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HARNACK INEQUALITY FOR NONUNIFORM DEGENERATED ELLIPTIC EQUATIONS OF SECOND ORDER

Abstract

In the present paper the class of elliptic equations of second order of nondivergent structure with the nonuniform power degeneration is considered. It is assumed, that a part of eigen values of the matrix of the main part of the equation can unboundedly increase and the another part tends to zero. For the nonnegative solutions of such equations the inequality of Harnack type has been proved.

It is will known, that if the matrix $\|a_{ij}(x)\|$ satisfies Cordes condition, then for nonnegative solutions of the equation (1) the Harnack type inequality [1-2] is true. Meaning of the Cordes condition is in the «smallness» of scattering of eigen values of the matrix $\|a_{ij}(x)\|$. The analogous fact, for arbitrary uniform elliptic equations of second order with generally speaking, disconnected coefficients, has been set up in [3-4]. In [5-6] this result was transferred to quasilinear parabolic equations of second order of nondivergent structure. As to the elliptic equations with nonuniform generation we must note the paper [7], in which a class of such equations with the eigen values of the matrix of the main part tending to zero has been considered and the analogue of Harnack inequality for their nonnegative solutions has been proved.

The aim of the present paper is the proof of the inequality of Harnack type for the solutions of elliptic equations of second order of nondivergent structure with the nonuniform power generation at some point of the considered domain. It is assumed that a part of eigen values of the matrix of the main part of the equation can unboundedly increase by tending to this point, and the another part tends to zero. Note that for the solution of the indicated class of equations the integral apriori estimation of the Hölder norm has been set up in [8].

Let D be a bounded domain of n dimensional Euclidean space $\mathbf{R}_n$, $n \ge 3$.

Consider in D an elliptic equation of the form:

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x) = 0, \qquad (1)$$

where for $x \in D$ the matrix $\|a_{ij}(x)\|$ is uniformly positively determined $a_{ij}(x) = a_{ji}(x)$, $(i,j = 1,...,n)$, the coefficients $b_i(x)$ $(i = 1,...,n)$ and $c(x)$ are bounded, however $c(x) \le 0$.

Let $\alpha = (\alpha_1,...,\alpha_n)$ be a vector and

$$-2 < \alpha_i \le 2, \quad (i = 1,...,n) \qquad (2)$$

Assume, that $0 \in D$ and for any $x \in D$ and $\xi \in \mathbf{R}_n$ the condition

$$\mu \sum_{i=1}^{n} \lambda_i(x)\xi_i^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \mu^{-1}\sum_{i=1}^{n} \lambda_i(x)\xi_i^2 \qquad (3)$$

is fulfilled.

$\mu \in (0,1]$ is a constant $\lambda_i(x) = |x|_\alpha^{\alpha_i}$, $|x|_\alpha = \sum_{i=1}^{n} |x_i|^{2/(2+\alpha_i)}$.

We will also assume, that

$$-d \le c(x) \le 0, \quad |b_i(x)| \le b_0, \quad (i=1,...,n), \tag{4}$$

where d and b_0 are positive constants.

Let's introduce the following notations:

for $x^0 \in \mathbf{R}_n$

$$E_R^{x^0}(k) = \left\{ x : \sum_{i=1}^{n} \frac{(x_i - x_i^0)^2}{R^{\alpha_i}} < (kR)^2 \right\} \text{ is an ellipsoid .}$$

We'll also introduce the following ellipsoids:

$$E_1 = E_R^{x^1}(4), \quad E_2 = E_R^{x^1}(1),$$

where $x^1 \in \partial E_R^0(5)$.

Lebesgue measure of the set G we'll denote by $mes\,G$.

1^0. The lemma on increasing of positive solutions.

Lemma 1. *Let in ellipsoid E_1 the nonnegative, L-superelliptic function $u(x)$, continuous in $\overline{E}_1$ be defined. Then there exist the positive constants η_1 and δ_1, depending only on μ,α and n, such that, if $R \le 1$ and*

$$mes(E_1 \cap \{u \ge 1\}) \ge \delta_1 mes\, E_1$$

then

$$\inf_{E_2} u \ge \eta_1 .$$

Proof. Let us denote by D the following set

$$D = \{x : x \in E_1, u(x) < 1\}.$$

Then by the condition of the lemma:

$$mes\, D \le (1 - \delta_1) mes E_1 .$$

Obviously, that

$$u(x)|_{\Gamma} = 1 ,$$

where Γ is a part of the boundary ∂D, lying strictly inside E_1.

Two cases are possible by constructing the domain D:

a) $D \cap E_2 = \varnothing$, b) $D \cap E_2 \ne \varnothing$. In the case a) the lemma has been proved. Let the case b) fulfilled. In this case we get that all conditions of lemma 4 from [8] are fulfilled.

Then according to this lemma there exists positive constant η_1 depending on μ,α and n such, that

$$\inf_{D \cap E_2} u \ge \eta_1 .$$

On the other hand it is obvious, that

$$\inf_{E_2 \setminus D} u \ge 1 .$$

Consequently,

$$\inf_{E_2} u \ge \eta_1 .$$

Lemma has been proved.

