MECHANICS

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THE TORSIONAL VIBRATIONS OF A NON-LINEAR BIOELASTIC ROUND BAR

Abstract

The equation of the torsional vibrations of a finite cylindrical body subject to the biofactor of the lag type has been derived. The solution is based on the method of the small parameter. The recurrent system of the solving equation has been got. The amplitude-frequency curves for the first two approximations have been constructed.

In the present paper on the basis of the model [1] the problem about the torsional vibrations of the nonlinear elastic round bar of finite length with the biological reaction is considered.

The equation of the torsional motion, as known, has the form:

$$\rho J_{p_0} \frac{\partial^2 \varphi}{\partial t^2} = \frac{\partial M}{\partial x}, \qquad (1)$$

where M is torque, $J_{p_0} = \frac{\pi}{2} a^4$, a is the radius of the cross-section of a bar, φ is the angle of swing of bar.

To account the biofactor following to [1] in the equation of motion (1) replace the moment M, reduced by $M + R$, where $R = -AM(t-\tau)$ is the term, accounting the presence of the biofactor.

The sign minus account the reaction of the biofactor to the external effort. Besides that, the parameters A and τ of the biofactor must satisfy the inequalities

$$0 < A < 1 \text{ and } 0 < \tau \le 1 .$$

Note, that the parameter τ is the response lag to the external action. Subject to the trifle of the lag, we'll get

$$M + R = M(t) - AM(t-\tau) = (1-A)M(t) + A\tau \frac{\partial M}{\partial t}. \qquad (2)$$

Following to Couderery [2] we adopt the following non-linear low, connected the torque and the angle of swing of the sector of the bar.

$$M(t) = GJ_{p_0}\left[1 + \gamma_2 \frac{J_{p_2}}{J_{p_0}}\left(\frac{\partial \varphi}{\partial x}\right)^2\right]\frac{\partial \varphi}{\partial x}, \qquad (3)$$

where G is a shear modulus. J_{p_0}, J_{p_2} are the moment of inertion of axial section, which in particular, for the hollow cylinder have the form:

$$J_{p_0} = \frac{\pi}{2}\left(a^4 - b^4\right); \quad J_{p_2} = \frac{2}{9}\pi\left(a^6 - b^6\right).$$

Here γ_2 is the parameter of non-linearity. Subject to (3) in (2), and the latter in the equation of motion (1) finally we'll obtain the solving equation of the non-linear motion of the bioelastic bar in the form:

$$\rho J_{p_0}\frac{\partial^2\varphi}{\partial t^2}=\frac{\partial}{\partial x}\left\{(1-A)GJ_{p_0}\left[1+\gamma_2\frac{J_{p_0}}{J_{p_2}}\left(\frac{\partial\varphi}{\partial x}\right)^2\right]\frac{\partial\varphi}{\partial x}+\right.$$

$$\left.+A\tau GJ_{p_0}\frac{\partial}{\partial t}\left(\left[1+\gamma_2\frac{J_{p_2}}{J_{p_0}}\left(\frac{\partial\varphi}{\partial x}\right)^2\right]\frac{\partial\varphi}{\partial x}\right)\right\}. \tag{4}$$

On the basis of the equation (4) the problem about the torsional vibrations of the bar of the finite length, one end-wall of which is free from efforts and the another subjected to the torsional vibration of the given amplitude and frequency, has been considered.

Corresponding record of the boundary conditions has the form:

$$\varphi(0,t)=A_0\cos\omega t,\ \ \varphi(l,t)=0 \tag{5}$$

here A_0 is the amplitude given in the left end-wall, ω is the frequency of the vibration given in the same place.

The experimental data indicates the smallness either of the parameter γ_2 or of its component. This permits us to represent this parameter in the form $\gamma_2=\gamma_2'\lambda$, where λ is the dimensionless small parameter. Namely the presence of the small parameter in the non-linear low permits Couderery be content with the cubic dependence between the stress and elasticity. This makes possible to use the method of the small parameter to search the solution of problem. Let's decompose the desired function $\varphi(x,t)$ in series by the small parameter of non-linearity

$$\varphi(x,t)=\sum_m\lambda^m\varphi_m(x,t). \tag{6}$$

Subject to the decomposition (6) in (4) and comparing the terms of the same smallness order for the bioelastic material of bar, we get the system of the recurrent differential equations in the form:

$$L\varphi_m(x,t)=f_m(\varphi_0,\dots,\varphi_{m-1})\quad m=0,1,2,\dots, \tag{7}$$

where the operator L has the following representation:

$$L=(1-A)GJ_{p_0}\frac{\partial^2}{\partial x^2}+A\tau GJ_{p_0}\frac{\partial^3}{\partial x^2\partial t}-\rho J_{p_0}\frac{\partial^2}{\partial t^2}. \tag{8}$$

For the first two approximations:

$$f_1(\varphi_0)=3(1-A)GJ_{p_2}\gamma_2'\frac{\partial^2\varphi_0}{\partial x^2}+6A\tau GJ_{p_2}\gamma_2'\frac{\partial\varphi_0}{\partial x^2}\cdot\frac{\partial\varphi_0}{\partial x}\cdot\frac{\partial^2\varphi_0}{\partial x\partial t}+$$

$$+3A\tau GJ_{p_2}\gamma_2'\left(\frac{\partial\varphi_0}{\partial x}\right)^2\frac{\partial^3\varphi_0}{\partial x^2\partial t},$$

$$f_2(\varphi_0,\varphi_1)=A(1-A)GJ_{p_2}\gamma_2'\frac{\partial\varphi_0}{\partial x}\cdot\frac{\partial\varphi_1}{\partial x}\cdot\frac{\partial^2\varphi_0}{\partial x^2}+$$

