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ON THE SOLVABILITY CONDITIONS OF THE BOUNDARY VALUE PROBLEMS FOR ONE CLASS OPERATOR-DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

Abstract

Sufficient conditions in terms of the coefficients of elliptic type operator-differential equation of the second order are obtained in the paper. These conditions provide solvability of some boundary value problems for this equation.

Let H be a separable Hilbert space, A is positively-defined self-adjoint operator in H. Denote by H_γ $(\gamma \geq 0)$ the scale of Hilbert spaces, generated by the operator A, i.e. $H_\gamma = D(A^\gamma)$, $(x,y)_\gamma = (A^\gamma x, A^\gamma y)$.

Denote by $L_2(R_+;H)$ the set of measurable vector-functions $f(t)$ with values from H, for which

$$\|f\|_{L_2(R_+;H)} = \int_0^\infty \|f(t)\|^2 dt < \infty .$$

Introduce the following Hilbert spaces [1]:

$W_2^2(R_+;H) = \{u : A^2 u \in L_2(R_+ : H),\, u'' \in L_2(R_+;H)\}$,

$\overset{\circ}{W}{}_2^2(R_+;H:0) = \{u : u \in W_2^2(R_+ : H),\, u(0) = 0\}$,

$\overset{\circ}{W}{}_2^2(R_+;H:1) = \{u : u \in W_2^2(R_+ : H),\, u'(0) = 0\}$,

$\overset{\circ}{W}{}_2^2(R_+ : H;0:1) = \{u : u \in W_2^2(R_+ : H),\, u(0) = 0,\ u'(0) = 0\}$

with the norm

$$\|u\|_{W_2^2(R_+;H)} = \left(\|u''\|^2_{L_2(R_+;H)} + \|A^2 u\|^2_{L_2(R_+;H)}\right)^{1/2} .$$

Here and further the derivatives are considered in sense of distributions theory [1]. Consider the operator-differential equation

$$P(d/dt\, u) = -(d/dt - \omega_1 A)(d/dt - \omega_2 A)u(t) + A_1 \frac{du}{dt} = f(t), \quad t \in R_+ = (0,\infty) \tag{1}$$

with one of the initial boundary conditions

$$u(0) = 0 \tag{2}$$

or

$$u'(0) = 0 . \tag{3}$$

Here A is positively-defined self-adjoint operator, A_1 is linear, generally speaking, unbounded operator in H, the numbers $\omega_1 < 0$, $\omega_2 > 0$.

Definition 1. *The problem (1), (2)((1), (3)) is called regularly solvable, if for each vector-function* $f(t) \in L_2(R_+;H)$ *there is unique vector-function* $u(t) \in W_2^2(R_+;H)$, *which satisfies the equation (1) almost everywhere in* R_+, *and the boundary conditions (2) ((3)) are fulfilled in sense of convergence of the norm of the space* $H_{3/2}(H_{1/2})$, *and the inequality*

$$\|u\|_{W_2^2} \le const\|f\|_{L_2}$$

takes place.

Sufficient solvability conditions of the problems (1), (2) and (1), (3) are shown in given paper. Note, that analogous problems for $\omega_1 = -1,\ \omega_2 = 1$ are studied, for example, in the works [2-4].

In proof of the main theorem on the regular solvability of the boundary value problems (1), (2) ((1), (3)) there are obtained the exact estimations of the norms of intermediate derivative $A\frac{d}{dt}$ through the main part of the equation (1) in the spaces $\overset{\circ}{W}{}_2^2(R_+;H:0,1)$, $\overset{\circ}{W}{}_2^2(R_+;H:0)$ and $\overset{\circ}{W}{}_2^2(R_+:H:1)$. These estimations also have independent mathematical interest. First of all consider the equation

$$P_0(d/dt, A)u = -(d/dt - \omega_1 A)(d/dt - w_2 A)u(t) = f(t), \quad t \in R_+, \tag{4}$$

with boundary condition (2) $(\omega_1 < 0, \omega_2 > 0)$.

