IBADOV N.V.

CONVOLUTION EQUATIONS IN SPACES OF FUNCTIONS OF GIVEN INCREASE NEAR THE BOUNDARY
II
COUNTABLE SYSTEMS OF INHOMOGENEOUS CONVOLUTION EQUATIONS

Abstract

The paper is devoted to the study of infinite system of convolution equations in some class of analytic functions of given increase near the boundary.

Introduction.

The paper is devoted to the study of infinite system of convolution equations in some class of analytic functions of given increase near the boundary.

Convolution equations, whose partial cases are infinite order differential equations with constant coefficients and other functional equations, have been studied by many mathematicians. Such equations appear, first of all in applied problems (see for instance [1-3]), secondly, in a complex analysis by solving many theoretical problems. For example in a theory of one complex variable functions homogeneous convolution equations are used to study Dirichlet series, completeness problems of holomorphic functions theory and others (see [4-6]).

One of principal problems in a convolution equations theory is the investigation of a solvability problem of an inhomogeneous convolution equation, and also systems of such equations. Speaking about these papers, devoted to this problem, papers [3-16] must be particularly distinguished. In particular in [2] a general solvability test of an inhomogeneous system of convolution equations in an entire functions space was obtained. This test has been investigated in [14-16].

The paper consists of two paragraphs. At the first paragraph some necessary information are given.

In Paragraph 2, the results obtained at the first part of this paper are applied the problem on solvability of countable inhomogeneous system of convolution equations on the space $H(p,\mathbf{D})$, and also for the uniqueness of the solution of the indicated system.

1. Preliminary information.

1.1. Spaces $H(p,\mathbf{D})$.

Let $\mathbf{D}$ be a bounded convex domain on the plane $\mathbf{C}$, $0\in\mathbf{D},\mathbf{d}(z)$ be the distance from the point $z\in\mathbf{D}$ to the boundary $\mathbf{D}$, $H(\mathbf{D})$ be a space of functions analytic in $\mathbf{D}$, with a uniform convergence topology in compact. For $p>0$ introduce the set $H(p,\mathbf{D})$ of functions $f(z)$, satisfying the condition

$$|f(z)|\le C(f,B)\exp\left\{B\frac{1}{(d(z))^p}\right\},\ \ \forall B>0. \tag{1.1}$$

Take the sequence $B_n\downarrow 0$ and introduce Banach spaces

$$Q_n=\left\{f(z)\in H(D):\|f\|_n=\sup_{z\in D}|f(z)|\exp\left[-B_n\frac{1}{(d(z))^p}\right]<\infty\right\}.$$

The set $H(p,\mathbf{D})$ coincides, obviously, with intersection of spaces Q_n, and on the space $H(p,\mathbf{D})$ one can consider a topology of projective limit of Banach spaces Q_n:

$$H(p,\mathbf{D})=\lim_{n\to\infty}\mathbf{PrQ_n}.$$

By $H^*(p,\mathbf{D})$ denote a space adjoint to $H(p,\mathbf{D})$, in which a strong topology is introduced. Describe the space $H^*(p,\mathbf{D})$ in terms of Laplace transformation:

$$\hat{S}(\lambda)=(S,\exp(\lambda z)),\qquad S\in H^*(p,\mathbf{D}).$$

Let $K(\varphi)$ be a support function of the compactum $\overline{\mathbf{D}}$ and $q\in(0,1)$. Consider the sequence of normed spaces E_n of entire functions $\varphi(\lambda)$ with norm

$$\|\varphi\|_n=\sup|\varphi(\lambda)|\exp\left[-h(\arg\lambda)|\lambda|+B_n|\lambda|^q\right],\quad h(\theta)=K(-\theta).$$

By $P_{(q)}(D)$ denote an inductive limit of norm spaces E_n:

$$P_{(q)}(\mathbf{D})=\lim_{n\in\infty}\mathbf{indE_n}.$$

It is valid

Theorem 1.1. *The space $H^*(p,D)$ is topologically isomorphic to the space $P_{(q)}(\mathbf{D})$, where $q=p/(p+1)$.*

The proof of theorem 1.1 is given in paper [16].

1.2. Generating ring $E_{[q,0]}$.

By $E_{[q,0]}$ denote a ring of all entire functions of q order and of zero type, i.e. satisfying the inequality

$$|\varphi(\lambda)|\le C\exp\left\{\varepsilon|\lambda|^q\right\},\qquad \forall\varepsilon>0,\ \lambda\in\mathbf{C}.$$

In Paragraph 2, part I of the present paper the following problem is studied: When does the given system of functions $\varphi_1,\varphi_2,\dots,\varphi_s,\dots\in E_{[q,0]}$ generate the whole of the ring $E_{[q,0]}$?

