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ON SOME PROPERTIES OF THE SOLUTION OF INTEGRAL EQUATIONS SYSTEM WITH TWO VARIABLE LIMITS

Abstract

In this paper the integral equations with two variable limits are investigated. To this end the corresponding theorem on integral inequalities has been proved. With the help of this theorem the existence and uniqueness of solution and some its properties have been proved.

Let X and Y be some Banach spaces. We'll denote by X_a and Y_a spaces of continuous abstract functions determined on $[-1,1+a]$ with values from X and Y correspondingly, and S and S^* are balls in these spaces with radius r and centers at the points x_0 and y_0, correspondingly.

Consider a system of the integral equations

$$x(t) = x_0 + \int_{a-t}^{t} K_1[t,s;x(s),y(s)]ds,$$
$$y(t) = y_0 + \int_{-1}^{1+a} K_2[t,s;x(s),y(s)]ds, \tag{1}$$

in topological product $X_a \times Y_a$.

In this paper the questions of the existence and uniqueness of the solution of the system (1), and the differentiability of the solution by parameter are investigated.

The following theorem is of great significant in investigating these questions.

Theorem 1. *Let continuous scalar functions $u(t), v(t)$ $(-1 \le t \le 1+a)$ satisfy the system of in equalities*

$$u(t) \le u_0(t) + sign\left(t - \frac{a}{2}\right) \int_{a-t}^{t} \{L_{11}(s)u(s) + L_{12}(s)v(s) + \psi_1(t,s)\}ds,$$
$$v(t) \le v_0(t) + \int_{-1}^{1+a} \{L_{21}(s)u(s) + L_{22}(s)v(s) + \psi_2(t,s)\}ds, \tag{2}$$

where $u_0(t), v_0(t)$, $(-1 \le t \le 1+a)$ are continuous, $L_{ij}(t)$ $(i,j=1,2; -1 \le t \le 1+a)$ are nonnegative summable functions, and $\psi_i(t,s)$ $(i,j=1,2;\ -1 \le t,\ s \le 1+a)$ are continuous by t and summable by s functions; let, besides, the inequalities

$$\alpha = \left(1 - \int_{-1}^{1+a} L_{21}(s)\Phi_1(s)ds\right)^{-1} > 0,\ \ \beta = \left(1 - \int_{-1}^{1+a} L_{22}(s)ds\right)^{-1} > 0$$

be fulfilled.

Then

$$u(t) \le F_1(t) + \alpha\Phi_1(t)\int_{-1}^{1+a} L_{21}(s)F_1(s)ds,$$

$$v(t) \le f_2(t) + \alpha\beta \int_{-1}^{1+a} L_{21}(s)F_1(s)ds,$$

hold, where $F_1(t)$, $\Phi_1(t)$, $f_2(t)$ are known functions whose expressions are indicated in the proof of the theorem.

Proof. Denote

$$\varphi_1(t)=u_0(t)+sign\left(t-\frac{a}{2}\right)\int_{a-t}^{t}[L_{12}(s)v(s)+\varphi_1(t,s)]ds\,,$$
$$\varphi_2(t)=v_0(t)+\int_{-1}^{1+a}[L_{21}(s)u(s)+\varphi_2(t,s)]ds. \tag{3}$$

Then the system of inequalities (2) gets the following form

$$u(t)\le\varphi_1(t)+sign\left(t-\frac{a}{2}\right)\int_{a-t}^{t}L_{11}(s)u(s)ds\,,$$
$$v(t)\le\varphi_2(t)+\int_{-1}^{1+a}L_{22}(s)v(s)ds. \tag{4}$$

Applying the theorem on integral inequalities with two variable limits [1,2], to the first inequality of (4) we have

$$u(t)\le\varphi_1(t)+sign\left(t-\frac{a}{2}\right)\int_{a/2}^{t}[L_{11}(s)\varphi_1(s)+L_{11}(a-s)\varphi_1(a-s)]\exp\times$$
$$\times\left\{sign\left(t-\frac{a}{2}\right)\int_{s}^{t}[L_{11}(\tau)+L_{11}(a-\tau)d\tau]\right\}ds. \tag{5}$$

