^H\Pi = \left\{{}^H\underset{\alpha}{\varphi}\right\}$ *on* $T_q^p(M_n)$ *along the cross-section* $\sigma_\xi(M_n)$.

Now, on putting $B_{\bar{j}} = C_{\bar{j}}$, we write the adapted (B,C)-frame of $\sigma_\xi(M_n)$ as $B_J = \{B_j, B_{\bar{j}}\}$. We define a coframe $\widetilde{B}^J$ of $\sigma_\xi(M_n)$ by $\widetilde{B}^I(B_J) = \delta_J^I$. If $B_J = B_J^K \partial_K$, then we have

$$B_J^K \widetilde{B}_K^I = \delta_J^I, \tag{3.11}$$

where $\widetilde{B}_K^I = \widetilde{B}^I(\partial_K)$. From (2.7), (2.8) and (3.11), we see that covector fields $\widetilde{B}^I$ have components

$$\widetilde{B}^i = \left(\widetilde{B}_K^i\right) = \left(\delta_k^i, 0\right), \tag{3.12}$$
$$\widetilde{B}^{\bar{i}} = \left(\widetilde{B}_K^{\bar{i}}\right) = \left(-\partial_k \xi_{i_1 \ldots i_q}^{j_1 \ldots j_p}, \delta_{i_1}^{k_1} \ldots \delta_{i_q}^{k_q} \delta_{l_1}^{j_1} \ldots \delta_{l_p}^{j_p}\right)$$

with respect to the natural coframe $\{dx^k, dx^{\bar{k}}\}$.

Taking into account of representation ${}^H\varphi = {}^H\widetilde{\varphi}_I^J B_J \otimes \widetilde{B}^I$ and

$${}^H\varphi_L^K = {}^H\varphi\left(dx^K, \partial_L\right) = {}^H\widetilde{\varphi}_I^J B_J \otimes \widetilde{B}^I\left(dx^K, \partial_L\right) =$$
$$= {}^H\widetilde{\varphi}_I^J dx^K(B_J)\widetilde{B}^I(\partial_L) = {}^H\widetilde{\varphi}_I^J dx^K\left(B_J^H \partial_H\right)\widetilde{B}_L^I =$$
$$= {}^H\widetilde{\varphi}_I^J B_J^H \delta_H^K \widetilde{B}_L^I = {}^H\widetilde{\varphi}_I^J B_J^K \widetilde{B}_L^I,$$

and also (2.7), (2.8), (3.9) and (3.12), we have along $\sigma_\xi(M_n)$ the formulas

$${}^H\varphi_l^k = \varphi_l^k, \quad {}^H\varphi_{\bar{l}}^k = 0,$$
$${}^H\varphi_{\bar{l}}^{\bar{k}} = \begin{cases} \varphi_{s_1}^{l_1}\delta_{s_2}^{l_2}\ldots\delta_{s_p}^{l_p}\delta_{k_1}^{r_1}\ldots\delta_{k_q}^{r_q}, & p \geq 1, \\ \delta_{s_1}^{l_1}\ldots\delta_{s_p}^{l_p}\varphi_{k_1}^{r_1}\delta_{k_2}^{r_2}\ldots\delta_{k_q}^{r_q}, & q \geq 1. \end{cases}$$
$${}^H\varphi_l^{\bar{k}} = -\left(\Phi_\varphi \xi\right)_{lk_1\ldots k_q}^{l_1\ldots l_p} + \varphi_l^m \partial_m \xi_{k_1\ldots k_q}^{l_1\ldots l_p} - \begin{cases} \varphi_m^{l_1}\partial_l \xi_{k_1\ldots k_q}^{m\ldots l_p}, & p \geq 1 \\ \varphi_{k_1}^m \partial_l \xi_{m\ldots k_q}^{l_1\ldots l_p}, & q \geq 1. \end{cases}$$

Thus, ${}^H\varphi$ has along the cross-section $\sigma_\xi(M_n)$ components of the form