When a given system of functions $(f_1(z),f_2(z),\dots,f_n(z))$ is finite, in paper [21] this problem has been studied in the ring $H_p(\Omega)$ of functions $f(z)$, holomorphic in $\Omega\subset\mathbf{C}^n$ and satisfying the estimate

$$|f(z)|\le C_1\exp\{C_2p(z)\},\quad C_1=C_1(f),\ C_2=C_2(f),$$

where $p(z)$ is some nonnegative function given in Ω. The ring $H_p(\Omega)$, as it is not difficult to see, is the inductive limit of norm spaces

$$H_p^n(\Omega)=\left\{f\in H(\Omega):\|f\|\right\}=\sup\frac{|f(z)|}{\exp(\ln p(z))}$$

where $C_n\to\infty$, $n\to\infty$.

In paper [16] the problem, when does the given finite system of functions $\varphi_1,\varphi_2,\dots,\varphi_5\in E_{[q,0]}$ generate the whole of the ring $E_{[q,0]}$ is studied? Unlike $H_p(\Omega)$ the ring $E_{[q,0]}$ is the projective limit of spaces

$$E_{[q,\varepsilon_m]}=\left\{\varphi:\|\varphi\|_m=\sup\frac{|\varphi|}{\exp\{\varepsilon_m|\lambda|^q\}}<\infty\right\},$$

where $\varepsilon_m \downarrow 0$, $m=1,2,\ldots$. Note, that the sequence of Banach spaces $E_{[q,\varepsilon_1]}, E_{[q,\varepsilon_2]},\ldots,E_{[q,\varepsilon_m]},\ldots$ satisfies the relations $E_{[q,\varepsilon_1]}\supset E_{[q,\varepsilon_2]}\supset\ldots\supset E_{[q,\varepsilon_m]}\supset\ldots$. Therefore

$$E_{[q,0]}=\lim_{m\to\infty}\operatorname{Pr}E_{[q,\varepsilon_m]}.$$

Theorem 1.2. *The elements* $\varphi_1,\varphi_2,\ldots,\varphi_s\in E_{[q,0]}$ *generate the ring* $E_{[q,0]}$ *if and only if for any* $\varepsilon>0$ *the inequality*

$$|\varphi_1(\lambda)|+\ldots+|\varphi_s(\lambda)|\geq C_\varepsilon\exp\left[-\varepsilon|\lambda|^q\right]. \tag{1.2}$$

is fulfilled.

The proof of Theorem 1.2 is given in paper [16].

2. Inhomogeneous countable systems of convolution equations.

Every function $\varphi\in E_{[q,0]}$ by virtue of Theorem 1.1 generates some functional $\mu\in H^*(p,\mathbf{D})$. Therefore it determines some convolution operator

$$M_\varphi: H(p,\mathbf{D})\mapsto H(p,\mathbf{D}),$$

having the form

$$M_\varphi[f]=M_\varphi * f=\int f(z+t)d\mu. \tag{2.1}$$

Let B_n be the sequence of positive numbers, $B_n\to 0$. Introduce a sequence of spaces

$$\mathcal{H}_n^p=\{f(z)=(f_1(z),f_2(z),\ldots,f_m(z),\ldots): f_j(z)\in H(p,\mathbf{D})\}$$

with norm

$$\|f\|_n^2=\sum_{j=1}^{\infty}\int_D|f_j(\xi)|^2\exp\left[-2\left(B_n\frac{1}{(d(z))^p}\right)\right]d\lambda(\xi)<\infty,\quad n=1,2,\ldots.$$

Note, that $\mathcal{H}_n^p$ is a Hilbert space.

A projective limit of the space $\mathcal{H}_n^p$ denote by $\mathcal{H}^p$. Let $F=(F_1,F_2,\ldots,F_m,\ldots)$ be a functional belonging to the adjoint to $\mathcal{H}^p$ space $(\mathcal{H}^p)^*$. Then, there will be found such a number B_k for some element $h_F(z)=(h_1(z),h_2(z),\ldots,h_m(z),\ldots)\in\mathcal{H}_k^p$ the equality

$$(F,f)=\sum_{j=1}^{\infty}(F_j,f_j)=\sum_{j=1}^{\infty}\int_{\mathbf{D}}f_j(\xi)\overline{h}_j(\xi)\exp\left\{-2\left[B_k\left(\frac{1}{(d(z))^p}\right)\right]\right\}d\lambda(\xi) \tag{2.2}$$

will be valid, where $f(z)$ is an arbitrary element from $\mathcal{H}^p$.

By $\mathcal{L}$ denote a mapping which to each $F=(F_1,F_2,\ldots,F_m,\ldots)\in(\mathcal{H}^p)^*$ puts in correspondence the element

$$\hat{g}_F(z)=\left(F,e^{<\lambda,z>}\right)=(\hat{g}_{1F}(z),\hat{g}_{2F}(z),\ldots,\hat{g}_{mF}(z),\ldots)$$

where $\hat{g}_{jF}(z)$, $j \geq 1$ is the Laplace transformation of the functional F_j, i.e. $\hat{g}_{jF}(z) = (F_j, \exp\{<\lambda, z>\})$. Note that all $\hat{g}_{jF}(z)$ are entire functions from the space $P_{(q)}(\mathbf{D})$ (see paragraph 2, part I).

