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THE LOCALIZATION OF SPECTRUM AND ITS APPLICATION. I

Abstract

In the present paper the localization relation for residual spectrum and compression have been get by numerical range of operator in reflexive Banach space, which are applied to the investigations of properties of various class of operators.

In connection with localizability of compression spectrum, some questions of geometry of Banach spaces are considered, which are used for investigations of spectral properties of generalized self-adjoint and generalized unitary operators in Banach spaces.

Preface.

The main aim of our work, which consists of a few parts, is to give the expanded and complemented exposition around one observation from [17e]. The question is to obtain the localizability of the residual spectrum of the operator of the interior of its numerical range in the reflexive Banach space and about it's various applications in the theory of operators. These are the following questions: the structure of the spectrum of some classes of operators; the criterion of the closeness of the numerical range of the various classes of operators in the terms of spectrum; the representation of the spectrum of any operator through the interior of the numerical range and the investigation of the structure of the spectrum of Hermitian operators; the approximability of the inverse operator by the polynomials of the invertible operator. Besides in connection with the localization of the parts of spectrum another three questions will be considered: two-sided estimation of Gershgorin domain known from the matrix analyses by the numerical ranges, the relation of the spectrum of the operator-function with their numerical range and, finally, some problems from geometry of the reflexive Banach spaces and their applications to the investigations of the properties of the generalized selfadjant and unitary operators .

The state of the question and the comments on each of these problems are given in the corresponding parts of the exposition.

In the present part of the work, §1 is about the localization theorems of the residual spectrum and the compression spectrum of the operator [theorems 1.1 and 1.2] in a Banach space with the preliminary description of the location of the compression spectrum in the Bauer's numerical range of this operators (propositions 1.1. and 1.2). At the same place these localization theorems are used for the determination of the structure of the spectrum of various operators (corollaries 1.2-1.4), and also to obtain the sufficient condition of the finiteness of the operator in the terms of the location of the compression spectrum in the numerical range (proposition 1.3).

In §2 the criterion of the solvability of three problems from geometry of Banach spaces: on non-triviality of the orthogonal complement, on orthogonal decomposition of space and about the representation of linear functionals which set the correspondence between the space and its dual (theorem 2.1) is led. The description of a class of Banach spaces identified with their dual is given as a corollary (theorem 2.2). This permits to consider the generalization of the selfadjoint and unitary operators in such Banach spaces and to obtain a series properties, of their spectrum by analogy with the case of Hilbert spaces (theorems 2.3-2.5).

The rest from the above enumerated questions would be discussed in subsequent three parts of the work.

Preliminaries and notation.

For the convenience let's remind some notions co-ordinating the terminus and designations. Further a Banach space X above the field of the complex members $\mathcal{C}$ and algebra $\beta(X)$ of the bounded linear operators in X are considered.

For the operator $T \in \beta(x)$ a spectrum, the point spectrum, the compression spectrum, the residual spectrum, the continuous spectrum, approximate point spectrum are denoted by $\sigma(T), \sigma_p(T), \sigma_\gamma(T), \sigma_r(T), \sigma_c(T)$ *and* $\sigma_\pi(T)$, respectively, and by KerT and RanT the kernel and the range T, correspondingly

The set $V(T) = \{\lambda \in \mathcal{C} : \lambda = f(Tx), x \in S(x), f \in D(x, X)\}$, where $D(x, X) = \{f \in S(X^*) : f(x) = 1\}$ at $x \in S(X)$, and $S(Y)$ is the unit sphere of the space Y and * is the Banach conjugation, is called the Bauer's or spatial numerical range of the operator $T \in \beta(x)$. $W_s(T) = \{\lambda \in \mathcal{C} : \lambda = S[Tx, x], x \in S(x)\}$, is called Lumer's numerical range for $T \in B(x)$ responding to the semininer product (s.i.p.) s. generating the norm in X.

The set $\mathcal{V}(T) = \left\{\lambda \in \mathcal{C} : \lambda = F(T),\ F \in \beta(x)^*, \|F\| = F(I) = 1\right\}$, where I is a unit operator in X, is called algebraic numerical range of the operator $T \in \beta(x)$. It is known that in the general Banach space s.i.p. s giving its norm, isn't unique and for any such s the inclusions $W_s(T) \subset V(T) \subset \mathcal{V}(T)$ where $\overline{co} W_s(T) = \overline{co} V(T) = \mathcal{V}(T)$, are true. If X is the Hilbert space H with inner product <,>, then for $T \in \beta(H)$ the equalities $W_s(T) = V(T) = W(T)$ and $\mathcal{V}(T) = \overline{W}(T)$, where $W(T) = \{\lambda \in \mathcal{C} : \lambda = <Tx, x>, x \in S(H)\}$ is the Hausderff numerical range [chapter 17], are true.