Analogously, from the second inequality of (4) we have

$$v(t)\le\varphi_2(t)+\beta\int_{-1}^{1+a}L_{22}(s)\varphi_2(s)ds. \tag{6}$$

Allowing for the notations (3) in (5) and (6) we get

$$u(t)\le f_1(t)+sign\left(t-\frac{a}{2}\right)\int_{a-t}^{t}L_{12}(s)v(s)ds+sign\left(t-\frac{a}{2}\right)\times$$
$$\times\int_{a/2}^{t}\left\{[L_{11}(s)+L_{11}(a-s)]sign\left(t-\frac{a}{2}\right)\int_{a-s}^{s}L_{12}(\tau)v(\tau)d\tau\right\}\exp\times \tag{7}$$
$$\times\left\{sign\left(t-\frac{a}{2}\right)\int_{s}^{t}[L_{11}(\tau)+L_{11}(a-\tau)d\tau]\right\}ds,$$
$$v(t)\le f_2(t)+\beta\int_{-1}^{1+a}L_{21}(s)u(s)ds\,, \tag{8}$$

where

$$f_1(t)=u_0(t)+sign\left(t-\frac{a}{2}\right)\int_{a-t}^{t}\psi_1(t,s)ds+sign\left(t-\frac{a}{2}\right)\times$$
$$\times\int_{a/2}^{t}\left\{L_{11}(s)\left[u_0(s)+sign\left(s-\frac{a}{2}\right)\int_{a-s}^{s}\psi_1(s,\tau)d\tau\right]+\right.$$
$$\left.+L_{11}(a-s)\left[u_0(a-s)+sign\left(s-\frac{a}{2}\right)\int_{a-s}^{t}\psi_1(a-s,\tau)d\tau\right]\right\}\exp\times$$

$$\times\left\{sign\left(t-\frac{a}{2}\right)\int_s^t[L_{11}(\tau)+L_{11}(a-\tau)]d\tau\right\}ds,$$

$$f_2(t)=v_0+\int_{-1}^{1+a}\psi_2(t,s)ds+\beta\int_{-1}^{1+a}L_{22}(s)\left[v_0(s)+\int_{-1}^{1+a}\psi_2(s,\tau)d\tau\right]ds.$$

By virtue of (8) from (7) we obtain

$$u(t)\le F_1(t)+\Phi_1(t)\int_{-1}^{1+a}L_{21}(s)u(s)ds, \tag{9}$$

where

$$F_1(t)=f_1(t)+sign\left(t-\frac{a}{2}\right)\int_{a-t}^{t}L_{12}(s)f_2(s)ds+sign\left(t-\frac{a}{2}\right)\times$$

$$\times\int_{a/2}^{t}\left\{[L_{11}(s)+L_{11}(a-s)]sign\left(s-\frac{a}{2}\right)\left(\int_{a-s}^{s}L_{12}(\tau)f_2(\tau)d\tau\right)\exp\times\right.$$

$$\left.\times\left\{sign\left(t-\frac{a}{2}\right)\int_s^t[L_{11}(\tau)+L_{11}(a-\tau)d\tau]\right\}ds,\right.$$

$$\Phi_1(t)=\beta\left\{sign\left(t-\frac{a}{2}\right)\int_{a-t}^{t}L_{12}(s)ds+sign\left(t-\frac{a}{2}\right)\times\right.$$

$$\times\int_{a/2}^{t}[L_{11}(s)+L_{11}(a-s)]sign\left(s-\frac{a}{2}\right)\left(\int_{a-s}^{s}L_{12}(\tau)d\tau\right)\times$$

$$\left.\times\exp\left\{sign\left(t-\frac{a}{2}\right)\int_s^t[L_{11}(\tau)+L_{11}(a-\tau)d\tau]\right\}ds\right..$$