Introduce the space $\mathcal{P}_k(\mathbf{D})$ as a sequence set of entire functions $\varphi(\lambda) = (\varphi_1(\lambda), \varphi_2(\lambda), \ldots, \varphi_m(\lambda), \ldots)$ for which the estimate

$$(Q_2(\varphi)(\lambda))^2 \equiv \sum_{j=1}^{\infty} |\varphi_j(\lambda)|^2 \leq A(f, m) \exp\left\{2\left[H_{\overline{\mathbf{D}}}(\lambda) - B_k |\lambda|^q\right]\right\} \qquad (2.3)$$

is fulfilled.

Note, that from the last equation it follows that φ_j, $j = 1,2,\ldots$, lies on the space E_k.

Assume $\mathcal{P}(\mathbf{D}) = \lim_{k\to\infty} \in \mathcal{P}_k(\mathbf{D})$.

Furthermore we assume that the bounds of the domain are twice differentiable and its curvature is strongly determined from 0 and ∞.

Theorem 2.1. *The mapping* $\mathcal{L}$ *establishes one-to-one correspondence between the spaces* $(H^p)^*$ *and* $P(\mathbf{D})$.

Proof. Let $F = (F_1, F_2, \ldots, F_m, \ldots) \in (\mathcal{H}^p)^*$. Then there will be found such integer $k > 0$, that for some element $h(z) = (h_1(z), h_2(z), \ldots, h_m(z), \ldots)\ \mathcal{H}_k^p$, the equality (2.2) will be valid, where $f \in \mathcal{H}^p$. Therefore, by virtue of Holder inequality, we have

$$|\hat{g}_{jF}|^2 \leq C(k) \int_{\mathbf{D}} |h_j(\lambda)|^2 \exp\left\{-2B_k\left(\frac{1}{d(z)}\right)^p\right\} d\lambda(z) \exp\left\{2\left[H_{\overline{\mathbf{D}}} - B_m |\lambda|^q\right]\right\},$$

where

$$B_m \leq B_n^{\frac{1}{p+1}} \frac{p+1}{p^{\frac{p}{p+1}}},\ j = 1,2,\ldots,$$

and $\hat{g}_{jF}(z) = (F_j \exp<\lambda, z>)$, $j = 1,2,\ldots$. From the last estimate with regard to belonging $h(z)$ to the set $\mathcal{H}_k^p$ we obtain

$$(Q_2(\mathcal{L}(F))(z))^2 \leq \mathcal{A}_{1k} \exp\left\{2\left|H_{\overline{\mathbf{D}}}(\lambda) - B_m |\lambda|^q\right|\right\},\ z \in \mathbf{C} \quad m = 1,2,\ldots.$$

Therefore, the element $\hat{g}_F(z) = \mathcal{L}(F)$ belongs to the set $\mathcal{P}(\mathbf{D})$.

Conversely, let $\varphi(\lambda)$ be an arbitrary element from $\mathcal{P}(\mathbf{D})$. Then for some integer k, the estimate (2.3) is fulfilled. From this estimate by Theorem 12.3 from [12] it follows the existence of functionals F_j, whose Laplace transformation coincides with corresponding function $\varphi(\lambda)$. Consider the element $F = (F_1, F_2, \ldots, F_m, \ldots)$. It is not difficult to establish that $F \in (\mathcal{H}^p)^*$. and $\mathcal{L}(f) = f$. Show that the found element $F \in (\mathcal{H}^p)^*$ is unique. Let $\mathcal{L}(F^1) = \mathcal{L}(F^2) = \varphi$ $F \in (\mathcal{H}^p)^*$, where $F^1, F^2 \in (\mathcal{H}^p)^*$, $\varphi(z) \in \mathcal{P}(\mathbf{D})$. Then

$$\mathcal{H}(F^1 - F^2) = 0. \qquad (2.4)$$

Since a linear span of the set $\{\exp<\xi,z>\}$, $z\in\mathbf{C}$, forms a dense subset $H(p,\mathbf{D})$, then from (2.4) it follows that $(F^1-F^2,\varphi)=0$ for any element $\varphi(\lambda)\in\mathcal{P}(\mathbf{D})$. Consequently, $F^1=F^2$. Theorem 2.1 has been proved. Describe the above introduced space $\mathcal{P}(\mathbf{D})$. It holds

Theorem 2.2. *The element* $\varphi(z)=(\varphi_1(z),\varphi_2(z),...,\varphi_m(z),...)$, $\varphi_j(z)\in H(\mathbf{C})$ *if and only if it belongs to the set* $\mathcal{P}(\mathbf{D})$, *when for some integer* $\widetilde{k}$ *the estimate*

$$\sum_{j=1}^{\infty}\int_{\mathbf{C}}|\varphi_j(\xi)|^2\exp\left\{-2\left[H_{\overline{D}}(\xi)-B_{\widetilde{k}}|\lambda|^q\right]\right\}d\lambda(\xi)<\infty \tag{2.5}$$

is fulfilled.