The detailed information about the geometric and spectral properties of the numerical ranges one can find in [2] and [1, chapter 17].

Banach space X is called rotund or strictly convex, if for any $x, y \in S(X)$ from the equality $\|x + y\| = 2$ follows $x = y$ or by the equivalent manner if for any $f \in S(X^*)$ there exists at most one $x \in S(X)$ such that $f(x) = 1$. But if for any sequences $x_n, y_n \in S(X)$ from $\|x_n + y_n\| \to 2$ it follows $\|x_n - y_n\| \to 0$ for $n \to \infty$, then X is called a uniformly rotund or uniformly convex Banach space. X is called reflexive, if the canonical imbedding $\Lambda : X \to X^{**}$ is surjective, where $\hat{x}(f) = f(x)$ for any $f \in X^*$. It is known that any uniformly rotund Banach space is reflexive [3, pg. 182] and obviously rotund. The inverse, generally speaking, is wrong. For example 2-dimensional space with max-norm is reflexive, but isn't rotund and all the more so isn't uniformly rotund. Banach space X is called smooth, if for any $x \in S(x)$ is found at most one $f \in S(X^*)$ for which $f(x) = 1$.

For the set M in $\mathscr{C}$ by $\operatorname{int} M$ is designated its interior, by ∂M - the border of the set M, i.e. the set of all boundary points for M lying in M; coM is a convex hull, $\overline{co}M$ is a closed convex hull, $\overline{M}$ is a closure. The set M is called codense if $M = \partial M$.

More complete information about geometry of Banach spaces have been represented in [4].

§0. Survey of the localization theorems.

The localization of spectrum (and its parts) by the numerical range, being one of the most important relations between these two numerical characteristics of the operator, has not only theoretical interest, since find application in the various applied problems. This question, and in general the theory of the numerical ranges in itself take their beginning from O. Toeplitz's (1918) work, where the notion of the numerical range has been derived and together with other results the following historically first localization theorem has been proved.

Theorem A (Toeplitz's O.). *For the linear operator T in the finite dimensional unitary space:*

$$co\sigma_p(T) \subset W(T). \tag{1}$$

After the convexity $W(T)$ was proved by Hausdorff F. (1919y.), the relation (1) is turned out to be a simple corollary of the evident fact: $\sigma_p(T) \subset W(T)$ and nevertheless remains as the base of the proof of many localization theorems. For example, from (1) it follows obviously, that

$$co\sigma_\gamma(T) \subset W(T). \tag{2}$$

After ten years Wintner A. (1929 y.) (for the coordinated space l_2) and Stone M. (1930 g.) (in the abstract Hilbert space) proved:

Theorem B (Wintner-Stone) *For any $T \in \beta(H)$ in a Hilbert space H:*

$$co\sigma(T) \subset \overline{W}(T). \tag{3}$$

In the history of development of the theory of the numerical ranges of operators G. Lumer's (1961) and f. Bauer's (1962) fundamental works installing the bases of this theory for the case of Banach spaces became stake. In particular using the notion s.i.p. (see p.22) by G. Lumer it has been obtained:

Theorem C (Wintner-Lumer). *For $T \in \beta(X)$ for any s.i.p. s generating the norm in X the relation*

$$\sigma_\pi(T) \subset \overline{W_s}(T) \tag{4}$$

is fulfilled. As for $\sigma(T)$, then in general case $\overline{W_s}(T)$ not localized it, but it is true:

Theorem D (Lumer). *For any s.i.p. s compatible with the norm in X for $T \in \beta(X)$ the inclusion*

$$\sigma(T) \subset \overline{co}W_s(T) = \mathscr{V}(T) \tag{5}$$

is fulfilled.

The following progress in the considered question was proved by Williams (1967y.).

Theorem E (Williams). *For any $T \in \beta(X)$*

$$\sigma(T) \subset \overline{V}(T) \tag{6}$$

is just.

In view of the absence of the convexity of the numerical range $V(T)$ in a Banach space the analogues of the localization relations (1) and (3) were turned out to be Zenger's (1968) and Crab's (1969) non-trivial theorems.