Denote by P_0 the operator, acting from the space $\overset{\circ}{W}{}_2^2(R_+;H:0)$ to $L_2(R_+;H)$ by the following way:

$$P_0 u \equiv P_0(d/dt, A)u, \quad u \in \overset{\circ}{W}{}_2^2(R_+;H,0).$$

It takes place

Theorem 1. *Operator* P_0 *carries out the isomorphism from the space* $\overset{\circ}{W}{}_2^2(R_+;H)$ *on the space* $L_2(R_+;H)$.

Proof. Obviously, the equation $P_0 u = 0$ has only zero solution. From the other side, applying Plancherel theorem, it is easy to obtain that any $f(t) \in L_2(R_+;H)$ vector-function

$$u_1(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} P_0^{-1}(-i\xi, A)\left(\int_0^{\infty} f(\xi) e^{-i\xi(t-s)} ds\right) d\xi$$

belongs to the space $W_2^2(R;H)$ and satisfies the equation (4) almost everywhere. Denote by $\omega(t)$ narrowing $u_1(t)$ on $[0,\infty)$ and we search the regular solution of the equation (4) in the form:

$$u(t) = \omega(t) + e^{\omega_1 t A}\varphi, \quad \varphi \in H_{3/2}.$$

From the condition (2) we obtain that $u(t) = \omega(t) - e^{\omega_1 t A}\omega(0)$. As $\omega(t) \in W_2^2(R_+ : H)$, then according to the theorem on tracks [1, p.36] $\omega(0) \in H_{3/2}$. Consequently, $e^{\omega_1 t A}\omega(0) \in W_2^2(R_+;H)$ and because of it $u(t) \in \overset{\circ}{W}{}_2^2(R_+;H:0)$.

From the other side according to the theorem on intermediate derivatives [1, p.29]

$$\|P_0 u\|_{L_2}^2 \le 2\left(\|u''\|_{L_2}^2 + |\omega_1\omega_2|^2\|A^2 u\|_{L_2}^2 + |\omega_1 + \omega_2|^2\left\|A\frac{du}{dt}\right\|_{L_2}^2\right) \le const\|u\|_{W_2}^2.$$

Consequently, according to Banach theorem on the inverse operator P_0 is the isomorphism between the spaces $\overset{\circ}{W}{}_2^2(R_+;H:0)$ and $L_2(R_+ : H)$. Theorem is proved.

Thus, the norms $\|u\|_{W_2^2(R_+;H)}$ and $\|P_0u\|_{L_2(R_+;H)}$ are equivalent in the space $\overset{\circ}{W}{}_2^2(R_+;H:0)$. That is why, according to the theorem on intermediate derivatives the number

$$N_1(R_+;0)=\sup_{0\neq u\in \overset{\circ}{W}{}_2^2(R_+;H:0)}\left(\|Au'\|_{L_2}\cdot\|P_0u\|_{L_2}^{-1}\right) \qquad (5)$$

is finite.

It takes place

Theorem 2. *Let* A *be positively-defined self-adjoint operator, the operator* A_1A^{-1} *is bounded in* H *and*

$$\|A_1A^{-1}\|<N_1^{-1}(R_+,0).$$

Then the problem (1)-(2) is regularly solvable.

Proof. We write the problem (1), (2) in the form of operator equation $(P_0+P_1)u=f$, where $P_1u=A_1u'$, $f\in L_2(R_+;H)$, $u\in\overset{\circ}{W}{}_2^2(R_+;H:0)$. According to theorem 1 operator P_0^{-1} is bounded from $L_2(R_+;H)$ to $\overset{\circ}{W}{}_2^2(R_+;H:0)$. After substitution $u=P_0^{-1}\vartheta$ we obtain the equation $(E+P_1P_0^{-1})\vartheta=f$ in $L_2(R_+;H)$. As