Proof. Let $\varphi(\lambda)\in\mathcal{P}(\mathbf{D})$. Then for some integer k the estimate (4.3) is valid i.e.

$$(Q_2(\varphi)(\lambda))^2\equiv\sum_{j=1}^{\infty}|\varphi_j(\lambda)|^2\le A(f,m)\exp\left\{2\left[H_{\overline{\mathbf{D}}}(\lambda)-b_k|\lambda|^q\right]\right\}.$$

Consider the number $\widetilde{k}$ such that $k<\widetilde{k}$. From the inequality (2.3) we get

$$\sum_{j=1}^{\infty}\int_{\mathbf{C}}|\varphi_j(\xi)|^2\exp\left\{-2\left[H_{\overline{D}}(\xi)-B_{\widetilde{k}}|\xi|^q\right]\right\}d\lambda(\xi)<a\int_{\mathbf{C}}\exp\left\{2\left(B_{\widetilde{k}}-B_k\right)|\xi|^q\right\}d\lambda(\xi).$$

Since $B_{\widetilde{k}}-B_k<0$, then the last integral converges, therefore we obtain the estimate (2.5). The necessity has been proved.

Prove the sufficiency. Let for the element $\varphi(\lambda)=(\varphi_1(\lambda),\varphi_2(\lambda),...,\varphi_m(\lambda),...)$, where $\varphi_j(\lambda)\in H(\mathbf{C})$ be fulfilled the estimate (2.5). Then for any j, $1\le j<\infty$ it is fulfilled for some $C>0$ the estimate

$$\int_{\mathbf{C}}|\varphi_j(\xi)|^2\exp\left\{-2\left[H_{\overline{\mathbf{D}}}(\xi)-B_{\widetilde{k}}|\xi|^q\right]\right\}d\lambda(\xi)<C<\infty. \tag{2.6}$$

Since $\partial\mathbf{D}$ is twice differentiable, then the function $H_{\overline{\mathbf{D}}}(\xi)-B_{\widetilde{k}}|\xi|^q$ is strictly subharmonic and therefore it satisfies conditions 1), 2) and 3), imposed on the function $\psi(z)$ from paragraph 2, part I. Therefore, taking into account the estimate (2.6) and the inequality (2.5) we obtain the inequality

$$|Q_2(\varphi)(\lambda)|^2\le C\exp\left\{2\left[H_{\overline{\mathbf{D}}}-B_n|\lambda|^q\right]\right\}$$

with some constants n and C. Therefore, 2.2 has been proved.

Pass to the definition of convolution in spaces $H(p,\mathbf{D})$. Let $\mathbf{G}$ be a bounded convex domain in $\mathbf{C}$ and $\varphi(\lambda)\in P_{(q)}(\mathbf{G})$.

Note that a differentiation operation in $H(p,\mathbf{D})$ may be considered as a dual operation to a multiplication operation in the space $P_{(q)}(\mathbf{D})$. By T_φ denote a multiplication operation by the function φ.

Definitions. *A convolution operator in the space* $H(p,\mathbf{D}+\mathbf{G})$, *generated by the function* $\varphi(\lambda)$ *we call the operation, dual to the multiplication operation* T_φ *of the elements of the space* $P_{(q)}(\mathbf{D})$ *by the function* $\varphi(\lambda)$.

Lemma 2.1. *The multiplication operation T_φ by the function $\varphi(\lambda)$ linearly and continuously acts from the space $P_{(q)}(\mathbf{D})$ in $P_{(q)}(\mathbf{D}+\mathbf{G})$.*

Proof. Since $\varphi(\lambda)\in P_{(q)}(\mathbf{D})$, then for $\varphi(\lambda)$ we obtain the estimate

$$|\varphi(\lambda)|\le C\exp\left\{H_{\mathbf{G}}(\arg\lambda)|\lambda|-B|\lambda|^q\right\},\qquad B>0,$$

where $H_{\mathbf{G}}(-\arg\lambda)$ is a support function of the compactum $\overline{\mathbf{G}}$. Let $\psi(\lambda)\in P_{(q)}(\mathbf{D})$. Show that $\psi(\lambda)\varphi(\lambda)\in P_{(q)}(\mathbf{D}+\mathbf{G})$. For $\psi(\lambda)$ we have the estimate

$$|\psi(\lambda)|\le C_1\exp\left\{H_D(\arg\lambda)|\lambda|-B_1|\lambda|^q\right\}.$$