Theorem F (Zenger).

$$co\,\sigma_p(T) \subset V(T), T \in \beta(X). \tag{7}$$

Theorem G (Crabb).

$$co\sigma(T) \subset \overline{V}(T),\ T \in \beta(X). \tag{8}$$

Also note that for the residual spectrum and the compression spectrum theorem the folloving localization theorems have been proved by Meng (1963) and Koehler (1971).

Theorem H (Meng). *For any* $T \in \beta(H)$:

$$\sigma_r(T) \subset \operatorname{int} W(T). \tag{9}$$

Theorem J (Koehler D.). *In a smooth uniformly rotund Banach space X for any* $T \in \beta(X)$ *the inclusion*

$$\sigma_\gamma(T) \subset W_s(T) \tag{10}$$

where s is only possible s.i.p. in X, generating its norm, is fulfilled.

§1. The localization of the residual spectrum and spectral properties of some operators.

1.1. In the present point the analogues of the localization relations (9) and (2) are proved for Banach space. Being violated in arbitrary Banach space they prove to be turned out to be true under reflexivity of space. There localization theorems find application in various questions of the theory of operators and closely related with geometry of space.

The following localization relation in a Hilbert pace has been obtained by Meng [8], the proof of which uses the Hilbert property of space.

Theorem 1.1. *For any operator* $T \in \beta(x)$ *in reflexive Banach space X the residual spectrumlies in the interior of Bauer's numerical range*

$$\sigma_r(T) \in \operatorname{int} V(T) \tag{11}$$

with possible equality in the relation (11), and there exist a irreflexive Banach space X and the operator $T \in \beta(X)$ *for which the inclusion (11) isn't fulfilled.*

Short proof. The proof of the theorem consists of statements describing the location of the compression spectrum in Bauer's numerical range. The first of them strengthing the result of work [9, theorem 1] one can prove by the method of orthogonal complement [see Point 2.2], and by transfering to the adjoint space (see, further theorem 1.2). We give the proof showing that facts from geometry of Banach spaces that lie on the base of the both above mentioned approach.

Proposition 1.1. *For any operator* $T \in \beta(X)$ *in a reflexive Banach space X, its compression spectrum is contained in Bauer's numerical range.*

The proof of the statement is obtained with the help of the lemma about annihilator for a subspace [5, pg. 72, theorem 3.5] or [6, pg. 77]and Mazur lemma on that any linear functional attains its norm on a unit sphere of a reflexive space X.

Before we give the second auxiliary statement note the essence of the reflexivity condition of the space and attainability of the numerical range by the compression spectrum in proposition 1.1. Besides on additional smoothness of space it follows the result obtained in [9, theorem 1] under more strong restriction of smoothness and the uniform rotundness.

Remark also that if one shall use the numerical range of adjoint operator, then one can remove the reflexivity condition of the space in the proposition 1.1.

If proposition 1.1 shows that the compression spectrum located either in the interior of the numerical range or on border, then the following statement describes the points from the compression spectrum which are the border points for the numerical range.

Proposition 1.2. *For any operator $T \in \beta(x)$ in the reflexive Banach space X the points of the compression spectrum lying on the border of the Bauer's numerical range, are the eigenvalues*

$$\sigma_\gamma(T) \cap \partial V(T) \subset \sigma_p(T).$$

Scheme of the proof. If zero belongs both to the compression spectrum and to the border of the numerical range of some operator, then it'll be eigenvalue of the adjoint operator lying on the border of the Bauer's numerical range.

Therefore Crabb-Sinclair theorem [2, §20 theorem 9] yields that the kernel of the adjoint operator is orthogonal to range by Birkhoff and therefore as it is easy to see and also to the closure of the range of the adjoint operator. Hence by virtue of the reflexivity of Banach space it follows that zero is an eigenvalue of the operator itself.

Now the localization relation (11) for the residual spectrum diretly follows from statements 1 and 2 by virtue of disjointness of the pointwise and the residiual spectrums [6, pg 620].

Example of the attainnability of the interior of the numerical range of operator by its residual spectrum gives one-sided shift in the space l_2 [1, problem 67].

Finally, the essence of the reflexivity condition of Banach space for justice of the statement of the theorem follows from the following example. Let's consider the irreflexive Banach space X of all continuous complex valued functions determined on the segment [0,1] with sup-norm, and the function $y(t) = t$ for each $t \in [0.1]$. Let's consider the operator $T \in \beta(x)$ of multiplication $Tx = tx$ of any function $x(t) \in X$ by t [6, pg 621]. It is easy to show that zero lies the residual spectrum of this operator and the numerical range is not contained in its interior. Thus all statements of theorem 1 are instilled.