$$\|P_1P_0^{-1}\vartheta\|_{L_2}=\|P_1u\|_{L_2}\le\|A_1A^{-1}\|\cdot\|Au'\|\le N_1(R_+;0)\|A_1A^{-1}\|\|P_0u\|_{L_2}=N_1(R_+;0)\|A_1A^{-1}\|\cdot\|\vartheta\|_{L_2},$$

then if the inequality $N_1(R_+;0)\|A_1A^{-1}\|<1$ is fulfilled, the operator $E+P_1P_0^{-1}$ is invertible and we can find $u(t)$:

$$u(t)=P_0^{-1}\left(E+P_1P_0^{-1}\right)^{-1}f(t).$$

From here it follows that $\|u\|_{W_2^2}\le const\|f\|_{L_2(R_+;H)}$. Theorem is proved.

Thus, for finding solvability conditions of the problem (1), (2) we must find the exact value of $N_1(R_+;0)$ or estimate it from above.

First of all we calculate the number

$$N_1(R_+;0;1)=\sup_{0\neq u\in \overset{\circ}{W}{}_2^2(R_+;H:0;1)}\left(\|Au'\|_{L_2}\cdot\|P_0u\|_{L_2}^{-1}\right). \qquad (6)$$

Preliminarily we'll prove auxiliary statement.

Lemma 1. *For any* $u\in W_2^2(R_+;H)$ *and* $\beta\in\left(0,(\omega_2-\omega_1)^2\right)$ *it takes place the identity*

$$\|P_0(d/dt:A)u\|_{L_2}^2-\beta\|Au'\|_{L_2}^2=Q(\beta,\varphi_0,\varphi_1)+\|\phi_1(d/dt:\beta,A)u\|_{L_2}^2, \qquad (7)$$

where $\varphi_0=A^{3/2}u(0)$ $\varphi_1=A^{1/2}u'(0)$,

$$\phi_1(\lambda;\beta,A)=\lambda^2E+\sqrt{(\omega_2-\omega_1)^2-\beta}\,\lambda A-\omega_1\omega_2A, \qquad (8)$$

$$Q(\beta;\varphi_0,\varphi_1)=4|\omega_1\omega_2|\operatorname{Re}(\varphi_1,\varphi_0)+\left(\sqrt{(\omega_2-\omega_1)^2-\beta}+(\omega_1+\omega_2)\right)\|\varphi_1\|^2+$$

$$+|\omega_1\omega_2|\left(\sqrt{(\omega_2-\omega_1)^2-\beta}-(\omega_1+\omega_2)\right)\|\varphi_0\|^2. \qquad (9)$$

Proof. Obviously, for all $u\in W_2^2(R_+;H)$, the equality

$$\|P_0(d/dt)u\|_{L_2}^2 = \|u''\|_{L_2}^2 + |\omega_1\omega_2|^2 \|A^2u\|_{L_2}^2 + (\omega_1+\omega_2)^2\|Au'\|_{L_2}^2 +$$
$$+2\omega_1\omega_2 \operatorname{Re}(u'',A^2u)_{L_2} - 2(\omega_1+\omega_2)\operatorname{Re}(u'',A^2u')_{L_2} - 2\omega_1\omega_2(\omega_1+\omega_2)\operatorname{Re}(Au',A^2u)_{L_2} \quad (10)$$

is true.

From the other side with integrating by part it is easy to verify the correctness of the following equalities

$$\operatorname{Re}(u'',A^2u)_{L_2} = -\|Au'\|_{L_2}^2 - \operatorname{Re}(\varphi_1,\varphi_0),$$
$$2\operatorname{Re}(u'',Au')_{L_2} = -\|\varphi_1\|^2,\ \ 2\operatorname{Re}(Au',A^2u)_{L_2} = -\|\varphi_0\|^2. \quad (11)$$

Taking into account the equalities (11) in (10) we obtain:

$$\|P_0(d/dt)u\|_{L_2}^2 = \|u''\|_{L_2}^2 + |\omega_1\omega_2|^2\|A^2u\|_{L_2}^2 + (\omega_1^2+\omega_2^2)\|Au'\|_{L_2}^2 + (\omega_1+\omega_2)\|\varphi_1\|^2 -$$
$$-\omega_1\omega_2\operatorname{Re}(\varphi_1,\varphi_0) + \omega_1\omega_2(\omega_1+\omega_2)\|\varphi_0\|^2. \quad (12)$$