Then the function $\psi(\lambda)\varphi(\lambda)$ is estimated as follows:

$$|\psi(\lambda)\varphi(\lambda)|=|\psi(\lambda)||\varphi(\lambda)|\le C_1C\exp\left\{[H_{\mathbf{D}}(\arg\lambda)+H_{\mathbf{G}}(\arg\lambda)]|\lambda|-(B_1+B)|\lambda|^q\right\}=$$
$$C_2\exp\left\{H_{\mathbf{D}+\mathbf{G}}(\arg\lambda)|\lambda|-\widetilde{B}|\lambda|^q\right\},$$

where $\widetilde{B}=B_1+B,\ \ H_{\mathbf{D}+\mathbf{G}}(\arg\lambda)=H_{\mathbf{D}}(\arg\lambda)+H_{\mathbf{G}}(\arg\lambda)$. Consequently, $\psi(\lambda)\varphi(\lambda)\in P_{(q)}(\mathbf{D}+\mathbf{G})$. The linearity of operation T_φ is obvious. Now prove the continuity of the operation T_φ. Take in $P_{(q)}(\mathbf{D})$ the sequence $\psi_n(\lambda)\in P_{(q)}(\mathbf{D})$ and $\psi(\lambda)\in P_{(q)}(\mathbf{D})$. Let $\psi_n(\lambda)$ be converged in $P_{(q)}(\mathbf{D})$ to $\psi(\lambda)$. Since.

$$|\psi_n-\psi|\le\varepsilon_n\exp\left\{H_{\mathbf{D}}(\arg\lambda)|\lambda|-B_1|\lambda|^q\right\},\quad \varepsilon_n\to 0,\quad n\to\infty,$$

and

$$|\varphi(\lambda)|\le C_1\exp\left\{H_{\mathbf{G}}(\arg\lambda)|\lambda|-B|\lambda|^q\right\},\quad B>0,$$

then for the difference $\varphi(\lambda)\psi_n(\lambda)-\varphi(\lambda)\psi(\lambda)$ we obtain the following estimate

$$|\varphi(\lambda)\psi_n(\lambda)-\varphi(\lambda)\psi(\lambda)|\le\varepsilon_nC_1\exp\left\{H_{\mathbf{G}+\mathbf{D}}(\arg\lambda)|\lambda|-\widetilde{B}|\lambda|^q\right\}=$$
$$=\widetilde{C}_n\exp\left\{H_{\mathbf{G}+\mathbf{D}}(\arg\lambda)|\lambda|-\widetilde{B}|\lambda|^q\right\},$$

where $\widetilde{B}=B_1+B, C_n\to 0,\ \ n\to\infty$. Hence, it follows that the multiplication operation T_φ of by the function φ continuously acts from $P_{(q)}(\mathbf{D})$ in $P_{(q)}(\mathbf{D}+\mathbf{G})$.

Let $\varphi(\lambda)=(\varphi_1(\lambda),\varphi_2(\lambda),\ldots,\varphi_m(\lambda),\ldots)\in\mathcal{E}^p_{[q,0]}$. Determine the operator M_φ acting by the rule: if $y(z)\in H(p,\mathbf{D})$, then

$$M_\varphi[y]=\begin{pmatrix}M_{\varphi_1}[y]\\ \cdots\cdots\\ \cdots\cdots\\ \cdots\cdots\\ M_{\varphi_m}[y]\\ \cdots\cdots\end{pmatrix},$$

where $M_{\varphi_j},\ j\ge 1$ are convolution operators in $H(p,\mathbf{D})$ with characteristic functions $\varphi_j(\lambda)$ correspondingly.

Theorem 2.3. *The operator M_φ is a linear and continuous mapping from the space $H(p,\mathbf{D})$ in $\mathcal{H}^p$.*

Proof. Let $\mathcal{K}$ be a set of continuous on $\mathbf{C}$ functions $k(\xi)$ increasing on infinity rapidly than any function of the form $\exp\{H_{\mathbf{D}}(\lambda)-B_n|\lambda|^q\}$, i.e.

$$\lim_{|\lambda|\to\infty}\frac{\exp\{H_{\mathbf{D}}(\lambda)-B_n|\lambda|^q\}}{k(\lambda)}=0.$$

It is known that (see [27]) for the arbitrary function $y(z)$ from $H(p,\mathbf{D})$ may be found such a complex valued measure of the bounded variation $\mu(\xi)$ and the function $k(\xi)\in K$, that the representation

$$y(z)=\int_{\mathbf{C}}\exp<z,\xi>\frac{d\mu(xi)}{k(\xi)}. \qquad (2.7)$$

is valid.