By the way note that in the last example the pointwise and the continuous spectrums operator are absent and its Bauer's numerical range coinciding to the approximate point and the residual spectrums is equal to the segment [0.1].

Corollary 1.1 *For any operator $T \in \beta(x)$ in the reflexive Banach space X the points of spectrum lying on the boundary of Bauer's numerical range is contained either in the pointwise or in the continuous spectrums of the operator T.*

In Hilbert space due to the convexity of the numerical range the fact that the convex hull of the compression spectrum in Hausdorff numerical range, 'll be trivial. In spit of the absence of the convexity condition of Bauer's numerical range of operator in Banach space [2, §11] the nontrivial Zenger theorem [2, §19] gives the following strengthening of proposition 1.1.

Theorem 1.2. *In the reflexive Banach space X the convex hull of the compression compression of any operator $T \in \beta(x)$ is contained in its Bauer's numerical range.*

Note that in the proof of theorem 1.1 instead of proposition 1.1 one can use William's localization theorem [2, §10, theorem 1] whose proof is in simplified reflexive Banach space due to proposition 1.1, being as the copy of the situation in a Hilbert space.

Note also that the substitution of the numerical range of operator in theorem 1.2 to the numerical range of the adjoint operator permits to omit the reflexivity condition of the space.

1.2. Let's apply the theorems about the localization of the residual spectrum and the compression spectrum to the investigation of the structure of spectrum of some classes of operators and to the deducting of a sufficient condition of the finiteness of the operator in terms of the location of the compression spectrum of this operator in its numerical image.

H.Weyl's classic theorem reads that for a selfadjant operator in a Hilbert space any point of the spectrum is approximate eigenvalue. Furthormore the residual spectrum of such an operator is empty. For Hermitian operator s by Lumer and p-selfadjoint operators $1 < p < \infty$ the correctness of Weyl theorem is noted for determined classes of Banach spaces in [17a] (see p.2.3). From theorem 1 it directly follows the generalization of mentioned Weyl's theorem.

Corollary 1.2 *If in the reflexive Banach space X the operator $T \in \beta(x)$ has the co dense numerical range $V(T)$, then the residual spectrum $\sigma_r(T)$ is empty.*

In particular, we following Banach analogue of Weyl theorem.

Corollary 1.3. *The residual spectrum of Hermitian operator in the reflexive Banach space is empty.*

Now let's consider some properties of the spectrum of the unimodular compression in Banach space. Remind that the operator $T \in \beta(x)$ in Banach space X is called the unimodular contraction, if T is a contraction, i.e. its norm $\|T\| \le 1$ and its spectrum is unimodular, i.e. lies on the unit circumference. It is evident that the spectrum of such an operator coincides with the boundary of the approximate pointwise spectrum. Besides the following corollary is true.

Corollary 1.4. *Let T be the unimodalar compression in a Banach space X. Then: a) under the reflexivity of X the residual spectrum $\sigma_r(T)$ is empty, b) in the rotund (or smooth reflexive) X the points of the spectrum $\sigma(T)$ lying in the numerical range $V(T)$ be the eigenvalues.*

Proof. Use theorem 1 about localization of the residual spectrum, Wintner-Lumer theorem [2, §9, theorem 8] about principle points of the numerical range and Williams localization theorem [2, §10, theorem1], take into account the fact that spectral and numerical radiuses of any operator do not exceed its norm.

Note that corollary 1.4 is true for wider class of the unimodalar numerical contraction.

The important strong subclass of the unimodalar compression form the isoabelian (by Stampfly) [14] operators which are Banach analogue of unitary operators in a Hilbert space. Some properties of such operators are represented in [17f]. In connection with theorem 1.1 note the following fact

Corollary 1.5. *In reflexive Banach space X any isometry* $T \subset \beta(x)$ *is an isoabelian operator, provided that* $0 \notin \operatorname{int} V(T)$.

Let's conclude this point with the sufficiency condition of finiteness of operator in the reflexive Banach space in terms of the location of the compression spectrum in the numerical range. At first let's bring the notion of finite operator reduced in Hilbert space by J.Williams to Banach space.

The operator $T \in \beta(x)$ in Banach space X we'll call finite if for any operator $H \in \beta(x)$ the condition $0 \in \mathcal{V}([T, A])$, where [T,A]=TA-AT, and $\mathcal{V}([T, A])$ is the algebraic numerical range of the comutator [T,A], is fulfilled.