Lemma 2. *Let in the ellipsoid E_1 the nonnegative, L-superelliptic function $u(x)$, continuous in $\overline{E}_1$ be defined. Then there exists the positive constant η_2, such that, if $R \le 1$ and*

$$\inf_{E_R^{x^1}(\varepsilon)} u \geq 1,$$

then

$$\inf_{E_2} u \geq \eta_2,$$

where $\varepsilon \in (0,1]$ *and* $\eta_2 = \eta_2(\mu,\alpha,n,\varepsilon)$.

Proof. Consider the set

$$D = \{x : x \in E_1,\ u(x) < 1\}.$$

If $D \cap E_2 = \varnothing$, then it means that $u(x) \geq 1$ for $x \in E_2$, i.e. in this case lemma has been proved. Assume, that $D \cap E_2 \neq \varnothing$. Consider in D the following function:

$$v(x) = 1 - u(x).$$

Then

$$Lv(x) = c(x) - Lu(x) \geq c(x) = c(x)\sup_D v(x).$$

From the condition of the lemma it follows, that

$$H = E_2 \setminus D \supset E_R^{x^1}(\varepsilon).$$

On the other hand it is obvious, that

$$v(x)\big|_\Gamma = 0,$$

where Γ is a part of the boundary ∂D, lying strictly inside E_1.

So we obtain that for the function $v(x)$ all the conditions of corollary 1 to lemma 2 from [8] are fulfilled. Then there exists the positive constant $\overline{\eta_1}$, depending on μ,α,n and ε such, that

$$\sup_D v(x) \geq (1 + \overline{\eta_1}) \sup_{D\cap E_2} v(x).$$

or

$$\inf_{D\cap E_2} u(x) \geq \frac{\overline{\eta_1}}{1+\overline{\eta_1}} = \eta_2.$$

Obviously, that

$$\inf_{E_2\setminus D} u(x) \geq 1.$$

So we have

$$\inf_{E_2} u \geq \eta_2$$

and lemma has been proved.

Lemma 3. *Let in the ellipsoid* E_1 *the nonnegative,* L*-superelliptic function* $u(x)$, *continuous in* $\overline{E_1}$ *be defined. Then there exist positive constants* η_3 *and* δ_2 *depending only on* μ,α *and* n, *such that, if* $R \leq 1$ *and*

$$mes(E_2 \cap \{u \geq 1\}) \geq \delta_2 mes E_2,$$

then

$$\inf_{E_2} u \geq \eta_3.$$

Proof. Let's introduce the following set

$$D = \{x : x \in E_1,\ u(x) < 1\}.$$

If $D \cap E_2 = \varnothing$, then in this case lemma has been proved.

Assume, that $D \cap E_2 \neq \varnothing$.

Let

$$H=\{x:x\in E_2,\,u(x)\geq 1\},\quad i.e.\quad H=E_2\setminus D.$$

By the condition of the lemma:

$$mesH\geq\delta_2 mesE_2.$$

From the construction of the domain D follows, that

$$u(x)|_{\Gamma}=1,$$

where Γ is a part of boundary ∂D, lying strictly inside E_1.

Consequently, all the conditions of lemma 5 in [8] are fulfilled. Then there exists the positive constant η_3 depending on μ,α and n, such that

$$\inf_{D\cap E_2} u\geq\eta_3.$$

On the other hand

$$\inf_{E_2\setminus D} u\geq 1.$$

Consequently,

$$\inf_{E_2} u\geq\eta_3.$$

Lemma has been proved.

Lemma 4. *Let in E_1 the solution of the equation (1) be defined. Further let $R\leq 1$ and $u(x^1)\geq 1$. If for some constant $M>1$*

$$\sup_{E_2} u\leq M,$$

then

$$\inf_{E_2} u\geq\eta_4>0,$$

where $\eta_4=\eta_4(\mu,\alpha,n,M)$.

Proof. Let's introduce the following function

$$v(x)=\frac{M-u(x)}{M-1/2}. \tag{5}$$

Hence we've

$$v(x)\geq 0 \text{ on } E_2 \text{ and } Lv(x)\leq 0 \text{ in } E_2.$$

It easy to see that

$$\inf_{E_R^{x^1}(1/4)} v(x)\leq v(x^1)\leq\frac{M-1}{M-1/2}<1. \tag{6}$$

Applying lemma 1 to the ellipsoid E_2 we will obtain:

$$mes(E_2\cap\{v(x)\geq 1\})<\delta_1 mesE_2.$$

Indeed, otherwise will get contradiction with (6). Consequently

$$mes(E_2\cap\{v(x)\leq 1\})\geq(1-\delta_1)mesE_2.$$

Allowing for (5) we've:

$$mes\left(E_2\cap\left\{u(x)\geq\frac{1}{2}\right\}\right)\geq(1-\delta_1)mesE_2.$$

Then according to lemma 3:

$$\inf_{E_2} u\geq\frac{1}{2}\eta_3=\eta_4.$$

Lemma has been proved.