Applying the operator M_{φ_j} to the last equality we obtain

$$M_{\varphi_j}[y](z)=\int_{\mathbf{D}}\varphi_j(\xi)\exp<z,\xi>\frac{d\mu(\xi)}{k(\xi)}.$$

To prove that $M_\varphi[y]$ belongs to $\mathcal{H}^p$, it is necessary to show that $M_\varphi[y]\in\mathcal{H}_n^p$ for any integer n. Fix some integer n. We have

$$\sum_{j=1}^{\infty}\int_{\mathbf{D}}\left|M_{\varphi_j}[y](z)\right|^2\exp\left\{-2\left(B_n\frac{1}{(d(z))^p}\right)\right\}d\lambda(z)=$$

$$=\sum_{j=1}^{\infty}\int_{\mathbf{D}}\left|\int_{\mathbf{C}}\varphi_j(\xi)\exp\{(z,\xi)\}\frac{d\mu(\xi)}{k(\xi)}\right|^2\exp\left\{-2\left(b_n\frac{1}{(d(z))^p}\right)\right\}d\lambda(z)\le$$

$$\le\sum_{j=1}^{\infty}\int_{\mathbf{D}}\left[\int_{\mathbf{C}}|\varphi_j(\xi)|^{\tilde{p}}\frac{|d\mu(\xi)|}{k^{\tilde{p}}(\xi)}\right]^{\frac{2}{\tilde{p}}}$$

$$\left[\int_{\mathbf{C}}\frac{|\exp(z,\xi)|^{\tilde{q}}|d\mu(\xi)|}{k^{\tilde{q}}(\xi)}\right]^{\frac{2}{\tilde{q}}}\exp\left\{-2\left(B_n\frac{1}{(d(z))^p}\right)\right\}d\lambda(z)\le \qquad (2.8)$$

$$\le\tilde{C}\int_{\mathbf{D}}d\lambda(z)\left\{\int_{\mathbf{C}}\exp\left[H_D(\xi)-A_n|\xi|^q\right]\right\}^{\tilde{q}}\frac{|d\mu(\xi)|}{k^{\tilde{q}}(\xi)}<\infty,\qquad \frac{1}{\tilde{p}}+\frac{1}{\tilde{q}}=1.$$

At the last inequality it was used the fact, that $k(\xi)$ on the infinity increases rapidly than any function of the form $\exp\{H_{\mathbf{D}}(\xi)-a|\xi|^q\}$. Thus, for any integer n, the norm $\|M_\varphi[y]\|_n$ in the space H_n^p is finite, i.e. $M_\varphi[y](z)$ belongs to $\mathcal{H}^p$. From the inequality (2.8), the continuity of the operator $M_\varphi[y](z)$ follows, and its linearity is obvious. By the same, theorem 2.3 has been proved.

For $\varphi(z)=(\varphi_1(z),\varphi_2(z),\ldots,\varphi_m(z),\ldots)\in\mathcal{E}_{[q,0]}^l$ consider a countable system of inhomogeneous convolution equations

$$M_\varphi[y]=g, \qquad (2.9)$$

where $g(z)=(g_1(z),g_2(z),...,g_m(z),...)\in \mathcal{H}^p$. By $R(\varphi)$ denote a relation module generated by the element $\varphi(z)$, in $\mathcal{P}(\mathbf{D})$, i.e. $R(\varphi)$ is a totality of all elements $\psi(z)=(\psi_1(z),\psi_2(z),...,\psi_m(z),...)\in \mathcal{P}(\mathbf{D})$ for which it holds the identity

$$\sum_{j=1}^{\infty}\varphi_j(z)\psi_j(z)=0$$

in addition the convergence of the series at the left hand side of the last equality at the topology $P_{(q)}(\mathbf{D})$ is assumed. Let $R^*(\varphi)=\{S=(S_1,S_2,...,S_m,...)\in(\mathcal{H}^p)^*:\mathcal{L}(S)\in R(\varphi)\}$. It is not difficult to see, that a necessary solvability condition of the system (2.9) is the following equality

$$(S,g)=0 \tag{2.10}$$

for any $S\in R^*(\varphi)$. A set of all elements $g(z)\in\mathcal{H}^p$, satisfying (2.10) denote by G. It is obvious, that the set G is closed in $\mathcal{H}^p$. Let M_φ^* be any operator, conjugated with M_φ, acting from G^* in $H^*(p,\mathbf{D})$. From the definition of the adjoint operation we have: for any $f(z)\in H(p,\mathbf{D})$ and $S\in G^*$ the equality $(M_\varphi^*[S],f)=(S,M_\varphi[f])$ is valid.

If at the last expression as $f(z)$ to take $\exp<\xi,z>$, then we obtain

$$\hat{M}_\varphi^*[S]=\sum_{j=1}^{\infty}\hat{S}_j(\xi)\varphi_j(\xi). \tag{2.11}$$

Thus, the operator M_φ^* generates the operator $\hat{M}_\varphi^*$ its equivalent in a topological sense. Besides, from (2.11) and from theorem 2.1, the equality

$$im\,\hat{M}_\varphi^*=I(\varphi_1,\varphi_2,...,\varphi_m,...) \tag{2.12}$$

follows.