Proposition 1.3. *If the compression spectrum of the operator* $T \in \beta(x)$ *in the reflexive Banach space doesn't lie wholly in the interior of the numerical range* $V(T)$, *then T is a finite operator.*

The proof is obtained with the help of the propositins 1 and 2, of Crabb-Sinclair theorem [2, §20, corollary 10] and lemma 3 from [2, §20].

§2. Geometry of Banach spaces and self adjointness (unitarily).

2.1. In this point some questions of geometry of Banach spaces arised in connection with the localization of the compression spectrum of operator by the numerical range, are considered and their applications to the theory of operators in Banach space are given.

Comparison of more general result (see, proposition 1.1, from §1, and also [17 b]) with Koehler D theorem [9, theorem 1] based in J.Ciles's paper [10] shows that the restrictions imposed by them to the geometry of space are redundant. This leads us to the reconsideration and development of their results. By such way we came to Banach variant of classic facts of geometry of Hilbert spaces (Riesz-Levi theorem about the orthogonal complement, Riesz-Rellich theorem about the orthogonal decomposition, and Riesz-Frechet theorem about the representation of functional).

It was clarificatied that these theorems related with such properties of Banach spaces as reflexivity rotundness and smoothness. The reflexivity of space replying to the existence appear to be equivalent to the justice of each of Riesz-Levi, Riesz-Rellich and Riesz-Frechet generalized theorems and each of the roundness and smoothness properties of space replies for its certain uniqueness in these theorems.

The corollary of this is that the property of self-adjointness of Hilbert space is correct in the class of reflexive rotund smooth Banach spaces Besides in this class of spaces wider than in Koehler D. [9],it appears possible to consider the generalized self-adjoint and unitary operators and to prove for them a series of properties operators of the same name in Hilbert spaces. Note that, all these are achieved by the adaptation of the standard classic technique of the theory of Hilbert spaces.

The choice of terminology in formulated further problems is acquitted by excursion to their source in the articles of classics F.Riesz, M.Frechet, B.Hvi and F.Rellich [6, pg 440].

Parallel with Hilbert space it is known also the classes of Banach spaces, in which these problems are solvable. For example, according to J.Giles [10, pg 440 and 441] in smooth uniformly rotund Banach spaces the problems about Riesz-Levi normal and about Riesz-Frechet representation, are solvable. The main result of article [11] is the relaxation of conditions on geometry of space in Giles theorem about Riesz-Frechet representation. These questions were considered also in [12], but their class of Banach spaces are narrower than in Giles. The more general result about Riesz-Frechet representation, belongs to G.Faulkner [13, theorem 2]

2.2. For to the formulation and solution of the considered problems on geometry of Banach space we need the following notions.
At fist let's remind the orthogonality variant in Banach space introduced by G.Birkhoff and which is one of the characteristics of orthogonality in a Hilbert space [5, pg 330, theorem 12.2].

Let $x,y \in X$ be a complex Banach space wit norm ν. They say that x is orthogonal (by Birkhoff) to the vector y and write $x \perp_B y$, if for $\lambda \in \mathcal{C}$ the inequality $\nu(x) \le \nu(x+\lambda y)$ is fulfilled. For any subset M in X the set $M^{\perp_B} = \{x \in X : x \perp_B y \text{ for any } y \in M\}$ is called its orthogonal complementation (by Birkhoff) in X. It is easy to see that $M^{\perp_B}$ is a nonempty closed set and $\lambda y \in M^{\perp_B}$ for any $y \in M^{\perp_B}$ and $\lambda \in \mathcal{C}$. It is obvious that $x^{\perp_B} = \{\theta\}$, $\{\theta\}^{\perp_B} = X$ and $M \cap M^{\perp_B} = \{\theta\}$, where θ is a zero vector in X. Another orthogonality in a Banach space X is generated by semi-inner production (s.i.p.) (by Lumer) giving generating, compatible with the norm ν in X [3, pg 345]. For $x,y \in X$ they say that x is normal to y (or is orthogonal with respect to s.i.p.s) if $s[y,x]=0$ (sometimes shortly [y,x]=0) and write $x \perp_s y$ or $x \perp_{[,]} y$. For the set M from X the set $M^{\perp_s} = \{x \in X : [y,x]=0 \text{ for all } y \in M\}$ is called orthogonal complement of M with respect to s.i.p.s.[,]. It is easy to see that $M^{\perp_s}$ has the above anumerated simple properties of set $M^{\perp_B}$. It is not difficult to verify that for any $x,y \in X$ the relation $x \perp_s y$ implies $x \perp_B y$ for any s.i.p.s., generating the norm ν. Inversely, as G. Faulnher [13, theorem 1] shoved, if $x \perp_B y$, then there exists s.s.p.s. giving the norm in X with respect to which $x \perp_s y$.