$$+3(1-A)GJ_{p_2}\gamma_2'\left(\frac{\partial\varphi_0}{\partial x}\right)^2\frac{\partial^2\varphi_1}{\partial x^2}+6A\tau GJ_{p_2}\gamma_2'\frac{\partial^2\varphi_0}{\partial x^2}\cdot\frac{\partial\varphi_1}{\partial x}\cdot\frac{\partial^2\varphi_0}{\partial x\partial t}+$$

$$+6A\tau GJ_{p_2}\gamma_2'\frac{\partial\varphi_0}{\partial x}\cdot\frac{\partial^2\varphi_1}{\partial x^2}\cdot\frac{\partial^2\varphi_1}{\partial x\partial t}+6A\tau GJ_{p_2}\gamma_2'\frac{\partial\varphi_0}{\partial x}\cdot\frac{\partial\varphi_1}{\partial x}\cdot\frac{\partial^3\varphi_0}{\partial x^2\partial t}+$$

$$+6A\tau GJ_{p_2}\gamma_2'\frac{\partial\varphi_0}{\partial x}\cdot\frac{\partial^2\varphi_0}{\partial x^2}\cdot\frac{\partial^2\varphi_1}{\partial x\partial t}+$$

$$+3A\tau GJ_{p_2}\left(\frac{\partial\varphi_0}{\partial x}\right)^2\frac{\partial^3\varphi_1}{\partial x^2\partial t}. \tag{9}$$

The decomposition (6) permitted us to split the boundary conditions (5) to the following

$$\begin{cases}\varphi_0(0,t)=A_0\cos\omega t, & \varphi_0(l,t)=0,\\ \varphi_m(0,t)=0, & \varphi_m(l,t)=0 \quad m=1,2,\dots\end{cases} \tag{10}$$

The solution of the boundary-value problems (1)-(10) is seeked in the form:

$$\varphi_m(x,t)=\sum_k\left[\Phi_{m,k}(x,\omega)e^{-i\omega tk}+\overline{\Phi_{m,k}(x,\omega)}e^{ik\omega t}\right]. \tag{11}$$

Let's reduce here the obtained form of the solution for zero and the first two approximations

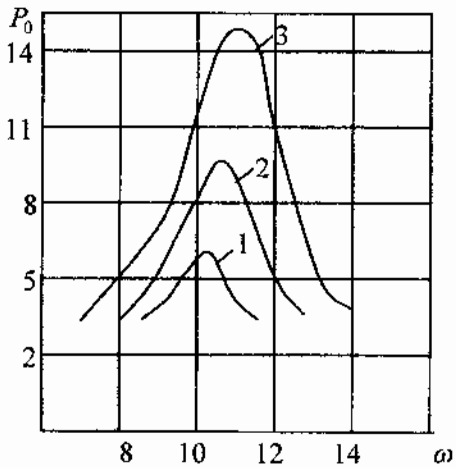

Fig.1. The influence of biofactor (zero approximation)
$1-\tau=6\cdot10^{-4}$, $2-\tau=2\cdot10^{-4}$, $3-\tau=10^{-4}$.

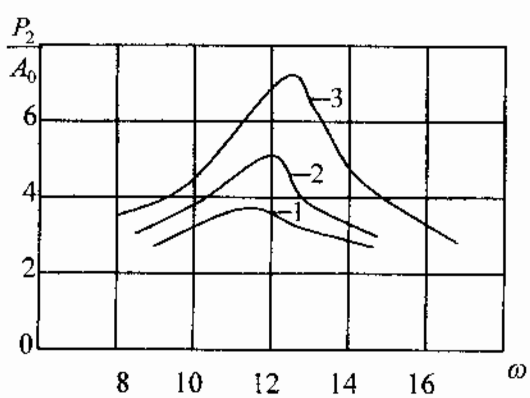

Fig.2. The influence of biofactor (second approximation)
$1-\tau=6\cdot10^{-4}$, $2-\tau=2\cdot10^{-4}$, $3-\tau=10^{-4}$.

$$\varphi_0(x,t)=P_0(x,\omega)\cos[\omega t-Q_0(x,\omega)],$$
$$\varphi_1(x,t)=P_1(x,\omega)\cos[2\omega t-Q_1(x,\omega)]+V_1(x,\omega),$$
$$\varphi_2(x,t)=P_2(x,\omega)\cos[\omega t-Q_2(x,\omega)]+P_3(x,\omega)\times$$
$$\times\cos[3\omega t-Q_3(x,\omega)], \qquad (12)$$

where $P_m(x,\omega)$, $Q_m(x,\omega)$, $V_1(x,\omega)$ are the known functions of the parameters of problem, whose expressions are not led here, because of their inconvenience.

The dependencies of the amplitude $P_m(x,w)$ on the frequency, so-called, the amplitude-frequency characteristics rate for the different values of the parameters of biofactor, in particular, for the response lag τ.

The results of this calculations have been reduced in the form of graphics.

In the fig.1 and 2 the amplitude-frequency curves for zero P_0u and for the second approximation P_2 for three values of the parameter of the response $\tau\left(\tau=10^{-4};\ \tau=2\cdot10^{-4};\ \tau=6\cdot10^{-4};\ \right)$ have been reduced. The solution (12) and the numerical calculations show, that the non-linearity reduces to the stimulation of higher harmonics and the saturation of the initial sinusoidal form of wave.

The influence of the biofactor manifest itself in that that the corresponding amplitude frequency curves become hollow from the increasing of the parameter of the response lag τ but the major resonance moves to the less frequencies side

References

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Received February 24, 2000; Revised October 25, 2000.
Translated by Nazirova S.H.