Theorem 2.4. *The system (2.9) is solvable in the space* $H(p,\mathbf{D})$ *for any* $g(z)\in G$ *if and only if the equality*

$$I(\varphi_1,\varphi_2,...,\varphi_m,...)=\overline{I(\varphi_1,\varphi_2,...,\varphi_m,...)} \tag{2.13}$$

is fulfilled.

Proof. The necessity. By virtue of closeness of the set G we have

$$\operatorname{im}M_\varphi=\overline{\operatorname{im}M_\varphi}. \tag{2.14}$$

Consequently, by theorem 3.5 from [12] the equality

$$\operatorname{im}M_\varphi^*=\overline{\operatorname{im}M_\varphi^*}$$

is valid.

Taking into account a topological equivalence of the operators M_φ^* and $\hat{M}_\varphi^*$ and the equality (2.12), from the last relation we obtain (2.13).

The sufficiency. Since the previous arguments are invertible, than from (2.13) the validity of the equality (2.14) follows. For the proof of sufficiency, it is necessary to establish that

$$\operatorname{im}M_\varphi=G. \tag{2.15}$$

For this purpose consider a set $\tilde{G}$ of elements from $\mathcal{H}^p$ of the form $(\tilde{g}_1(z),...,\tilde{g}_m(z),...)$ where $\tilde{g}_j(z)=\varphi_j(\xi)\exp<\xi,z>,\xi\in\mathbf{D}$. It is obvious, that $\tilde{G}\subset G$. And what is more, the closure of the set $\widetilde{G}$ coincides with G. Indeed, let S from $(\mathcal{H}^p)^*$ be so that $(S,l)=0$ for all $l\in\widetilde{G}$. Then $S\in\mathcal{R}^*(\varphi)$, and consequently, $(S,g)=0$ for all $g\in G$. Thus, $\widetilde{G}$ is dense in G. Therefore, from the equality (2.14) the relation (2.15) follows, i.e. theorem 2.4 has been proved completely.

Study a uniqueness problem for the solution of convolution equations system (2.9). In a general case the indicated system has a non-unique solution, for instance, in the case when the functions $\varphi_1(z),\varphi_2(z),...,\varphi_m(z),...$ have general zeros. Note that for the dimension $n>1$ the solution of the system (2.9) may be non-unique and when the functions $\varphi_1(z),\varphi_2(z),...,\varphi_m(z),...$ have no general zeros (this follows from the conclusions of paper [20]). Find a condition, when a countable system of convolution has a unique solution.

Theorem 2.5. *The system (2.9) has a unique solution in the class* $H(p,\mathbf{D})$ *for any* $g\in G$ *if and only if the equality*

$$\overline{I(\varphi_1,\varphi_2,...,\varphi_m,...)}=\mathcal{P}(\mathbf{D}) \tag{2.16}$$

is fulfilled.

Proof. It is obvious that the system (2.9) has a unique solution for any $g\in G$ if and only if when the inhomogeneous system

$$M_\varphi[y]=0 \tag{2.17}$$

has only a zero solution. Let W be a set of entire solutions of the last system, and $W^\perp$ be a set of all functionals from $H^*(p,\mathbf{D})$ vanishing in W. Since a set of Laplace transformations of functionals from $W^\perp$ coincides with $\overline{I(\varphi_1,\varphi_2,...,\varphi_m,...)}$, then the system (2.17) has only a zero solution if and only if the equality (2.16) is fulfilled. Theorem 2.5 has been proved.

Thus, from theorems 2.4 and 2.3 it follows that the solution $H(p,\mathbf{D})$ of the system (2.9) is unique for any $g\in G$ if and only if the equality

$$\overline{I(\varphi_1,\varphi_2,...,\varphi_m,...)}=\mathcal{P}(\mathbf{D})$$

holds.

Consequently, taking into account theorem 2.3, part I we obtain the following result

Theorem 2.6. *Let* $\varphi(\lambda)$ *belong to* $\mathcal{E}^r_{[q,0]}$ *and* $r>1$ *be some even number. In order that the system (2.9) have a unique solution in the class* $H(p,\mathbf{D})$ *for any* $g\in G$ *it is necessary and sufficient that the estimate*

$$Q_r(\varphi)\geq A_2(\varepsilon)\exp\{-\varepsilon|\lambda|^q\},\qquad \lambda\in\mathbf{C}$$

be fulfilled, where $\varepsilon>0$ *is an arbitrary number and* $A_2(\varepsilon)>0$ *is some constant.*

References

[1]. Maher K. *On special functional equation* // J. London Math. Soc., 1940, v.15, p.115-123.
[2]. Гарднер М.Ф., Бэрнс Дж.Л. *Переходные процессы в линейных системах.* М.: Гостехиздат, 1961.
[3]. Leontief W. *Lags and the stability of dynamic systems: rejoinder// Econometrica.* 1961, v.29, №4, p.674-675.