Similarly to Hilbert space the natural question about the nontrivality of orthogonal complementation to its subspace of Banach space X and the question related with it about orthogonal decomposition X arise. Besides, the problem of transfer of the representation of functionals to Banach space established the natural correspondences between the vector and functionals with respect to inner production, is legitimate.

Let's formulate these problems. Let X be a Banach space with the norm ν.

1) we will call X decomposable by Riesz-Rellich, if for any its nontrivial subspace M and for any $x \in X$ there will found such s.i.p [,] generating the norm y that $x = y + z$, where $y \in M,\ z \in M^{\perp}$.

2)we will say that for X Riesz-Frechet representation is holds, if for any $f \in X^*$ there exists such a vector $y \in X$ and such s.i.p. [,] compatible with the norm ν, that for all $x \in X$ the equality

$$f(x) = [x,y]$$

is correct, where the norms f and y coincide.

3) we will say that Riesz-Levi problem about normal is solvable in X, if for some s.i.p.s [,] giving the norm ν for any nontrivial subspace M there exists nonzero $x \in M^{\perp_s}$.

The following criterion gives the full answer to all three formulated problems showing once more that the reflexive Banach spaces have many good properties of Hilbert spaces and therefore they are important in applications.

Theorem 2.1 *For Banach space X the following statements: a) X is reflexive; b) x is decomposable by Riesz-Rellich; c) Riesz-Frechet representation holds in X; d) Riesz-Levi problem about normal is solvable in X, are equivalent.*

Scheme of proof. *The following four propositions 2.1-2.4. make the equivalence of the statements a)-d).*

Proposition 2.1. *The reflexivity of Banach space X implies the decomposition X.*

The proof of the decomposition by Riesz-Rellich of the space X follows from the existence of the element of the best approximation for any $x \in X \setminus M$ in any inproper subspace M and from theorem 1 [13] on that for any orthogonal to M by Birkhoff vector s.i.p. generating norm in X with respect to which this vector is normal to M, is found.

From this proposition, using lemma 5 [10, pg. 441] we obtain.

Corollary 2.1. *In a reflexive rotund Banach space X for any subspace M and for any $x \in X$ there exists (generally speaking non unique) s.i.p. s, generating norm in X and there exist unique $y \in M$ and $z \in M^{\perp_s}$ that $x = y + z$. If besides space X is also smooth, then s.i.p.s determined uniquely.*

Proposition 2.2. *In any Banach space X from its decomposability by Riesz-Relix follows the justice of Riesz-Frechet representation.*

Proof is as in the scheme of arguments in a Hilbert space. One can be convinced in the existence of vector $y \in X$ representing the given functional $f(x) = [x,y]$ for any $x \in X$, from Riesz-Rellich decomposition of space X with participation of kernel of the functional f. Coincidence of norms f and y follows from the justice of Cauchy inequality for s.i.p.

From the statements 2.1. and 2.2. follows that in reflexive Banach space Riesz-Frechet representation [13, theorem 2] holds, moreover corollary2.1 gives the following statement.

Corollary 2.2. *If X is a reflexive rotund Banach space, then the element $y \in x$ representing the given functional $f \in X^*$, by Riesz-Frechet is determined uniquely.*

In particular, we have the following result of paper [11, proposition 7].

Corollary 2.3. *In a reflexive rotund smooth Banach space X for any $f \in X^*$ there exist the unique vector $y \in X$ and unique s.i.p [,] giving norm in X, such that $f(x) = [x,y]$, and the norms f and are equal.*

More particular case of corollary 2.3 is theorem 6 [10].

Proposition 2.3. From the justice of Riesz-Frechet representation in space X follows the solvability of the problem about Riesz-Levi normal in X.

Proof is in the verifying of the fact that the suitable nonzero vector normal to arbitrary nontrivial subspace M is vector representing by Riesz-Frechet functional annihilating M.

Proposition 2.4. *In any space X from the solvability of the problem about Riesz-Levi normal follows the reflexivity X.*

Proof follows from that any functional f achieves its norm on any unit vector which is normal to the kernel of functional f [4,pg 26 theorem 3] with the followig application of James theorem [4, pg. 16, theorem 3] on sufficient condition of the reflexivity of Banach space. Theorem 2.1 has been proved.

Note that in the chain of conditions equivalent to the reflexivity of space one can include the feasibility of the generalized Lax-Milgram representation. About theorem of Lax-Milgram in a Hilbert space see [3, pg. 134].

