HARNACK INEQUALITY FOR NONUNIFORM DEGENERATED ELLIPTIC EQUATIONS OF SECOND ORDER

Abstract

In the present paper the class of elliptic equations of second order of nondivergent structure with the nonuniform power degeneration is considered. It is assumed, that a part of eigen values of the matrix of the main part of the equation can unboundedly increase and the another part tends to zero. For the nonnegative solutions of such equations the inequality of Harnack type has been proved.

It is well known, that if the matrix \( A(x) \) satisfies Cordes condition, then for nonnegative solutions of the equation (1) the Harnack type inequality [1-2] is true. Meaning of the Cordes condition is in the smallness of scattering of eigen values of the matrix \( A(x) \). The analogous fact, for arbitrary uniform elliptic equations of second order with generally speaking, disconnected coefficients, has been set up in [3-4]. In [5-6] this result was transferred to quasilinear parabolic equations of second order of nondivergent structure. As to the elliptic equations with nonuniform generation we must note the paper [7], in which a class of such equations with the eigen values of the matrix of the main part tending to zero has been considered and the analogue of Harnack inequality for their nonnegative solutions has been proved.

The aim of the present paper is the proof of the inequality of Harnack type for the solutions of elliptic equations of second order of nondivergent structure with the nonuniform power generation at some point of the considered domain. It is assumed that a part of eigen values of the matrix of the main part of the equation can unboundedly increase by tending to this point, and the another part tends to zero. Note that for the solution of the indicated class of equations the integral a priori estimation of the Hölder norm has been set up in [3].

Let \( D \) be a bounded domain of \( n \) dimensional Euclidean space \( \mathbb{R}^n, n \geq 3 \). Consider in \( D \) an elliptic equation of the form

\[
Lu = \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} a_{\alpha\beta}(x) \frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u(x) = 0,
\]

where for \( x \in D \) the matrix \( A(x) \) is uniformly positively determined, \( a_{i\alpha}(x) = a_{\alpha i}(x) \), \((i, \alpha = 1, \ldots, n)\), the coefficients \( b_i(x) \) \((i = 1, \ldots, n)\) and \( c(x) \) are bounded, however \( c(x) \leq 0 \).

Let \( \alpha = a_{i_1 \ldots i_n} \) be a vector and

\[
-2 < \alpha_i \leq 2, \quad (i = 1, \ldots, n)
\]

Assume, that \( 0 \in D \) and for any \( x \in D \) and \( \xi \in \mathbb{R}^n \), the condition

\[
\mu \sum_{i=1}^{n} \lambda_i(x) \xi_i^2 \leq \sum_{i=1}^{n} a_{i\alpha}(x) \xi_i \xi_\alpha \leq \mu \sum_{i=1}^{n} \lambda_i(x) \xi_i^2
\]

is fulfilled.

\[
\mu \in (0, 1) \quad \text{and} \quad \lambda_i(x) \geq \frac{c}{\mu}, \quad |x|^{-2} = \sum_{i=1}^{n} |x_i|^2 D x_i^3
\]

We will also assume, that...
where $d$ and $h_i$ are positive constants.

Let's introduce the following notations:

$$E_i^x(k) = \left\{ x : \sum_{j=1}^{n} \frac{(x_j - k_j)^2}{R^2} \leq (kr)^2 \right\}$$

is an ellipsoid.

We'll also introduce the following ellipsoids:

$$E_1 = E_i^x(1), \quad E_2 = E_i^x(2).$$

where $x' \in \partial E^x_2(5)$.

Lebesgue measure of the set $G$ we'll denote by $\text{mes}G$.

1'. The lemma on increasing of positive solutions.

Lemma 1. Let in ellipsoid $E_i$ the nonnegative, $L$-superelliptic function $u(x)$, continuous in $E_i$ be defined. Then there exist the positive constants $\eta_i$ and $\delta_i$, depending only on $\mu_c$ and $n$, such that, if $R \leq 1$ and

$$\text{mes}(E_i \cap \{x < 1\}) > \delta_i \text{mes}E_i,$$

then

$$\inf_{E_i} u \geq \eta_i.$$

Proof. Let us denote by $D$ the following set

$$D = \{ x \in E_i : u(x) < 1 \}.$$

Then by the condition of the lemma:

$$\text{mes}D \leq (1 - \delta_i)\text{mes}E_i,$$

Obviously, that

$$u(\partial E_i) = 1,$$

where $\Gamma$ is a part of the boundary $\partial D$, lying inside $E_i$.

Two cases are possible by constructing the domain $D$:

a) $D \cap E_2 = \emptyset$, b) $D \cap E_x \neq \emptyset$. In the case a) the lemma has been proved. Let the case b) fulfilled. In this case we get that all conditions of lemma 4 from [8] are fulfilled.