As

$$\|\phi_1(d/dt:\beta:A)u\|_{L_2}^2 = \left\|u'' + \sqrt{(\omega_2-\omega_1)^2-\beta}\,Au' + |\omega_1\omega_2|A^2u\right\|_{L_2}^2 =$$
$$= \|u''\|_{L_2}^2 + |\omega_1\omega_2|^2\|A^2u\|_{L_2}^2 + \left((\omega_2-\omega_1)^2-\beta\right)\|Au'\|_{L_2}^2 + 2|\omega_1\omega_2|\operatorname{Re}(u'',A^2u)_{L_2} +$$
$$+2|\omega_1\omega_2|\sqrt{(\omega_2-\omega_1)^2-\beta}\,\operatorname{Re}(Au',A^2u)_{L_2},$$

taking into consideration the equalities (11) and (12) in last identity we finish the lemma proof.

From this lemma we obtain the following

Corollary 1. *The following identities take place:*

a) for $u\in \overset{\circ}{W}{}_2^2(R_+;H;0;1)$ *and* $\beta\in\left(0,(\omega_2-\omega_1)^2\right)$

$$\|P_0u\|_{L_2}^2 - \beta\|Au'\|_{L_2}^2 = \|\phi_1(d/dt:\beta:A)u\|_{L_2}^2; \quad (13)$$

b) for $u\in W_2^2(R_+;H;0)$ *and* $\beta\in\left(0,(\omega_2-\omega_1)^2\right)$

$$\|P_0u\|_{L_2}^2 - \beta\|Au'\|_{L_2}^2 = \|\phi_1(d/dt:\beta:A)u\|_{L_2}^2 + \left(\sqrt{(\omega_2-\omega_1)^2-\beta} + (\omega_1+\omega_2)\right)\|\varphi_1\|^2; \quad (14)$$

c) for $u\in \overset{\circ}{W}{}_2^2(R_+;H;1)$ *and* $\beta\in\left(0,(\omega_2-\omega_1)^2\right)$

$$\|P_0u\|_{L_2}^2 - \beta\|Au'\|_{L_2}^2 = \|\phi_1(d/dt:\beta:A)u\|_{L_2}^2 + |\omega_1\omega_2|\left(\sqrt{(\omega_2-\omega_1)^2-\beta} + (\omega_1+\omega_2)\right)\|\varphi_0\|^2. \quad (15)$$

Now we'll find the exact value for $N_1(R_+;0;1)$.

Theorem 3. *The number* $N_1(R_+;0;1) = (\omega_2-\omega_1)^{-1}$, $(\omega_1<0,\ \omega_2>0)$.

Proof. In the equality (13) passing to the limit for $\beta\to(\omega_2-\omega_1)^2$, we obtain that $\|P_0u\|_{L_2} \ge (\omega_2-\omega_1)\|Au'\|_{L_2}$ for all $u\in \overset{\circ}{W}{}_2^2(R_+;H;0;1)$, i.e. $N_1(R_+;0;1)\le(\omega_2-\omega_1)^{-1}$.

We'll prove that $N_1(R_+;0,1) = (\omega_2-\omega_1)^{-1}$.

