M.ALI SARIGÖL, JAFAROV S.Z.

APPROXIMATION CLASSES OF FUNCTIONS ON R AND H TYPE CONTINUUMS

Abstract

In this paper the constructive description of continuous functions classes on continuums of a more general type complex plane is obtained.

The paper is devoted do the constructive description of continuous functions classes determined by local conditions in given continuums of a more general type complex plane.

In papers [5] and [8] using localized continuity module classes of functions $H^{z_0}_{\varphi,\Gamma}$ were considered. Note that when a continuum E belongs to the set of B^0_K type (see [3, p.440]) and in domains with a quasiconformal boundary for the class $H^{z_0}_{\varphi,\Gamma} \cap A(E)$ in paper [11] the direct local theorem has been obtained. The local problem is also solved in papers [7], [8] by J.I.Mamedkhanov.

A class of continuums R was introduced by V.V.Andriewsky. As we know [1], [2] a class of continuums R introduced by V.V.Andriewsky at present day is the widest from the classes of sets, on which the improved global estimate of polynomial approximation (in terms of the size δ_n) has been obtained. This class of sets is described by primitive geometric conditions. In order to obtain constructive characteristics of classes of functions H^{ω} in continuums of type H (see [1], [2]), V.V.Andriewsky uses S.B.Stechkin's [11] approximation characteristics.

We have obtained a localized direct theorem on polynomial approximation for the class $H^{z_0}_{\varphi,\Gamma} \cap A(E)$ in continuums of the class R with regard to the growth of derivatives of approximating polynomials. In the case when a continuum belongs to the class H, one can invert this theorem. Thus, on the sets of the class H we get the approximating characteristics of classes of functions $H^{z_0}_{\varphi,\Gamma} \cap A(E)$. The obtained theorems are the generalizations of some results by A.A.Musayev [1], J.I.Mamedkhanov [7], S.Z.Jafarov [16].

Main definitions and notations.

Let E be a finite continuum of a complex plane $\mathbf{C}$, $diam\,E > 0$, with simply connected component $\Omega = CE = \mathbf{C}|E$, where $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ and $\Gamma = \partial\Omega = \partial E$ is their common boundary. Consider the function $W = \phi(z)$, conformally and univalently mapping Ω onto $\Omega' = \{W : |W| > 1\}$ with normalization $\phi(\infty) = \infty$, $\lim\limits_{z\to\infty} \frac{1}{z}\phi(z) > 0$. Let $\psi = \phi^{-1}$, $\Gamma_R = \{\zeta : |\phi(\zeta)| = R\}$, $R > 1$.

For $\delta > 0$ put

$$U(z,\delta) = \{\zeta : |\zeta - z| < \delta\},\ d(z,E) = \inf_{\zeta\in E} |\zeta - z|,$$

$$E_\delta = \bigcup_{z\in E} U(z,\delta) = \{\zeta : d(\zeta,E) < \delta\}.$$

Let Γ be a Jordan curve. Denote by $D_{\alpha}^{\beta}(z_0,\Gamma)$ ($z_0 \in \Gamma$ is fixed), $(0<\alpha\le 1,\ \beta \ge 0)$ a class of functions for which

$$|f(z_1)-f(z_2)| \le c_f \max\left\{|z_0-z_1|^{\beta}, |z_0-z_2|^{\beta}\right\}|z_1-z_2|^{\alpha}, \quad \forall z_1, z_2 \in \Gamma .$$

It is obvious that when $\beta = 0$ we obtain a class $H^{\alpha}(\Gamma)$.

By $A(E)$ we denote a class of continuous on E and analytic in its internal points functions.

Besides, in papers by J.I.Mamedkhanov and V.V.Salayev [5], R.M.Rzayev [18] it was considered a localized continuity modulus in the form

$$\omega_f^{z_0}(\delta,\eta) = \sup_{\substack{|z-\tau|\le\delta \\ z,\tau\in\Gamma_\eta(z_0)}} |f(z)-f(\tau)|, \tag{1}$$

where $\delta,\eta>0$ and $\Gamma_\eta(z_0)=\{z\in\Gamma, |z-z_0|\le\eta\}$.

It is obvious that if d is a diameter of Γ, then

$$\omega_f^{z_0}(\delta,d)=\omega(f,\delta)=\sup_{\substack{|z-\tau|\le\delta \\ z,\tau\in\Gamma_\eta(z_0)}} |f(z)-f(\tau)|,$$

where $\omega(f,\delta)$ is an ordinary continuity modulus f on Γ.

By Q denote a class of positive functions $\varphi(\delta,\eta)$ determined for $0<\delta,\eta<+\infty$ and such that

1) $\varphi(\delta,\eta)$ doesn't decrease on each argument,
2) $\varphi(\delta,\eta)\cdot\delta^{-1}$ doesn't increase on δ,
3) $\exists \eta \in R_+ : \lim\limits_{\delta\to 0}\varphi(\delta,\eta)=0$,
4) $\varphi(\delta,2\eta)\le c\varphi(\delta,\eta)$, the constant c doesn't depend on δ and η.