2^0. Harnack inequality:

Theorem. *Let in the ellipsoid E_2 the positive solution $u(x)$ of the equation (1) be defined. If $R \le 1$ and the conditions (2)-(4) be fulfilled, then there exists the positive constant C_1 depending on μ, α and n, such that*

$$\sup_{E_R^{x^1}(1/16)} u(x) \le C_1 \inf_{E_R^{x^1}(1/16)} u(x).$$

Proof. Without loss of generality we'll assume that

$$\sup_{E_R^{x^1}(1/16)} u(x) = 1.$$

Let

$$\frac{1}{16} \le k < \frac{1}{8}.$$

Assume

$$P(k) = \sup_{E_R^{x^1}(k)} u(x), \quad q(k) = (2-16k)^{-s},$$

where $s > 0$.

Let's define k_0 as the greatest root of the equation $P(k) = q(k)$. Since $p(1/16) = q(1/16) = 1$, $q(k) \to \infty$ at $k \to 1/8$ and $u(x)$ is continuous and bounded in $E_R^{x^1}(k)$, then k_0 is correctly defined and $k_0 < 1/8$.

For some point $x_0 \in E_R^{x^1}(k_0)$, it is true the equality $u(x_0) = p(k_0) = q(k_0)$. Assume $k_1 = (1-8k_0)/16$. Consequently

$$E_R^{x_0}(k_1) \subset E_R^{x^1}\left(\frac{1+8k_0}{16}\right).$$

By the definition of k_0 we have

$$\sup_{E_R^{x_0}(k_1)} u(x) \le p\left(\frac{1+8k_0}{16}\right) < q\left(\frac{1+8k_0}{16}\right) = (1-8k_0)^{-s} = 2^s q(k_0).$$

Applying lemma 4 to the function $u(x)/q(k_0)$

$$\left(E_R^{x^1}(k) = E_R^{x_0}(2k_1) \subset E_R^{x^1}(1/8), M = 2^s\right)$$

we will obtain

$$\inf_{E_R^{x_0}(k_1)} u(x) \ge \eta_4 q(k_0).$$

It is easy to see that

$$x_0 \in E_R^{x^1}(k_0) \subset E_R^{x^1}(1/8) \text{ and}$$

$$E_R^{x^1}(1/16) \subset E_R^{x_0}(3/16) \subset E_R^{x_0}(3/8) \subset E_2.$$

Then according to lemma 2 we have $\left(E_R^{x^1}(k) = E_R^{x_0}(3/8), \varepsilon = 8k_1/3\right)$:

$$\inf_{E_R^{x^1}(1/16)} u(x) \ge \inf_{E_R^{x^0}(3/16)} u(x) \ge \inf_{E_R^{x_0}(3/8)} u(x) \ge \eta_2 \eta_4 q(k_0) > 0.$$

Let's denote $C_1 = (\eta_2 \eta_4 q(k_0))^{-1}$, and theorem has been proved.

Let for $\rho > 0$ $\quad D_\rho = \{x : x \in D,\ dist(x, \partial D) > \rho\}$.

Corollary. *Let in domain D the positive solution $u(x)$ of the equation (1) be defined for whose coefficients the conditions (2)-(4) have been fulfilled.*

Then for any $\rho > 0$ *there exist a positive constant* C_2 *depending only on* μ, α, n *and* ρ *such that*

$$\sup_{D_\rho} u(x) \le C_2 \inf_{D_\rho} u(x).$$

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References

[1]. Cordes H.O. *Die erste Randwerafgabe bei Differentialgleichungen zweiter ordnung in mehr als zwei Variabeln.* Math. Ann., 1956, v.131, №3, p.278-318.

[2]. Ландис Е.М. *Уравнения второго порядка эллиптического и параболического типов.* «Наука», 1971, 288 с.

[3]. Крылов Н.В., Сафонов М.В. *Некоторые свойства решений параболических уравнений с измеримыми коэффициентами.* Изв. АН СССР, 1980, сер. мат., т.44, №1, с.161-175.

[4]. Сафонов М.В. *Неравенство Харнака для эллиптических уравнений и гельдеровость их решений.* Записки научных семинаров ЛОМИ им.Стеклова, 1980, т.96, с.272-287.

[5]. Мамедов И.Т. *Об априорной оценке нормы Гельдера решений квазилинейных параболических уравнений с разрывными коэффициентами.* ДАН СССР, 1980, т.252, №5, с.1052-1054.

[6]. Мамедов И.Т. *Теорема об осцилляции решений параболических уравнений 2-го порядка с разрывными коэффициентами.* Изв. АН Аз.ССР, сер. ФТМН, 1983, №2, с.15-23.

[7]. Мамедов И.Т. *О внутренней гладкости решений эллиптических уравнений 2-го порядка с неравномерным степенным вырождением.* Труды Института математики и механики, т.III(XI), Баку, 1995, с.8-22.

[8]. Алыгулиев Р.М. *О внутренней гладкости решений вырождающихся эллиптических уравнений 2-го порядка с разрывным коэффициентами.* Доклады АН Азербайджана, 2000, т.LVI, №4-6, с.25-41.

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