To do it for any $\varepsilon>0$ it is sufficiently to construct the vector-function $u_\varepsilon(t)\in \overset{\circ}{W}{}_2^2(R_+;H;0;1)$ such that

$$\mathcal{E}(u_\varepsilon) \equiv \|P_0u\|_{L_2}^2 - \left((\omega_2-\omega_1)^2+\varepsilon\right)\|Au'_\varepsilon\|_{L_2}^2 < 0. \quad (16)$$

We'll search $u_\varepsilon(t)=g_\varepsilon(t)\psi_\varepsilon$, where $\psi_\varepsilon \in H_4$, and $g_\varepsilon(t)$ is scalar function from the class $W_2^2(R)$. For this function the inequality (16), according to Plancherel theorem, has the form

$$\mathcal{E}(u_\varepsilon)=\int_{-\infty}^{\infty} q(\xi,\psi_\varepsilon,A)\left|\hat{g}_\varepsilon(\xi)\right|^2 d\xi \quad (\|\psi_\varepsilon\|=1), \tag{17}$$

where

$$q(\xi,\psi_\varepsilon,A)=\left(\left(P_0(-i\xi,A)P_0^*(-i\xi,A)-\left((\omega_2-\omega_1)^2+\varepsilon\right)\xi^2A^2\right)\psi_\varepsilon,\psi_\varepsilon\right). \tag{18}$$

Obviously, if the operator A has eigen value μ and $A\psi_\varepsilon=\mu\psi_\varepsilon$ $(\|\psi_\varepsilon\|=1)$, then minimum of the function on ξ

$$q(\xi,\psi_\varepsilon)=P_0(-i\xi,\mu)\overline{P_0(-i\xi,\mu)}-\left((\omega_2-\omega_1)^2+\varepsilon\right)\xi^2\mu^2$$

is negative for $\varepsilon>0$.

If A has not eigen vector, then $\mu>0$ is its continuous spectrum and there is "almost eigen vector" ψ_ε $(\|\psi_\varepsilon\|=1)$ such that $A\psi_\varepsilon=\mu\psi_\varepsilon+\delta$, where δ is the vector with sufficiently small norm. In this case minimum of the function $q(\xi,\psi_\varepsilon)$ is also negative. As the function $q(\xi,\psi_\varepsilon)$ is continuous on ξ, then there is an interval $(\eta_0(\varepsilon),\eta_1(\varepsilon))$ in which $q(\xi,\psi_\varepsilon)<0$. Now let $\hat{g}_\varepsilon(\xi)$ is an arbitrary twice continuous function in R with support in the interval $(\eta_0(\varepsilon),\eta_1(\varepsilon))$. If $g_\varepsilon(t)$ is the inverse Fourier transformation of the function $\hat{g}_\varepsilon(\xi)$, then from the equality (17) it follows that

$$\mathcal{E}(g_\varepsilon(t)\psi_\varepsilon)=\int_{\eta_0(\varepsilon)}^{\eta_1(\varepsilon)} q(\xi,\psi_\varepsilon)\left|\hat{g}_\varepsilon(\xi)\right|^2 d\xi<0.$$

As $\mathcal{E}(\cdot)$ is continuous functional in the space $W_2^2(R;H)$, then from the theorem on density of the finite vector-functions in this space [1, p.23] it follows that there is finite function $u_N(t)\in W_2^2(R;H)$ with support in $(-N,N)\subset R$, such that $\mathcal{E}(u_N(t))<\varepsilon$. Supposing $u_\varepsilon(t)=u_N(t+2N)$ we obtain that $u_\varepsilon(t)\in \mathring{W}_2^2(R_+:H:0,1)$ and $\mathcal{E}(u_\varepsilon(t))<\varepsilon$. Theorem is proved.

As $\mathring{W}_2^2(R_+:H:0,1)\subset \mathring{W}_2^2(R_+:H:0)$, then it takes place

Corollary 2. *The number* $N_1(R_+;0)\geq(\omega_2-\omega_1)^{-1}$.

The case when $N_1(R_+;0)=(\omega_2-\omega_1)^{-1}$ is interesting.

It takes place

Theorem 4. *The number* $N_1(R_+:0)=(\omega_2-\omega_1)^{-1}$ *if and only if* $|\omega_2|\geq|\omega_1|$.