The class of functions $D_{\alpha}^{\beta}(z_0,\Gamma)$ in the term of the localized continuity modulus is described in the following form:

$$f\in D_{\alpha}^{\beta}(z_0,\Gamma) \Leftrightarrow \omega_f^{z_0}(\delta,\eta)=O(\delta^{\alpha}\cdot\eta^{\beta}), \quad \forall \delta,\eta : 0<\frac{\delta}{2}\le\eta\le d .$$

Definition 1. *Let* $\varphi\in Q$. *Let's denote*

$$H_{\varphi,\Gamma}^{z_0} \overset{df}{=} \left\{ f\in C(\Gamma) : \omega_f^{z_0}(\delta,\eta)\le C\varphi(\delta,\eta),\ \forall \delta,\eta : 0<\frac{\delta}{2}\le\eta \right\},$$

where $C(\Gamma)$ *is designated a class of continuous functions on* Γ.

Definition 2 [1], [2]. *We will say that* $E\in R$, *if some* $C=C(E)>1$ *for the points* $z\in\Gamma=\partial E$ *and* $\delta>0$ *the relation*

$$U(z,c\delta)\cap CE_\delta \ne \phi , \tag{2}$$

where $CE_\delta=\overline{\mathbf{C}}|E_\delta$ *is complement of the domain* E_δ, *is satisfied.*

Let's assume

$$\delta(u)=\sup_{\zeta\in \operatorname{int}\Gamma_{1+u}} d(\zeta,E),$$

$$\delta_n \overset{df}{=} \delta\left(\frac{1}{n}\right), \quad n=1,2,...,$$

where by $\operatorname{int}\Gamma$ the finite domain is designated, whose boundary coincides with Γ (Γ is a Jordon curve).

Let's denote

$$G_n \overset{df}{=} \{\zeta : d(\zeta,\Omega) < \delta_n\},$$
$$D_{u_0} = \operatorname{int}\Gamma_{1+u_0},\ u_0 > 0.$$

Further, by $c, c_1, \ldots$ we will denote constants which in different relations, generally speaking, different. We will also use a symbol $A \preceq B$ meaning that $A \le cB$, where $c = const > 0$ and in different relations generally speaking different. Following [1], [2] we'll write $E \in H$, if any points z and $\zeta \in E$ can be connected by $\gamma(z,\zeta) \subset E$ with the property

$$mes\,\gamma(z,\zeta) \le c|z-\zeta|,\ c = c(E) \ge 1. \tag{3}$$

Now we'll reduce the basic results.

Theorem 1. *Let* $E \in R$, $\varphi \in Q$ *and* $f \in H^{z_0}_{\varphi,\Gamma} \cap A(E)$. *Then there exists a sequence of polynomials* $P_n(z)$ *of a degree no more than* n, *for which at* $z \in \Gamma = \delta E$ *and* $n = 1,2,\ldots$ *the inequalities*

$$|f(z) - P_n(z)| \le c_1 \varphi(\delta_n, \delta_n + |z - z_0|), \tag{4}$$
$$|P_n'(z)| \le c_2 \varphi(\delta_n, \delta_n + |z - z_0|) \cdot \delta_n^{-1}, \tag{5}$$

where the constants $c_j \ge 0$, $j = 1,2$ *don't depend on* z, z_0 *and* n, *are true.*

Theorem 2. *Let* $E \in H$. *Then in order that* $f(z) \in H^{z_0}_{\varphi,\Gamma} \cap A(E)$ *it is necessary and sufficient the existence of a sequence of polynomials* $P_n(z)$ *of a degree no more than* n *satisfying for* $z \in \Gamma = \partial E$ *the following inequalities*

$$|f(z) - P_n(z)| \le c_2 \varphi(\delta_n, \delta_n + |z - z_0|), \tag{6}$$
$$|P_n'(z)| \le c_3 \varphi(\delta_n, \delta_n + |z - z_0|) \cdot \delta_n^{-1}, \tag{7}$$

where the constants c_2, c_3 *don't depend on* z, z_0 *and* n.

Auxiliary facts.

For proving the basic results we need the following auxiliary results.