$$\begin{cases} {}^H\varphi_l^k = \varphi_l^k, \quad {}^H\varphi_{\bar{l}}^k = 0, \quad {}^H\varphi_{\bar{l}}^{\bar{k}} = \begin{cases} \varphi_{s_1}^{l_1}\delta_{s_2}^{l_2}\ldots\delta_{s_p}^{l_p}\delta_{k_1}^{r_1}\ldots\delta_{k_q}^{r_q}, & p \geq 1, \\ \delta_{s_1}^{l_1}\ldots\delta_{s_p}^{l_p}\varphi_{k_1}^{r_1}\delta_{k_2}^{r_2}\ldots\delta_{k_q}^{r_q}, & q \geq 1 \end{cases} \\ {}^H\varphi_l^{\bar{k}} = \varphi_l^m\left(-\sum_{\lambda=1}^{p}\Gamma_{ms}^{l_\lambda}\xi_{(k)}^{l_1\ldots s\ldots l_p} + \sum_{\mu=1}^{q}\Gamma_{mk_\mu}^{s}\xi_{k_1\ldots s\ldots k_q}^{(l)}\right) + \varphi_m^{l_1}\left(-\sum_{\mu=1}^{q}\Gamma_{lk_\mu}^{s}\right. \\ \left. + \xi_{k_1\ldots s\ldots k_q}^{ml_2\ldots l_p} + \sum_{\lambda=2}^{p}\Gamma_{ls}^{l_\lambda}\xi_{(k)}^{ml_2\ldots s\ldots l_p} + \Gamma_{ls}^{m}\xi_{(k)}^{s\ldots l_p}\right), \quad p \geq 1 \\ {}^H\varphi_l^{\bar{k}} = \varphi_l^m\left(-\sum_{\lambda=1}^{p}\Gamma_{ms}^{l_\lambda}\xi_{(k)}^{l_1\ldots s\ldots l_p} + \sum_{\mu=1}^{q}\Gamma_{mk_\mu}^{s}\xi_{k_1\ldots s\ldots k_q}^{(l)}\right) + \varphi_{k_1}^m\left(\sum_{\lambda=1}^{p}\Gamma_{ls}^{l_\lambda}\right. \\ \left. \xi_{mk_2\ldots k_q}^{l_1\ldots s\ldots l_p} - \sum_{\mu=2}^{q}\Gamma_{lk_\mu}^{s}\xi_{mk_2\ldots s\ldots k_q}^{(l)} - \Gamma_{lm}^{s}\xi_{m\ldots k_q}^{(l)}\right), \quad q \geq 1 \end{cases} \tag{3.13}$$

with respect to the natural frame $\{\partial_k, \partial_{\bar{k}}\}$ of $\sigma_\xi(M_n)$ in $\pi^{-1}(U) \subset T_q^p(M_n)$ [7].

In particular, if we put $p=1,\ q=0\ (p=0,\ q=1)$ in (3.13), then ${}^H\varphi_L^K$ are the components of the horizontal lift of φ from a manifold $\Re$ to its tangent (cotangent) bundle with respect to the natural frame $\{\partial_k, \partial_{\bar{k}}\}$ of $\sigma_\xi(M_n)$ [2], [4, p.94] ([3], [4], p.281).

4. On a new class of the quasi-$\mathcal{A}$-holomorphic tensor fields.

Let M_n and N_m be two manifolds with algebraic structures $\Pi = \{\underset{\alpha}{\varphi}\}$ and $\widetilde{\Pi} = \{\underset{\alpha}{\psi}\}$, $\alpha = 1, \dots, m$ determined by the same associative commutative unital algebra $\mathcal{U}_m$. A differentiable mapping $f: M_n \mapsto N_m$ is called a quasi-$\mathcal{A}$-holomorphic mapping with respect to $(\Pi, \widetilde{\Pi})$ (see [8]), if at each point $P \in M_n$

$$df_P \circ \underset{\alpha}{\varphi}_P = \underset{\alpha}{\psi}_{f(P)} \circ df_P, \ \alpha = 1, \dots, m. \tag{4.1}$$

As the mapping $f: M_n \mapsto N_m$ $(m = n + n^{p+q})$ we take a cross-section $\sigma_\xi^\Pi : M_n \mapsto T_q^p(M_n)$ determined by the pure tensor field $\xi \in \mathcal{T}_q^p(M_n)$ with respect to Π-structure. A pure cross-section $\sigma_\xi^\Pi : M_n \mapsto T_q^p(M_n)$ can be locally expressed by (2.6). In (4.1), if $\widetilde{\Pi} = \{\underset{\alpha}{\psi}\}$ is the algebraic ${}^H\Pi$-structure defined in §3, the condition that the pure cross-section $\sigma_\xi^\Pi : M_n \mapsto T_q^p(M_n)$ be quasi-$\mathcal{A}$-holomorphic tensor field with respect to $(\Pi, {}^H\Pi)$ is locally given by

$$\underset{\alpha}{\varphi}_l^m \partial_m x^K = {}^H\underset{\alpha}{\varphi}_M^K \partial_l x^M, \ \alpha = 1, \dots, m, \tag{4.2}$$

where ${}^H\underset{\alpha}{\varphi}_M^K$ are components of ${}^H\underset{\alpha}{\varphi}$ along the pure cross-section $\sigma_\xi^\Pi(M_n)$ with respect to the natural frame $\{\partial_k, \partial_{\bar{k}}\}$.