Then according to this lemma there exists positive constant $\eta_i$ depending on $\mu_c$ and $n$ such that

$$\inf_{E_i} u \geq \eta_i.$$ 

On the other hand it is obvious, that

$$\inf_{E_i} u \geq 1.$$ 

Consequently,

$$\inf_{E_i} u \geq \eta_i.$$ 

Lemma has been proved.

Lemma 2. Let in the ellipsoid $E_i$ the nonnegative, $L$-superelliptic function $u(x)$, continuous in $E_i$ be defined. Then there exists the positive constant $\eta_2$, such that, if $R \leq 1$ and
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\[ \inf_{E_1} u \geq 1, \]
\[ \eta_1 \leq \eta_2, \]

where \( \epsilon \in (0, 1] \) and \( \eta_1 = \eta_2 (\mu, \alpha, \kappa, \epsilon) \).

**Proof.** Consider the set
\[ D = \{ x : x \in E_1, u(x) < 1 \}. \]

If \( D \cap E_1 = \emptyset \), then it means that \( u(x) \geq 1 \) for \( x \in E_1 \), i.e. in this case lemma has been proved. Assume, that \( D \cap E_1 \neq \emptyset \). Consider in \( D \) the following function:
\[ v(x) = 1 - u(x). \]

Then
\[ L v(x) = c(x) - L u(x) = c(x) \sup_{\partial D} v(x). \]

From the condition of the lemma it follows, that
\[ H = E_1 \setminus D \supset E_1^c (x) \]

On the other hand it is obvious, that
\[ v(x)|_{\partial D} = 0, \]

where \( \Gamma \) is a part of the boundary \( \partial D \), lying strictly inside \( E_1 \).

So we obtain that for the function \( v(x) \) all the conditions of corollary 1 to lemma 2 from [8] are fulfilled. Then there exists the positive constant \( \eta_2 \), depending on \( \mu, \alpha, \kappa \) and \( \epsilon \) such, that
\[ \sup_{\partial D} v(x) \geq \frac{1 + \eta_2}{1 + \eta_1} = \eta_2, \]

or
\[ \inf_{(E_1 \setminus \Gamma)} u(x) \geq \frac{1}{1 + \eta_1} = \eta_2. \]

Obviously, that
\[ \inf_{E_1} u(x) \geq 1. \]

So we have
\[ \inf_{E_1} u \geq \eta_2, \]

and lemma has been proved.

**Lemma 3.** Let in the ellipsoid \( E_1 \) the nonnegative, \( L \)-superelliptic function \( u(x) \), continuous in \( E_1 \) be defined. Then there exist positive constants \( \eta_3 \) and \( \delta_2 \) depending only on \( \mu, \alpha \) and \( n \), such that, if \( L u \leq 1 \) and
\[ \max_{E_1 \setminus \{ u \geq 1 \}} u \leq \delta_2 \max_{E_1}, \]

then
\[ \inf_{E_1} u \geq \eta_3. \]

**Proof.** Let’s introduce the following set
\[ D = \{ x : x \in E_1, u(x) < 1 \}. \]

If \( D \cap E_1 = \emptyset \), then in this case lemma has been proved.

Assume, that \( D \cap E_1 \neq \emptyset \).
Let \( H = \{ x : x \in E_1, u(x) \geq 1 \} \), i.e., \( H = E_2 \setminus D \).

By the condition of the lemma:
\[ \text{mes} H \geq \delta, \text{mes} E_1, \]

From the construction of the domain \( D \) follows, that
\[ u(x) = 1, \]

where \( \Gamma \) is a part of boundary \( \partial D \), lying strictly inside \( E_1 \).

Consequently, all the conditions of lemma 5 in [8] are fulfilled. Then there exists the positive constant \( \eta_1 \) depending on \( \mu, \alpha \) and \( n \), such that
\[ \inf_{x \in E_2} u \geq \eta_1. \]

On the other hand
\[ \inf_{x \in E_1} u > 1, \]

Consequently,
\[ \inf_{x \in E_1} u \geq \eta_2. \]

Lemma has been proved.

**Lemma 4.** Let in \( E_1 \) the solution of the equation (1) be defined. Further let \( E \subseteq 1 \) and \( u(\delta) > 1 \). If for some constant \( M > 1 \)
\[ \sup_{x \in E_1} u \leq M, \]

then
\[ \inf_{x \in E_1} u > \eta_1 > 0, \]

where \( \eta_1 = \eta_1(\mu, \alpha, n, M). \)

**Proof.** Let's introduce the following function
\[ v(x) = M - u(x). \]

Hence we've
\[ v(x) > 0 \text{ on } E_2 \text{ and } v(x) = 0 \text{ in } E_1. \]

It easy to see that
\[ \inf_{x \in E_1} v(x) \leq \frac{M - 1}{M - 1/2} < 1, \]

Applying lemma 1 to the ellipsoid \( E_1 \) we will obtain:
\[ \text{mes}(E_1) \cap \{ v(x) < \frac{1}{2} \} < \delta \text{ mes } E_1. \]