Lemma 1 [1]. *The imbedding* $H \subset R$ *holds.*

Lemma 2 [1]. *Let* $E \in R$, $u_0 > 0$ *be an arbitrary number. Then for any* $n = 1,2,\ldots$ *and* $\zeta \in G_n \cap D_{u_0}$ *there exists a polynomial kernel* $\pi_n(\zeta, z)$ *in the form*

$$\pi_n(\zeta,z) = \sum_{j=0}^{n} a_j(\zeta) z^j,\quad n = 1,2,\ldots \tag{8}$$

with summable by ζ *coefficients* $a_j(\zeta)$, $j = \overline{0,n}$ *for which at* $z \in E$ *and* $p = 0,1$ *the inequalities*

$$\left|\frac{\partial^p}{\partial z^p}\left[\frac{1}{\zeta - z} - \pi_n(\zeta,z)\right]\right| \le \frac{1}{|\zeta - z|^{p-1}}\left[\frac{\delta_n}{|\zeta - z| + \delta_n}\right]^2, \tag{9}$$
$$\left|\frac{\partial^p}{\partial z^p}\pi_n(\zeta,z)\right| \le (|\zeta - z| + \delta_n)^{-p-1}. \tag{10}$$

are fulfilled.

Let $f(z)\in H_{\varphi,\Gamma}^{z_0}\cap A(E)$. Then it can be continued on all extended complex plane $\mathbf{C}$, so that the condition

$$|f(z_1)-f(z_2)|\le c\varphi(|z_1-z_2|,\max\{|z_0-z_1|,|z_0-z_2|\}), \tag{11}$$

where $z_0\in\Gamma=\partial E,\ z_1,z_2\in\mathbf{C}$ is satisfied.

The proof of this statement is identical to the proof of analogous fact for H^{α} [4, p.206].

The following is true

Lemma 3. *Let* $E\in R,\ f(z)\in H_{\varphi,\Gamma}^{z_0}\cap A(E),\ z_0\in\Gamma$. *Then for any natural* n *there exists a continuous differentiable in* $\mathbf{C}$ *function* $f_n(z)$ *with the following properties*

$$|f(z)-f_n(z)|\le\varphi(\delta_n,|z-z_0|+\delta_n);\ \ z\in\Gamma, \tag{12}$$

$$\left|\frac{\partial f_n(z)}{\partial\bar z}\right|\le\varphi(\delta_n,|z-z_0|+\delta_n)/\delta_n,\ \ z\in\mathbf{C}, \tag{13}$$

$$|f_n(\zeta)-f(z)|\le\varphi(\delta_n,|z-z_0|+\delta_n);\ \ \zeta\in\partial U(z,\delta_n),\ \ z\in\partial E, \tag{14}$$

$$f(z)=f_n(z),\ \ z\in E\setminus G_n, \tag{15}$$

where in "$\le$" *the constants don't depend on* z_0,z,n.

Proof. Let's continue the function $f(z)$ on all extended complex plane $\mathbf{C}$ so that for extended function (we will remain for it the designation $f(z)$) the condition (11) be satisfied. For fixed $n=1,2,\ldots$ and δ_n we will construct δ_n-average of the function $f(z)$ assuming ([3, p.341])

$$f_n(z)=\frac{1}{\pi\delta_n^2}\iint\limits_{U(z,\delta_n)}f(\zeta)d\sigma_\zeta,\ \ z\in\mathbf{C}. \tag{16}$$

Let's prove the relation (12). For $z_0\in\partial E,\ z\in\mathbf{C}$ we have

$$\begin{aligned}|f(z)-f_n(z)|&=\frac{1}{\pi\delta_n^2}\left|\iint\limits_{U(z,\delta_n)}[f(z)-f(\zeta)]d\xi d\eta\right|\le\\ &\le\frac{1}{\pi\delta_n^2}\iint\limits_{U(z,\delta_n)}\omega_f(z_0:|z-\zeta|,\max\{|z-z_0|,|\zeta-z_0|\})d\xi d\eta.\end{aligned} \tag{17}$$

Let's consider two possible cases.

1) $|\zeta-z_0|\le|z-z_0|$; 2) $|\zeta-z_0|>|z-z_0|$.

In the first case we have

$$\begin{aligned}&\omega_f(z_0:|z-\zeta|,\max\{|z-z_0|,|\zeta-z_0|\})\le\\ &\le\varphi(\delta_n,|z-z_0|)\le\varphi(\delta_n,|z-z_0|+\delta_n).\end{aligned} \tag{18}$$

In the second case we will obtain

$$\begin{aligned}&|\zeta-z_0|\le|\zeta-z|+|z-z_0|\le\delta_n+|z-z_0|;\\ &\omega_f(z_0:|z-\zeta|,\max\{|z-z_0|,|\zeta-z_0|\})\le\omega_f(z_0:|z-\zeta|,|\zeta-z_0|)\le\\ &\le\omega_f(z_0:|z-\zeta|,|\zeta-z|+|z-z_0|)\le\varphi(\delta_n,\delta_n+|z-z_0|).\end{aligned} \tag{19}$$

Using (17), (18), (19) we have

$$|f(z)-f_n(z)|\le \varphi(\delta_n,\delta_n+|z-z_0|). \qquad (20)$$

Now we'll prove the relation (13). The function $f_n(z)$, $z=x+iy$ is continuous partially-differentiable in $\mathbf{C}$ and in this connection the relation ([3, p.342])

$$\frac{\partial f_n(z)}{\partial \bar z}=\frac{1}{2\pi i\delta_n^2}\int_{|\zeta-z|=\delta_n} f(\zeta)d\zeta \qquad (21)$$

is satisfied.