Multiplying the both sides of (9) by $L_{21}(t)$ and integrating from -1 to -1, we obtain.

$$\int_{-1}^{1+a}L_{21}(t)u(s)ds<\alpha\int_{-1}^{1+a}L_{21}(s)F_1(s)ds. \tag{9*}$$

Allowing for (9*) in (9) and (8) we have

$$u(t)\le F_1(t)+\alpha\Phi_1(t)\int_{-1}^{1+a}L_{21}(s)F_1(s)ds\ ,$$

$$v(t)\le f_2(t)+\alpha\beta\int_{-1}^{1+a}L_{21}(s)F_1(s)ds.$$

Q.E.D.

To prove the existence and uniqueness of the solution of the system (1) we'll construct for it sequential approximations by the following way:

$$x^{(0)}(t)=x_0(t),\qquad y^{(0)}(t)=y_0(t),$$

$$x^{(n)}(t)=x_0(t)+\int_{a-t}^{t}K_1\left[t,s;x^{(n-1)}(s),\ y^{(n-1)}(s)\right]ds,$$

$$y^{(n)}(t)=y_0(t)+\int_{-1}^{1+a}K_2\left[t,s;x^{(n-1)}(s),\ y^{(n-1)}(s)\right]ds,\ \ (n=1,2,...). \tag{10}$$

Theorem 2. *Let the operator*

$$K_i(t,s;x,y)\ \left(i=1,2;\ -1\le t,\ s\le 1+a,\ x\in S,\ y\in S^*\right)$$

be continuous and

$$\|K_i(t,s;x,y)\|\le M_i,\quad M_i(2+a)\le r\quad (i=1,2). \tag{11}$$

Let besides

$$\|K_i(t,s;x,y)-K_i(t,s;\overline{x},\overline{y})\|\le L_{i1}(t)\|x-\overline{x}\|+L_{12}(t)\|y-\overline{y}\|,$$
$$\left(i=1,2;\quad -1\le t,\ s\le 1+a,\ x,\overline{x}\in S,\ y,y\in S^*\right).$$

be fulfilled, where $L_{ij}(t)$ satisfy the conditions of theorem 1.

Then the system (1) has a unique solution and the sequential converge to this solution.

Applying theorem 1 it is easy to prove the uniqueness of the solution and also the fundamentality of the sequences $\{x^{(n)}(t)\}$, $\{y^{(n)}(t)\}$, determined by the equalities (10), that proves the existence of solution of the system (1).

Now let's pass to studying the solution of the system (1) by parameter, i.e. consider the system,

$$\begin{aligned} x(t)&=x_0(t)+\int_{a-t}^{t}K_1[t,s;x(s),y(s),\lambda]ds.\\ y(t)&=y_0(t)+\int_{-1}^{1+a}K_2[t,s;x(s),y(s),\lambda]ds.\end{aligned} \tag{12}$$

Here $\lambda\in E$ is a parameter, where E is some Banach space. We'll denote by S the ball from E with radius r and center at the point λ_0.

Theorem 3. *Let the operators* $K_i(t,s;x,y,\lambda)$

$$\left(i=1,2;\ -1\le t,\ s\le 1+a,\ x\in S,\ y\in S_1\right)$$

satisfy the conditions

$$\left\|K_i\left(t,s;\overline{x},\overline{y},\overline{\lambda}\right)-K_i\left(t,s;\overline{\overline{x}},\overline{\overline{y}},\overline{\overline{\lambda}}\right)\right\|\le L_{i1}(t)\left\|\overline{x}-\overline{\overline{x}}\right\|+L_{i2}(t)\left\|\overline{y}-\overline{\overline{y}}\right\|+\varphi_i(t)\left\|\overline{\lambda}-\overline{\overline{\lambda}}\right\|,$$

$(i=1,2)$ *where $L_{ij}(t)$ $(i,j=1,2)$ are nonnegative summable functions on $[-1,1+a]$, and also $\varphi_i(t)$ $(i=1,2)$ are continuous functions on $[-1,1+a]$.*

Then solution of the system (12) $[x(t,\lambda),y(t,\lambda)]$ determined on $[-1,1+a]$ continuously depends on parameter λ.