The part of the previous reflexivity criterion reestablishes the method of orthogonal complementation by proving the localizability of compression spectrum of operator by the scheme of work [9, theorem 1], where stronger conditions of uniform rotundness and the smoothness of space is required.

Example. Proof of the localization relation (10) by the orthogonal complement method.

If $\lambda \in \sigma_\gamma(T)$, then the closure of the range $R = \overline{R_{an}}(T-\lambda)$ of the operator $T-\lambda$ 'll be an proper subspace in a reflexive X. By virtue of solvability of problem about normal (theorem 2.1) there will be found the s.i.p. [,] generated norm in X, and unique vector $y \in X$, such that $[z,y]=0$ for any $z \in R$. Determining the functional $f \in X^*$ for any $z \in X$ by the formula $f(z)=[z,y]$, obviously we have $\|f\|=1, f(y)=1$, moreover $\lambda = f(Ty)$. So $\lambda \in V(T)$.

Now we'll get one more result from theorem2.1. To show deeper analogue with Hilbert space J.Giles [10, theorem 7] showed the selfadjointness of uniformly rotund and uniformly smooth Babach spaces. From the previous theorem 2.1 follows, that one can establish the complete duality between space and its dual in the following wider class of Banach spaces.

Theorem 2.2. *Any reflexive rotund smooth Banach space X with norm* v *is isometrically isomorphic to the dual Banach space* X^* *with dual norm* v^* *which is reflexive rotund and smooth.*

Scheme of proof copies reasonings from geometry of Hilbert spaces. Unique s.i.p. [,] in X giving its norm v determines the mapping $x \to f_x$ by the formula $f_x(y)=[y,x]$ which is bijective by virtue of Riesz-Frechet generalized theorem. Then'll be checked up that the formula $[f_x,f_y]_* = [y,x]$ correctly determines the function on $X^* \times X^*$,which is s.i.p. in X^* generating the norm v^*. As reflexivity property is invariant with respect to Banach conjugation, and the properties of roundness and smoothness are dual to each-other when reflexivity is available, then the well-known remark on the completeness of the space X^* will finish the proof.

2.3. In the given point let's briefly consider how geometric results from point 2.2 permit to apply reasonings of the theory of Hilbert spaces to the investigations of generalized self-adjoint and unitary operators in reflexive rotund smooth Banach spaces. More general results are cited in [17a,17f].

Parallel with Hermitan operators (by Vidav and Lumer) [2,ch.II] there exist another version of carring over of self-adjoint operators to general Banach space going up to J.Stampfli [14]. In uniformly rotund smooth space by investigating such operators D. Kochler [9] proposed the approach being the adaptation of methods form the theory of Hilbert spaces using J.Giles [10, theorem 6] theorem about Riesz-Frechet generalized representation of linear functional.

Our previous theorem [see, point 2 corollary 23] permits us beginning from the definition of generalized self-adjoint operators to repeat all stated in [9] for a wider class of reflexive smooth rotund Banach spaces. Not stopping on this, let's consider spectral properties of generalized self-adjoint and unitary operators which have been not reflected in D. Kochler's paper.

For self-adjoint and unitary operators in a Hilbert space Taylor-Halberg [16, theorem 21] possible states, that describe their thick structure of spectrum, are known. This gives in particular Weyl's classic theorem about emptiness of residual spectrum of such operators [5, pg 365]. In paper [17a] the similar results for Hermitian operators in Banach spaces (in detail about it see part II of the present paper), have been represented. At the same place [17a] validity of these facts (without mentioning of Taylor-Halberg states) has been noted, for the generalized p self-adjoint operators, $1 < p < \infty$, giving for $p = 2$ the class of Kochler operators in more general Banach spaces.

In the remaining part of the exposition X is a reflexive rotund smooth space.

Theorem 2.3. The generalized self-adjoint (unitary) operator $T \in \beta(X)$ has exactly three possible states I_1, II_2, III_3, where I_1 means that T is surjective and $0 \notin \sigma_\pi(T)$; II_2 means that T is not surjective, but $0 \notin \sigma_\gamma(T)$ and $0 \in \sigma_\pi(T) \setminus \sigma_p(T)$; and finally III_3 means that $0 \in \sigma_\gamma(T)$ and $0 \in \sigma_\rho(T)$.