Further reasoning by the analogy with the proof of the inequality (20) we'll obtain the following estimations:

$$\left|\frac{\partial f_n(z)}{\partial \bar z}\right|=\frac{1}{2\pi\delta_n^2}\left|\int_{|\zeta-z|=\delta_n}[f(\zeta)-f(z)]d\zeta\right|\le$$

$$\le\frac{1}{2\pi\delta_n^2}\int_{|\zeta-z|=\delta_n}\omega_f(z_0;|z-\zeta|,\ \max\{|z-z_0|,|\zeta-z_0|\})d\zeta\le \qquad (22)$$

$$\le\frac{\varphi(\delta_n,\delta_n+|z-z_0|)}{\delta_n},$$

as well for $z\in\partial E$, $\zeta\in\partial U(z,\delta_n)$, we have

$$|f_n(\zeta)-f(z)|\le|f_n(\zeta)-f(\zeta)|+|f(\zeta)-f(z)|\le$$

$$\le\varphi(\delta_n,\delta_n+|z-z_0|)+\omega_f(z_0:|z-\zeta|,\max\{|z-z_0|,|\zeta-z_0|\})\le$$

$$\le\varphi(\delta_n,\delta_n+|\zeta-z|+|z-z_0|)+\varphi(\delta_n,\delta_n+|z-z_0|)\le$$

$$\le\varphi(\delta_n,\delta_n+|z-z_0|). \qquad (23)$$

The equality (15) is proved in monograph [3, p.343]. now we'll reduce the following lemma.

Lemma 4 [1]. *Let E be an arbitrary finite continuum with a simply-connected complement and for polynomial $P_n(z)$ of a degree no more than n, $n=1,2,\dots$ for some $M_1=const>0$, $z_1\in\partial E$ and $\rho>0$ at every points $z\in\partial E$ the inequality*

$$|P_n(z)|\le M_1\left(1+\left|\frac{z-z_1}{\rho}\right|^{C_1}\right),\ C_1=const>0 \qquad (24)$$

be satisfied.

Then for every $C_2=const>0$ and $z\in U(z_0,\rho)\bigcap \operatorname{int}\Gamma_{1+\frac{C_2}{n}}$ the inequality

$$|P_n(z)|\le C_3M_1,\ \ C_3=C_3(C_1,C_2,E)>0\,. \qquad (25)$$

will be satisfied.

Proof of basic results.

The proof of theorem 1.

By virtue of Cauchy-Green formula and lemma 3 we have

$$f_n(z)=\frac{1}{2\pi i}\int_{\Gamma_{1+u_0}}\frac{f_n(\zeta)}{\zeta-z}d\zeta-\frac{1}{\pi}\iint_{D_{u_0}}\frac{\partial f_n(\zeta)}{\partial\bar\zeta}\frac{d\sigma_\zeta}{\zeta-z}=$$

$$=\frac{1}{2\pi i}\int_{\Gamma_{1+u_0}}\frac{f_n(\zeta)}{\zeta-z}d\zeta-\iint_{G_n\cap D_{u_0}}\frac{\partial f_n(\zeta)}{\partial\bar{\zeta}}\frac{d\sigma_\zeta}{\zeta-z}, \tag{26}$$

where $u_0>0$ is a fixed number, $z\in D_{u_0}$.

Starting from the relations (12) and (26), we'll set the unknown polynomial by the formula

$$P_n(z)=\frac{1}{2\pi i}\int_{\Gamma_2}f_n(\zeta)\pi_n(\zeta,z)d\zeta-\frac{1}{\pi}\iint_{G_n\cap D_1}\frac{\partial f_n(\zeta)}{d\bar{\zeta}}\pi_n(\zeta,z)d\sigma_\zeta, \tag{27}$$

where $z\in E$, $n=1,2,...$ and $\pi_n(\zeta,z)$ is a poynomial kernel from lemma 2 at $u_0=1$.

Let's prove the correctness of the inequality (4). Let's $z\in\partial E$ and $z_0\in\partial E$ be some fixed point.