Indeed, otherwise will get contradiction with (6). Consequently
\[ \text{mes}(E_1) \cap \{ v(x) < \frac{1}{2} \} \leq (1 - \delta) \text{ mes } E_2. \]

Allowing for (5) we've:
\[ \text{mes} \left( E_1 \cap \{ u(x) \geq 1/2 \} \right) \leq (1 - \delta) \text{ mes } E_2. \]

Then according to lemma 3:
\[ \inf_{x \in E_1} u \geq \frac{1}{2} \eta_2 = \eta_1, \]

Lemma has been proved.
2°. Harnack inequality:

**Theorem.** Let in the ellipsoid \( E \) the positive solution \( u(x) \) of the equation (1) be defined. If \( R \leq 2 \) and the conditions (2)-(4) be fulfilled, then there exists the positive constant \( C_1 \) depending on \( \mu, \sigma \) and \( n \), such that

\[
\sup_{E} u(x) \leq C_1, \quad \inf_{E} u(x) \geq \frac{1}{2}.
\]

**Proof.** Without loss of generality we’ll assume that

\[
\sup_{E} u(x) = 1.
\]

Let

\[
\frac{1}{16} \leq 4 < \frac{1}{8}.
\]

Assume

\[
\rho(x) = \sup_{E(x)} u(x), \quad q(x) = (2 - 16 \delta)^{-\frac{1}{2}}.
\]

where \( \delta > 0 \).

Let’s define \( k \) as the greatest root of the equation \( p(k) - q(x) \). Since \( p(1/16) - q(1/16) = 1, \) \( q(x) \to 0 \) as \( x \to 1/8 \) and \( \xi(x) \) is continuous and bounded in \( E(x) \), then \( k \) is correctly defined and \( k < 1/8 \).

For some point \( x \in E(x) \), it is true the equality \( u(x_k) = \rho(x_k) = q(x_k) \). Assume \( t = (1 - 8k) / 16 \). Consequently

\[
E(x_k) = E(x) \left( \frac{1 - 8k}{16} \right).
\]

By the definition of \( k \) we have

\[
\sup_{E(x)} u(x) \leq \rho \left( \frac{1 - 8k}{16} \right) < \rho \left( \frac{1 - 8k}{16} \right) = (1 - 8k) - t = 2^{-t} q(x_k).
\]

Applying lemma 4 to the function \( u(x) q(x_k) \)

\[
E(x_k) = E(x) \left( \frac{1}{16} \right), \quad M = 2^{t}
\]

we will obtain

\[
\inf_{E(x)} u(x) \geq \eta_q(x_k).
\]

It is easy to see that

\[
x_k \in E(x_k) = E(x) \left( \frac{1}{16} \right) \quad \text{and} \quad E(x_k) \left( \frac{1}{16} \right) \subset E(x_k) \left( \frac{1}{8} \right) \subset E(x_k) \left( \frac{1}{16} \right).
\]

Then according to lemma 1 and 2 we have

\[
\inf_{E(x_k) \left( \frac{1}{16} \right)} u(x) \geq \inf_{E(x_k) \left( \frac{1}{8} \right)} u(x) \geq \inf_{E(x_k) \left( \frac{1}{16} \right)} u(x) = \inf_{E(x_k) \left( \frac{1}{16} \right)} u(x) > 0.
\]

Let’s denote \( \xi(x_k) = \left( \mu \eta_q(x_k) \right)^{-\frac{1}{2}} \), and theorem has been proved.

**Corollary.** Let in domain \( D \) the positive solution \( u(x) \) of the equation (1) be defined for whose coefficients the conditions (2)-(4) have been fulfilled.

Let for \( \rho > 0 \)

\[
D_{\rho} = \{ x \in D, \text{dist}(x, \partial D) > \rho \},
\]

then

\[
\sup_{D_{\rho}} u(x) \leq C_1, \quad \inf_{D_{\rho}} u(x) \geq \frac{1}{2}.
\]
Then for any $p > 0$ there exist a positive constant $C_p$ depending only on $n, \alpha, \nu$ and $p$ such that
\[
\sup_{\mathbb{R}^n} (x) \leq C_p \inf_{\mathbb{R}^n} (x).
\]

Author expresses his grateful thanks to his supervisor prof. I.T.Mamedov for the formulation of the problem and for the valuable discussions of results.

References


[7]. Мамедов И.Т. О внутренней гладкости решений эллиптических уравнений 2-го порядка с меромодифицированным граничным условием. Труды Института математики и механики т.ШИКИ, Баку, 1995, с.8-22.

[8]. Ализуллы Р.М. О внутренной гладкости решений вырождающихся эллиптических уравнений 2-го порядка с разрывными коэффициентами. Доклады АН Азербайджана, 2000, т.LVI, №4-6, с.25-41.

9, F.Agayev str., 370141, Baku, Azerbaijan.
Tel.: 39-47-20.

Received December 12, 2000; Revised March 1, 2000.
Translated by Nazirova S.H.