Proof consists of three steps. Firstly we show that the kernel $\ker T$ orthogonally (with respect to s.i.p generating the norm in X) the closure $\overline{Ran\,T}$ of range of the operator T. Later we prove that the direct sum $KerT \oplus \overline{RanT}$ is closed in X, and finally that it is dense in the space X.

Ar corollary of this theorem one can show that for generalized self-adjoint (unitary) operators many properties of the same named operators in Hilbert spaces remain true. Anitting their enumeration, note for example two results: the analogue of Weyl theorem about essential spectrum and criterion of closeness of numerical range of unitary operators .

Theorem 2.4. *For the Weyl spectrum $\sigma_w(T)$ of the generalized self-adjoint (unitary) operators $T \in \beta(X)$ the Weyl's theorem holds, i.e.*

$$\sigma_w(T) = \sigma(T) - \pi_{00}(T),$$

where $\pi_{00}(T)$ is a set of isolated eigenvalues of finite geometrical multiplicity, and Weyl spectrum $\sigma_w(T) = \{\lambda \in \mathcal{C}: T - \lambda \notin \Phi_0\}$, where Φ_0 is a set of Fredholm operators of zero index, i.e. such $T \in \beta(x)$ for which $RanT$ is closed, $\dim KerT = co\dim RanT$ and are finite.

Scheme of the proof. Firstly using nowhere non-density spectrum of the operator T, applying the Werner's result [15, pg 469] we obtain for T fulfillment of Weyl type theorem, i.e. fulfillment of the relation $\sigma_w(T) = \sigma(T) - \hat{\pi}_{00}(T)$, where $\hat{\pi}_{00}(T)$ is a set of isolated eigenvalues of finite algebraic multiplicity [7, pg 229]. Then for the coincidence of the sets $\pi_{00}(T)$ and $\hat{\pi}_{00}(T)$ we are convinced that the first of them is included in the second one. It is obtained by considering restriction of the operator T on the range of the spectral projector corresponding to points from $\pi_{00}(T)$, and by applying theorem 6 from [14, pg 510] and theorem 20 from [6, pg 614].

Theorem 2.5. *Let $T \in \beta(X)$ be a generalized unitary operator. Then for closeness of Bauer numerical range $V(T)$ it is necessary and if T - convexoid [1, pg.118] it suffices that the spectrum $^{-}\sigma(T)$ coincides with pointwise spectrum $\sigma_p(T)$.*

Scheme of the proof. The necessity of the condition follows from the localization relation (6) §1 and Wintner-Lumer theorem about «principal» points of numerical range [2, §9 theorem 8]. Sufficiency follows from Krain-Milman theorem [5, pg 85 theorem 3.21] and from the localization relation (7) §1.

Let's make brief notes on previous three theorems. Theorem 2.3 is true in weak complete (reflexive) Banach space and there are four possible states in general Banach space. Theorem 2.4 is just in arbitrary Banach space and the supplemented version of theorem 2.5 remains true for weaker conditions on geometry of space.

These and other more general results on normal, selfadjoint and unitary operators in Banach space have been reflected in [17, e-f]. It is appropriate to note that unlike the Hilbert space case, in Banach space these three classes of operators are independent generally speaking between themselves. Also the closes of Hermitian and generalized self-adjoint operators are independent between themselves in general case.

As an example let's illustrate the application of theorem 2.1 to the proof by the method of Hilbert spaces of one property of spectrum of generalized self-adjoint operator. This property is used in problem of approximation of the inverse operator which 'll be discussed in subsequent part of the paper.

Proposition 2.5. *Injectivity of generalized self-adjoint operator T is equivalent to the injectivity of the adjoint operator T^*.*

Proof. If $0 \in \sigma_p(T^*)$, then $\overline{RanT}$ is proper subspace in X and by theorem 2.1 there exists a nonzero normal vector y with respect to s.i.p. [,]: $[Tx,y]=0$, that by virtue of the self-adjointness of T implies $[x,Ty]=0$ for any $x \in X$. Consequently, $Ty=0$, i.e. $0 \in \sigma_p(T)$. Inversely, if $0 \in \sigma_p(T)$, then $K = \ker T$ is an proper subspace in X and by theorem 2.1 we have Riesz-Rellich decomposition, i.e. $x = y + z$ for any $x \in X$, where $y \in K$ and $z \in K^{\perp}$. Hence it is easy to see, that $RanT$ lies in $K^{\perp}$. This by virtue of elosureness of the set $K^{\perp}$ and the strong inclusion $K^{\perp} \subset X$ implies the non-density $RanT$ in X, that is equivalent to the inclusion $0 \in \sigma_p(T^*)$. The statement has been proved.

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