Let's assume $V_n=G_n\cap D_1$. By virtue of (26), (27) we have

$$f(z)-P_n(z)=\frac{1}{2\pi i}\int_{\Gamma_2}f_n(\zeta)\left[\frac{1}{\zeta-z}-\pi_n(\zeta,z)\right]d\zeta+$$

$$+\frac{1}{\pi}\iint_{G_n\cap D_1}\frac{\partial f_n(\zeta)}{\partial\bar{\zeta}}\left[\pi_n(\zeta,z)-\frac{1}{\zeta-z}\right]d\sigma_\zeta+f(z)-f_n(z),$$

$$|f(z)-P_n(z)|\le|f(z)-f_n(z)|+\frac{1}{2\pi}\int_{\Gamma_2}|f_n(\zeta)|\left|\left[\frac{1}{\zeta-z}-\pi_n(\zeta,z)\right]\right||d\zeta|+$$

$$+\frac{1}{\pi}\iint_V\left|\frac{\partial f_n(\zeta)}{\partial\bar{\zeta}}\right|\left|\left[\frac{1}{\zeta-z}-\pi_n(\zeta,z)\right]\right|\equiv J_1+J_2+J_3. \tag{28}$$

According to lemma 2 and 3 we'll find

$$J_1=|f(z)-f_n(z)|\le\varphi(\delta_n,|z-z_0|+\delta_n), \tag{29}$$

$$J_2=\frac{1}{2\pi}\int_{\Gamma_2}|f_n(\zeta)|\left|\left[\frac{1}{\zeta-z}-\pi_n(\zeta,z)\right]\right||d\zeta|\le$$

$$\le\frac{1}{2\pi}\int_{\Gamma_2}\frac{\delta_n^2}{|\zeta-z|[|\zeta-z|+\delta_n]^2}|d\zeta|\le\delta_n^2\le\delta_n\le\varphi(\delta_n,\delta_n)\le \tag{30}$$

$$\le\varphi(\delta_n,\delta_n+|z-z_0|),$$

$$J_3=\frac{1}{\pi}\iint_{V_n}\left|\frac{\partial f_n(\zeta)}{\partial\bar{\zeta}}\right|\left|\left[\frac{1}{\zeta-z}-\pi_n(\zeta,z)\right]\right|d\sigma_\zeta=$$

$$=\frac{1}{\pi}\iint_{U_n}\left|\frac{\partial f_n(\zeta)}{\partial\bar{\zeta}}\right|\left|\left[\frac{1}{\zeta-z}-\pi_n(\zeta,z)\right]\right|d\sigma_\zeta+$$

$$+\frac{1}{\pi}\iint_{V_n\setminus U_n}\left|\frac{\partial f_n(\zeta)}{\partial\bar{\zeta}}\right|\left|\left[\frac{1}{\zeta-z}-\pi_n(\zeta,z)\right]\right|d\sigma_\zeta\le$$

$$\le\frac{1}{\pi}\iint_{U_n}\left|\frac{\partial f_n(\zeta)}{\partial\bar{\zeta}}\right|\frac{\delta_n^2}{|\zeta-z|[|\zeta-z|+\delta_n]^2}d\sigma_\zeta+$$

$$+\frac{1}{\pi}\iint_{V_n|U_n}\left|\frac{\partial f_n(\zeta)}{\partial\bar{\zeta}}\right|\frac{\delta_n^2}{|\zeta-z|\left[|\zeta-z|+\delta_n\right]^2}\equiv S_1+S_2\,, \tag{31}$$

where $U_n=U(z,\delta_n)$.

By virtue of relation (12) we'll obtain

$$S_1\le\iint_{U_n}\varphi(\delta_n,|\zeta-z_0|+\delta_n)/\delta_n\frac{d\sigma_\zeta}{|\zeta-z|}\le$$

$$\le\varphi(\delta_n,|\zeta-z_0|+\delta_n)/\delta_n\iint_{U_n}\frac{d\sigma_\zeta}{|\zeta-z|}\le\varphi(\delta_n,|\zeta-z_0|+\delta_n). \tag{32}$$

Now we'll estimate S_2.

By virtue of relation (12) we have

$$S_2\le\iint_{V_n|U_n}(\varphi(\delta_n,|\zeta-z_0|+\delta_n)/\delta_n)\frac{\delta_n^2}{|\zeta-z|^3}d\sigma_\zeta\le$$

$$\le\iint_{V_n|U_n}\varphi(\delta_n,|\zeta-z|+|z-z_0|+\delta_n)\cdot\delta_n\frac{d\sigma_\zeta}{|\zeta-z|^3}\le \tag{33}$$

$$\le\iint_{V_n|U_n}\varphi(|\zeta-z|,\,2|\zeta-z|+|z-z_0|)\cdot\delta_n\frac{d\sigma_\zeta}{|\zeta-z|^3}.$$

Here we have two possible cases:

a) $|\zeta-z|>|z-z_0|$; b) $|\zeta-z|\le|z-z_0|$.