Proof. Let $N_1(R_+;0)=(\omega_2-\omega_1)^{-1}$, then from the identity (14) it follows that for $\beta\in\left(0,(\omega_2-\omega_1)^2\right)$ and $u\in \mathring{W}_2^2(R_+;H:0)$ the inequality

$$\begin{aligned}\|\phi_1(d/dt:\beta:A)u\|_{L_2}^2+\left(\sqrt{(\omega_2-\omega_1)^2-\beta}+\omega_1+\omega_2\right)\|\varphi_1\|^2=\\=\|P_0u\|^2\left(1-\beta\|Au'\|_{L_2}^2\|P_0u\|_{L_2}^{-2}\right)\geq\|P_0u\|_{L_2}^2\left(1-\beta\cdot N_1^2(R_+;0)\right)=\\=\|P_0u\|_{L_2}^2\left(1-\beta(\omega_2-\omega_1)^{2}\right)>0.\end{aligned} \tag{19}$$

As both roots of characteristical equation $\phi_1(\lambda:\beta:\mu)=0$

$$\lambda_{1,2}(\beta;\mu)=\frac{1}{2}\left(\sqrt{(\omega_2-\omega_1)^2-\beta}\pm\sqrt{(\omega_2+\omega_1)^2-\beta}\right)\mu$$
$$\left(\omega_2>0,\ \omega_1<0,\ \beta\in\left(0,(\omega_2-\omega_1)^2\right)\right)$$

lie in the left semi-plane, then Cauchy problem

$$\phi_1(d/dt;\beta;A)u=0,\quad u(0)=0,\quad u'(0)=A^{-1/2}\varphi_1\ ,\ \varphi_1\in H \tag{20}$$

has solution $u(t;\beta:\varphi_1)\in \overset{\circ}{W}{}_2^2(R_+;H;0)$.

Writing this solution in the inequality (19) we obtain that for any $\beta\in\left(0:(\omega_2-\omega_1)^2\right)$ the number $\sqrt{(\omega_2-\omega_1)^2-\beta}+(\omega_1+\omega_2)>0$.

Passing to the limit for $\beta\to(\omega_2-\omega_1)^2$ we obtain $|\omega_2|\ge|\omega_1|$.

Now let $|\omega_2|\ge|\omega_1|$. Then the left part of the identity (14) is always positive and because of it for all $\beta\in\left(0,(\omega_2-\omega_1)^2\right)$ the inequality $\|P_0u\|_{L_2}^2\ge\beta\|Au'\|_{L_2}^2$ is true. Passing to the limit for $\beta\to(\omega_2-\omega_1)^2$ we obtain that $N_1(R_+;0)\le(\omega_2-\omega_1)^{-1}$. Consequently, with taking into account the corollary 2 $N_1(R_+;0)=(\omega_2-\omega_1)^{-1}$. Theorem is proved.

Theorem 5. *If* $|\omega_2|<|\omega_1|$, *then* $N_1(R_+;0)=2^{-1}|\omega_1\omega_2|^{-\frac{1}{2}}$.

Proof. Obviously, for $|\omega_2|<|\omega_1|$ the number $4|\omega_1\omega_2|\in\left(0,(\omega_2-\omega_1)^2\right)$. From theorem 4 it follows that $N_1(R_+;0)>(\omega_2-\omega_1)^{-1}$ (see corollary 1), i.e. $N_1^{-2}(R_+;0)\in\left(0,(\omega_2-\omega_1)^2\right)$. If $\beta\in\left(0,N_1^{-2}(R_+;0)\right)$ for the solution of Cauchy problem (13) from the identity (14) it follows that

$$\left(\sqrt{(\omega_2-\omega_1)^2-\beta}+\omega_1+\omega_2\right)\|\varphi_1\|^2=\|P_0u\|^2\left(1-\beta\|Au'\|_{L_2}^2\|P_0u\|_{L_2}^{-2}\right)\ge$$
$$\ge\|P_0u\|^2\left(1-\beta N_1^2(R_+;0)\right)>0.$$