[4]. Леонтьев А.Ф. *Дифференциальные уравнения бесконечного порядка и их применения* // Тр. 4 Всесоюзн. мат. съезда, 1961, Л.: Наука, 1964, т.2, с.269-328.

[5]. Леонтьев А.Ф.*О последовательности полиномов Дирихле*// Тр. 3 Всесоюзн. мат. съезда, 1956, М.: Изд-во АН СССР, 1958, с.218-226.

[6]. Леонтьев А.Ф. *Ряды Дирихле, последовательности полиномов Дирихле и связанные с ними функциональные уравнения* // Математический анализ. М.: ВИНИТИ, 1987, т.13, С.5-55. (Итоги науки и техники).

[7]. Muggli H. *Differentialgleichungen unendlich hoher ordnung mit konstanten koeffuzienten – commeht* //Math. Helv., 1938, BdII, p.151-179.

[8]. Гельфонд А.О. *Линейные дифференциальные уравнения бесконечного порядка с постоянными коэффициентами и асимптотические периоды целых функций экспоненциального типа многих переменных.* М., 1981, Деп. В ВИНИТИ 10.04.81, №1571-81; Рж. Мат., 1981, 11Б 225 Деп.

[9]. Леонтьев А.Ф. *О свойствах последовательностей полиномов Дирихле, сходящихся на интервале мнимой оси* // Изв. АН СССР, Сер. Мат. , 1965, т.29, №2, с.269-328.

[10]. Коробейник Ю.Ф. *О решениях некоторых функциональных уравнений в классах функций, аналитических в выпуклых областях* // Мат. сб., 1968, т.75, №2, с.225-234.

[11]. Dicson D.G. *Convolution equations and harmonic analysis in spaces of entire functions* // Trans. Amer. Math. Soc., 1973, v.184, №2, p.373-385.

[12]. Напалков В.В. *Уравнения свертки в многомерных пространствах.* М.: Наука, 1982.

[13]. Lang H. *On systems of convolution with functions of exponential type* // Rept. Dep. Math. Univ. Stoch., 1980, №16, p.1-7.

[14]. Филиппов В.Н. *О неоднородных системах уравнений свертки в пространстве голоморфных функций.* Рукопись деп. В ВИНИТИ 17 февр., 1982 , 16 с., №715-82 Деп., РЖ Мат., 1982 6Б105 ДЕП.

[15]. Филиппов В.Н. *Спектральный синтез в некоторых пространствах целых функций экспоненциального типа* // Мат. заметки., 1981, т.30, №4, с.527-534.

[16]. Ибадов Н.В. *Неоднородные системы уравнений свертки в одном классе аналитических функций.* Сиб. мат. журн., 1988, т.XXXIX, №1, с.39-49.

[17]. Ибадов Н.В. *Аппроксимационная задача для уравнения свертки* // Известия вузов, «Математика», Казань, 1987, №10, с.12-16.

[18]. Ибадов Н.В. *Об уравнениях свертки в пространствах функций заданного роста вблизи границы.* Док. АН зерб. ССР, Баку, 1986, №11.

[19]. Ибадов Н.В. *Уравнения свертки в пространствах функции* $H(\rho,\mathbf{D})$. Всесоюзная школа по теории операторов в функциональных пространствах, часть 3, Челябинск, 1986, с.47.

[20]. Гуревич Д.И. *Контрпримеры к проблеме Л.Шварца* // Функциональный анализ и его приложения, 1975, т.9, №2, с.29-35.

[21]. Hormander L. *Generators for some rings of analytic functions.* Bull. Amer. Math. Soc., 1967, v.73, №6, p.943-949.

[22]. Хермандер Л. *Введение в теорию функций нескольких комплексных переменных.* М.: Мир, 1968, 280 с.

[23]. Шабат Б.В. *Введение в комплексный анализ.* Часть 2, м.: Наука, 1976, 400 с.

[24]. Keller J.J., Taylor B.A. *An application of the Corona theorem to some rings of entire functions* // Bull. Amer. Math. Soc., 1967, v.73, №2, p.246-249.

[25]. Ронкин Л.И. *Введение в теорию функций многих переменных.* М.: Наука, 1971, 432 с.

[26]. Тимофеев А.Ю. *О представлении решения уравнения бесконечного порядка в виде суммы двух решений* // Математические заметки, 1982, т.31, №2, с.245-256.

[27]. Ehrenpreis L. *Fourier analysis in several complex variables.* New-York Wiley –Interscience publishers, 1970, 506 p.

Ibadov N.V.
Gandga State University named after H.Zardaby.
187, Shah Ismayil Khataii str., Gandga, Azerbaijan.
Tel.: 679-57 (89522).

Received December 20, 1999; Revised January 10, 2000.
Translated by Aliyeva E.T.