Let the case a) be true. Taking into account that $\varphi(\delta,\delta)\cdot\delta^{-l}$ almost decreases, we'll obtain

$$S_2\le\delta_n\iint_{V_n|U_n}\varphi(|\zeta-z|,|\zeta-z|)\frac{d\sigma_\zeta}{|\zeta-z|^3}\le\delta_n\iint_{V_n|U_n}\frac{\varphi(\delta_n,\delta_n)}{\delta_n^l}\frac{|\zeta-z|^l}{|\zeta-z|^3}d\sigma_\zeta\le$$

$$\le\frac{\varphi(\delta_n,\delta_n)}{\delta_n^{l-1}}\int_{\delta_n}^{\infty}\frac{dr}{r^{3-l-1}}\le\varphi(\delta_n,\,\delta_n+|z-z_0|). \tag{34}$$

Now let the case b) be true. Then taking into account that $\varphi(\delta,\eta)\cdot\delta^{-1}$ decreases by δ, we have

$$S_2\le\delta_n\iint_{V_n|U_n}\varphi(|\zeta-z|,2|z-z_0|)\frac{d\sigma_\zeta}{|\zeta-z|^3}\le\delta_n\varphi(\delta_n,|z-z_0|)\iint_{V_n|U_n}\frac{d\sigma_\zeta}{|\zeta-z|^3}\le$$

$$\le\varphi(\delta_n,\,\delta_n+|z-z_0|). \tag{35}$$

Using (34), (35) we have

$$S_2\le c\varphi(\delta_n,\,\delta_n+|z-z_0|), \tag{36}$$

where c doesn't depend on z_0,z,n.

By virtue of (31), (32), (36) we'll obtain that

$$J_3\le c_1\varphi(\delta_n,\,\delta_n+|z-z_0|), \tag{37}$$

where c_1 doesn't depend on z_0,z and n.

Comparing the estimations (28)-(30), (37) we'll pass to the inequality (4).

Now we'll prove the inequality (5).

According (27)

$$P_n'(z)=\frac{1}{2\pi i}\int_{\Gamma_2} f_n(\zeta)\frac{\partial \pi_n(\zeta,z)}{\partial z}d\zeta-\frac{1}{\pi}\iint_{G_n\cap D_1}\frac{\partial f_n(\zeta)}{\partial \overline{\zeta}}\frac{\partial \pi_n(\zeta,z)}{\partial z}d\sigma_\zeta . \tag{38}$$

Using the scheme of reasoning, used for the proof of statement (4) of theorem 1 we'll receive

$$|P_n'(z)|\le \int_{\partial u_n}\frac{|f_n(\zeta)-f(z)|}{|\zeta-z|^2}|d\zeta|+\frac{1}{\pi}\int_{\Gamma_2}|f_n(\zeta)|\left|\frac{\partial}{\partial z}\left[\frac{1}{\zeta-z}-\pi_n(\zeta,z)\right]\right||d\zeta|+$$

$$+\frac{1}{\pi}\iint_{U_n}\left|\frac{\partial f_n(\zeta)}{d\overline{\zeta}}\right|\left|\frac{\partial}{\partial z}-\pi_n(\zeta,z)\right|d\sigma_\zeta+\frac{1}{\pi}\iint_{V_n|U_n}\left|\frac{\partial f_n(\zeta)}{d\overline{\zeta}}\right|\left|\frac{\partial}{\partial z}-\pi_n(\zeta,z)\right|d\sigma_\zeta\equiv \tag{39}$$

$$=J_1+J_2+J_3+J_4 .$$

Applying lemma 3, reasoning analogously as was proved the inequality (4), as well taking into account that $\varphi(\delta,\delta)\cdot\delta^{-l}$ almost decreases we'll find

$$J_1=\int_{\partial u_n}\frac{|f_n(\zeta)-f(z)|}{|\zeta-z|^2}|d\zeta|\le\varphi(\delta_n,\delta_n+|z-z_0|)\int_{\partial u_n}\frac{1}{|\zeta-z|^2}|d\zeta|\le \tag{40}$$

$$\le\varphi(\delta_n,\delta_n+|z-z_0|)/\delta_n ;$$

$$J_2=\frac{1}{2\pi}\int_{\Gamma_2}|f_n(\zeta)|\left|\frac{\partial}{\partial z}\left[\frac{1}{\zeta-z}-\pi_n(\zeta,z)\right]\right||d\zeta|\le\int_{\Gamma_2}\frac{\delta_n^2}{|\zeta-z|^2\left[|\zeta-z|+\delta_n\right]^2}|d\zeta|\le \tag{41}$$

$$\le\delta_n^2\le\varphi(\delta_n,\delta_n+|z-z_0|)/\delta_n ;$$

$$J_3=\frac{1}{\pi}\iint_{U_n}\left|\frac{\partial f_n(\zeta)}{\partial\zeta}\right|\left|\frac{\partial}{\partial z}\pi_n(\zeta,z)\right|d\sigma_\zeta\le$$