In the case $K = k$, by virtue of (2.6) and (3.13) we get the identity $\varphi_l^k = \varphi_l^k$. When $K = \bar{k}$, by virtue of (2.6) and (3.13), (4.2) reduces to

$$\underset{\alpha}{\varphi}_l^m \partial_m x^{\bar{k}} - {}^H\underset{\alpha}{\varphi}_M^{\bar{k}} \partial_l x^M = \underset{\alpha}{\varphi}_l^m \partial_m \xi_{(k)}^{(l)} - {}^H\underset{\alpha}{\varphi}_m^{\bar{k}} \delta_l^m -$$

$$- {}^H\underset{\alpha}{\varphi}_{\bar{m}}^{\bar{k}} \partial_l \xi_{(m)}^{(n)} = \underset{\alpha}{\varphi}_l^m \partial_m \xi_{k_1 \dots k_q}^{l_1 \dots l_p} - \underset{\alpha}{\varphi}_l^m \Big(- \sum_{\lambda=1}^{p} \Gamma_{ms}^{l_\lambda} \xi_{(k)}^{l_1 \dots s \dots l_p} +$$

$$+ \sum_{\mu=1}^{q} \Gamma_{mk_\mu}^{s} \xi_{k_1 \dots s \dots k_q}^{(l)} \Big) - \underset{\alpha}{\varphi}_m^{l_1} \Big(- \sum_{\mu=1}^{q} \Gamma_{lk_\mu}^{s} \xi_{k_1 \dots s \dots k_q}^{ml_2 \dots l_p} +$$

$$+ \sum_{\lambda=2}^{p} \Gamma_{ls}^{l_\lambda} \xi_{(k)}^{ml_2 \dots s \dots l_p} + \Gamma_{ls}^{m} \xi_{(k)}^{sl_2 \dots l_p} \Big) - \underset{\alpha}{\varphi}_{n_1}^{l_1} \delta_{n_2}^{l_2} \dots \delta_{n_p}^{l_p} \delta_{k_1}^{m_1} \dots \delta_{k_q}^{m_q}$$

$$\partial_l \xi_{m_1 \dots m_q}^{n_1 \dots n_p} = 0$$

or

$$\underset{\alpha}{\varphi}{}_l^m \nabla_m \xi_{k_1\ldots k_q}^{l_1\ldots l_p} - \underset{\alpha}{\varphi}{}_m^{l_1} \nabla_l \xi_{k_1\ldots k_q}^{m l_2\ldots l_p} = 0, \qquad p \geq 1. \tag{4.3}$$

Similarly, we obtain

$$\underset{\alpha}{\varphi}{}_l^m \nabla_m \xi_{k_1\ldots k_q}^{l_1\ldots l_p} - \underset{\alpha}{\varphi}{}_{k_1}^m \nabla_l \xi_{m k_2\ldots k_q}^{l_1\ldots l_p} = 0, \qquad q \geq 1. \tag{4.4}$$

Thus quasi-$\mathcal{A}$-holomorphic tensor field with respect to $(\Pi, {}^H\Pi)$ is given by (4.3) (or (4.4)).

Let Π-structure be an almost integrable structure with respect to the connection ∇ (with zero torsion), i.e. $\nabla_\varphi = 0$, $\forall \varphi \in \Pi$. Then, we obtain (see [9]

$$\left(\Phi_\varphi \xi\right)_{l k_1\ldots k_q}^{l_1\ldots l_p} = \left(\widetilde{\Phi}_\varphi \xi\right)_{l k_1\ldots k_q}^{l_1\ldots l_p},$$

where $\widetilde{\Phi}_\varphi \xi$ is the Tachibana operator [10]:

$$\left(\widetilde{\Phi}_\varphi \xi\right)_{l k_1\ldots k_q}^{l_1\ldots l_p} = \varphi_l^m \partial_m \xi_{k_1\ldots k_q}^{l_1\ldots l_p} - \partial_l \overset{*}{\xi}{}_{k_1\ldots k_q}^{l_1\ldots l_p} +$$

$$+ \sum_{a=1}^{q} \left(\partial_{k_a} \varphi_l^r\right) \xi_{k_1\ldots r\ldots k_q}^{l_1\ldots l_p} + \sum_{b=1}^{p} \left(\partial_l \varphi_r^{l_b} - \partial_r \varphi_l^{l_b}\right) \xi_{k_1\ldots k_q}^{l_1\ldots r\ldots l_p},$$

where $\overset{*}{\xi}{}_{k_1\ldots k_q}^{l_1\ldots l_p}$ is tensor field defined in Remark 1. The equation

$$\left(\widetilde{\Phi}_\varphi \xi\right)_{l k_1\ldots k_q}^{l_1\ldots l_p} = 0 \tag{4.5}$$

is the equation characterizing the usual almost holomorphic tensor field [10], [11]. Thus, if Π-structure is almost integrable, then our quasi-$\mathcal{A}$-holomorphic tensor field with respect to $(\Pi, {}^H\Pi)$ coincides with the usual almost holomorphic tensor field. In general, quasi-$\mathcal{A}$-holomorphic tensor field with respect to $(\Pi, {}^H\Pi)$ satisfying (4.3) (or (4.4) does not satisfy (4.5)).

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