Consequently for $\beta\in\left(0,N_1^{-2}(R_+;0)\right)$ the function

$$\theta(\beta)=\sqrt{(\omega_2-\omega_1)^2-\beta}+(\omega_1+\omega_2)>0.$$

From the other side for $\beta\in\left(N_1^{-2}(R_+;0),(\omega_2-\omega_1)^2\right)$ according to definition of $N_1(R_+;0)$ there is the vector-function $\vartheta(t;\beta)\in W_2^2(R_+;0)$ such that $\|P_0\vartheta(t,\beta)\|_{L_2}^2<\beta\|A\vartheta'(t,\beta)\|_{L_2}^2$. Because of it for $\beta\in\left(N_1^{-2}(R_+;0),(\omega_2-\omega_1)^2\right)$ from the equality (14) it follows that

$$\|\phi_1(d/dt:\beta:A)\vartheta(t,\beta)\|_{L_2}^2+\left(\sqrt{(\omega_2-\omega_1)^2-\beta}+(\omega_1+\omega_2)\right)\|\varphi_{1,\beta}\|^2<0,$$

i.e. the function $\theta(\beta)=\sqrt{(\omega_2-\omega_1)^2-\beta}+(\omega_1+\omega_2)<0$ for $\beta\in\left(N_1^{-2}(R_+;0),(\omega_2-\omega_1)^2\right)$. As $\theta(\beta)$ is continuous function of the argument β, then $\theta\left(N_1^{-2}(R_+;0)\right)=0$, i.e. $N_1^{-2}(R_+;0)=4|\omega_1\omega_2|$. Consequently, $N_1(R_+;0)=2^{-1}|\omega_1\omega_2|^{-1/2}$. Theorem is proved.

Corollary 3. *The number* $N_1(R_+;0)$ *is defined by the following way*

$$N_1(R_+;0)=\begin{cases}(\omega_2-\omega_1)^{-1} & for\ |\omega_2|\ge|\omega_1|,\\ 2^{-1}|\omega_1\omega_2|^{-1/2} & for\ |\omega_2|<|\omega_1|.\end{cases}$$

From the theorem 2 and corollary 3 it follows the following theorem on the solvability conditions of the problem (1), (2).

Theorem 6. *Let A be self-adjoint positively-defined operator, the operator A_1A^{-1} is bounded in H and the conditions*

$$\left\|A_1A^{-1}\right\| < \begin{cases} (\omega_2-\omega_1) & for \quad |\omega_2| \ge |\omega_1| \\ 2|\omega_1\omega_2|^{1/2} & for \quad |\omega_2| < |\omega_1| \end{cases} \quad (\omega_1 < 0,\ \omega_2 > 0)$$

are satisfies. Then the problem (1), (2) is regularly solvable.

From this theorem it follows that the class of operators A_1 satisfying the solvability conditions of the problem (1), (2) is wide, if $|\omega_2| > |\omega_1|$.

Using the equality (15) we can analogously prove the following

Theorem 7. *The number*

$$N_1(R_+;1) = \sup_{0 \ne u \in \overset{\circ}{W}{}_2^2(R_+;H;1)} \left(\|Au'\|_{L_2} \cdot \|P_0u\|_{L_2}^{-1}\right)$$

is defined by the following way

$$N_1(R_+;1) = \begin{cases} (\omega_2-\omega_1)^{-1} & for \ |\omega_2| \le |\omega_1|, \\ 2^{-1}|\omega_1\omega_2|^{-1/2} & for \quad |\omega_2| > |\omega_1|. \end{cases}$$

Theorem 8. *Let A be self-adjoint positively defined operator, the operator A_1A^{-1} is bounded in H and the conditions*

$$\left\|A_1A^{-1}\right\| < \begin{cases} (\omega_2-\omega_1) & for \quad |\omega_2| \le |\omega_1| \\ 2|\omega_1\omega_2|^{1/2} & for \quad |\omega_2| > |\omega_1| \end{cases}$$

are satisfied. Then the problem (1), (3) is regularly solvable.

From this theorem it follows that the class of the operators A_1, satisfying solvability conditions of the problem (1), (3) is wide, if $|\omega_2| < |\omega_1|$.

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