$$\le\frac{\varphi(\delta_n,\delta_n+|z-z_0|)}{\delta_n}\iint_{U_n}\frac{d\sigma_\zeta}{\left[|\zeta-z|+\delta_n\right]^2}\le\frac{\varphi(\delta_n,\delta_n+|z-z_0|)}{\delta_n}; \tag{42}$$

$$J_4=\frac{1}{\pi}\iint_{V_n|U_n}\left|\frac{\partial f_n(\zeta)}{\partial\zeta}\right|\left|\frac{\partial}{\partial z}\left[\frac{1}{\zeta-z}-\pi_n(\zeta,z)\right]\right|d\sigma_\zeta\le$$

$$\le\iint_{V_n|U_n}\frac{\varphi(\delta_n,|z_0-\zeta|+\delta_n)}{\delta_n}\frac{\delta_n^2}{|\zeta-z|^2\left[|\zeta-z|+\delta_n\right]^2}d\sigma_\zeta\le$$

$$\le\frac{\varphi(\delta_n,|z-z_0|+\delta_n)}{\delta_n}\delta_n^2\iint_{V_n|U_n}\frac{d\sigma_\zeta}{|\zeta-z|^4}\le\delta_n\varphi(\delta_n,|z-z_0|+\delta_n)\int_{\delta_n}^{\infty}\frac{dr}{r^3}\le \tag{43}$$

$$\le\frac{\varphi(\delta_n,|z-z_0|+\delta_n)}{\delta_n};$$

Comparing the estimations (39)-(43) we'll pass to the inequality (5).

The theorem is proved.

The proof of theorem 2. Necessity of theorem 2 implies from lemma 1. Let's prove the sufficiency of conditions of theorem 2. Let z_1 and $z_2\in\Gamma=\partial E$, $|z_1-z_2|<\varepsilon$ (ε is sufficiently small). Let's connect them by the arc $\gamma(z_1,z_2)\subset E$ with the property

$$mes\gamma(z_1,z_2)\le|z_1-z_2| . \tag{44}$$

Let's find natural n from the condition $\delta_{n+1} < |z_1 - z_2| \le \delta_n$. Let's note that for the polynomial $P_n(\xi)$ satisfying inequalities (4) and (5), the following inequality

$$|P_n'(\xi)| \le \frac{\varphi(\delta,\eta)}{\delta}\left(1 + \left|\frac{\zeta - z_1}{\delta}\right|^{c_2}\right), \quad c_2 = const > 0, \tag{45}$$

where $|z_1 - z_2| = \delta$, $\eta = \max\{|z_1 - z_0|, |z_2 - z_0|\}$ holds.

Then by virtue of lemma 4 we'll find

$$|P_n'(\xi)| \le \frac{\varphi(\delta,\eta)}{\delta}, \quad \xi \in \gamma(z_1, z_2). \tag{46}$$

Using (4), (5) and (46) we have

$$|f(z_1) - f(z_2)| \le |f(z_1) - P_n(z_1)| + \int\limits_{\gamma(z_1,z_2)} |P_n'(\xi)|\,|d\xi| + |P_n(z_2) - f(z_2)| \le$$

$$\le \varphi(\delta_n, |z_1 - z_0| + \delta_n) + \frac{\varphi(\delta,\eta)}{\delta}\int\limits_{\gamma(z_1,z_2)} |d\xi| + \varphi(\delta_n, |z_2 - z_0| + \delta_n) \le \tag{47}$$

$$\le \varphi(\delta_n, \delta_n + \eta) + \varphi(\delta,\eta).$$

Let's consider two possible cases:

1) $\delta_n > \eta$, 2) $\delta_n \ge \eta$,

where $\eta = \max\{|z_1 - z_0|, |z_2 - z_0|\}$.

Let the case 1) be true. Then by virtue of property of the function φ, we have

$$\varphi(\delta_n, \delta_n + \eta) \le \varphi(\delta_n, 2\eta) = \varphi\left(\frac{\delta_n}{\delta}\delta, 2\eta\right) \le \frac{\delta_n}{\delta}\varphi(\delta, 2\eta) \le$$

$$\le \frac{\delta_n}{\delta_{n+1}}\varphi(\delta, 2\eta) \le \varphi(\delta,\eta). \tag{48}$$

Now let the case 2) be true. Then we have

$$\varphi(\delta_n, \delta_n + \eta) \le \varphi(\delta_n, 2\delta_n) = \varphi(2\delta_n, 2\delta_n) = \varphi\left(\frac{2\delta_n\delta}{\delta}, \frac{2\delta_n}{\delta}\delta\right) \le$$

$$\le \frac{\delta_n}{\delta}\varphi(\delta,\delta) \le \frac{\delta_n}{\delta_{n+1}}\varphi(\delta,\delta) \le \varphi(\delta,\eta). \tag{49}$$

Now taking into account (48) and (49) we'll receive that

$$\varphi(\delta_n, \delta_n + \eta) \le \varphi(\delta,\eta). \tag{50}$$

By virtue of (47) and (50) we have

$$|f(z_1) - f(z_2)| \le \varphi(\delta,\eta).$$

Hence we'll obtain that

$$\sup_{\substack{|z_1 - z| \le \delta \\ z_1, z_2 \in O_\eta(z_0) \cap \Gamma}} |f(z_1) - f(z_2)| \le c_1\varphi(\delta,\eta).$$

The theorem 2 is proved.

References

[1]. Андриевский В.В. *Аппроксимационная характеристика классов функций на континуумах комплексной плоскости.* Матем. сб., 1984, 125/167, №1/9, с.70-87.

[2]. Андриевский В.В. *Конструктивное описание классов функций на континуумах комплексной плоскости с учетом роста аппроксимационных полиномов.* –ИМ-83-12. Препринт института математики АН УССР, 1983.

[3]. Дзядык В.К. *Введение в теорию равномерного приближения функций полиномами.* М., Наука, 1977, 512 с.

[4]. Стейн И.М. *Сингулярные интегралы и дифференциальные свойства функций.* М.: Мир, 1973, 342 с.

[5]. Мамедханов Дж.И. *Об одном усилении теоремы Племеля-Привалова к задачи наилучшей аппроксимации. – Теория кубатурных формул и вычислительная математика* (труды конференции по дифференциальным уравнениям и вычислительной математике, Новосибирск, 1978). М., 1980, с.164-167.

[6]. Мамедханов Дж.И., Салаев В.В. *О новых классах функций, связанных с локальной структурой особых интегралов и некоторые аппроксимации в них.* Тезисы докл. Всесоюзного симпозиума по теории аппроксимации. Уфа, 1980, с.91-92.

[7]. Мамедханов Дж.И. *Аппроксимация в комплексной области и сингулярные операторы с ядром Коши.* - Афтореф. дис. на соиск. учен. степени докт. физ.-мат. наук. Тбилиси, 1985.

[8]. Мамедханов Дж.И. *Локальные теоремы теорий наилучшей аппроксимации.* ДАН Азерб. ССР, 1981, т.37, №8, с.14-19.

[9]. Мамедханов Дж.И., Мусаев А.А. *Локальная полиномиальная аппроксимация в областях с квазиконформной границей.* ДАН Азерб. ССР, 1986, т.4, №1, с.7-10.

[10]. Мусаев А.А. *Локальная полиномиальная аппроксимация на кривых, в комплексной плоскости.* В кн.: Труды 2-ой Саратовской зимней школы по теории функций и приближений, 1986, ч.3, с.40-43.

[11]. Стечкин С.Б. *О порядке наилучших приближений непрерывных функций.* Изв. АН СССР, сер. матем., 1951, т.15, №3, с.219-242.

[12]. Мусаев А.А. *О некоторых вопросах локализованной аппроксимации в комплексной плоскости.* Автореф. дисс. на соиск. учен. степени канд. физ.-мат. наук., Баку, 1987.

[13]. Дзядык В.К. *Аппроксимационная характеристика классов Липшица* $W^r H^1$ $(r=0,1,2,...)$ - Analysis Math., 1975, т.1, с.19-30.

[14]. Коновалов В.Н. *Аппроксимационные характеристики некоторых классов функций комплексной переменной.* В кн.: Методы теории приближения и их приложения. Киев, Институт математики АН УССР, 1982, с.54-66.

[15]. Джафаров С.З. *Аппроксимационная характеристика некоторых класов функций на континуумах комплексной плоскости.* РУК. деп. в ВИНИТИ 26.02.87, «14-29-В87, 15 с.

[16]. Джафаров С.З. *Аппроксимационная характеристика некоторых классов функций на континуумах комплексной плоскости.* Автореферат дис. канд. физ. мат. наук 01.01.01. АН Азерб. ССР, ИММ, Баку, 1988.

[17]. Тамразов П.М., Бардзинский В.В. *О комплексных конечно-разностных гладкостях и полиномиальных приближениях.* Препринт ИМ-76-7, Киев, 1976, 15 с.

[18]. Рзаев Р.М. *О некоторых метрических характеристиках функций.* Баку, 1981, 40 с. - Рукопись представлена институтом кибернетики АН Аз.ССР 20 июля 1981, №3674-81 Деп.

M.Ali Sarigöl, Jafarov S.Z.
Pamukkale University, Fen-Edebiyat faculty, Department of Mathematics.
Denizli, Turkey.

Received July 6, 2000; Revised November 15, 2000.
Translated by Mirzoyeva K.S.