Proof. Let $\left(\overline{x}(t,\overline{\lambda}),\overline{y}(t,\overline{\lambda})\right)$ and $\left(\overline{\overline{x}}(t,\overline{\overline{\lambda}}),\overline{\overline{y}}(t,\overline{\overline{\lambda}})\right)$ $(-1\le t\le 1+a)$ be two solutions of the system (12) corresponding to the values of the parameter $\lambda=\overline{\lambda}$ and $\lambda=\overline{\overline{\lambda}}$. Then we have

$$\left\|\overline{x}(t)-\overline{\overline{x}}(t)\right\|\le sign\left(t-\frac{a}{2}\right)\times$$

$$\times\int_{a-t}^{t}\left\{L_{11}(s)\left\|\overline{x}(s)-\overline{\overline{x}}(s)\right\|+L_{12}(s)\left\|\overline{y}(s)-\overline{\overline{y}}(s)\right\|+\varphi_1(s)\left\|\overline{\lambda}-\overline{\overline{\lambda}}\right\|\right\}ds,$$

$$\left\|\overline{y}(t)-\overline{\overline{y}}(t)\right\|\le\int_{-1}^{1+a}\left\{L_{21}(s)\left\|\overline{x}(s)-\overline{\overline{x}}(s)\right\|+L_{22}(s)\left\|\overline{y}(s)-\overline{\overline{y}}(s)\right\|+\varphi_2(s)\left\|\overline{\lambda}-\overline{\overline{\lambda}}\right\|\right\}ds,$$

i.e.

$$\left\|\overline{x}(t)-\overline{\overline{x}}\right\|\le\left\|\overline{\lambda}-\lambda\right\|\int_{a-t}^{t}\varphi_1(s)ds+sign\left(t-\frac{a}{2}\right)\int_{a-t}^{t}\left\{L_{11}(s)\left\|\overline{x}-\overline{\overline{x}}\right\|+L_{12}(s)\left\|\overline{y}-\overline{\overline{y}}\right\|\right\}ds,$$

$$\left\|\overline{y}(t)-\overline{\overline{y}}(t)\right\|\le\left\|\overline{\lambda}-\overline{\overline{\lambda}}\right\|\int_{-1}^{1+a}\varphi_2(s)ds+\int_{a-t}^{t}\left\{L_{21}(s)\left\|\overline{x}-\overline{\overline{x}}\right\|+L_{22}(s)\left\|\overline{y}-\overline{\overline{y}}\right\|\right\}ds. \qquad (13)$$

The obtained system of inequalities (13), corresponds to the partial case of system of inequalities of theorem (1), i.e.

$$\psi_i(t,s)=0,\ i=1,2;\ \ u_0(t)=\left\|\overline{\lambda}-\overline{\overline{\lambda}}\right\|\int_{a-t}^{t}\varphi_1(s)ds,\ v_0(t)=\left\|\overline{\lambda}-\overline{\overline{\lambda}}\right\|\int_{-1}^{1+a}\varphi_2(s)ds. \qquad (14)$$

Then applying theorem 1 to inequalities (13) and allowing for (14) we have

$$\left\|\overline{x}(t)-\overline{\overline{x}}(t)\right\|\le\left\|\overline{\lambda}-\overline{\overline{\lambda}}\right\|\left[F_1^*(t)+\alpha\Phi_1(t)\int_{-1}^{1+a}L_{21}(s)F_1^*(s)ds\right],$$

$$\left\|\overline{y}(t)-\overline{\overline{y}}(t)\right\|\le\left\|\overline{\lambda}-\overline{\overline{\lambda}}\right\|\beta\left[\int_{-1}^{1+a}\varphi_2(s)ds+\alpha\int_{-1}^{1+a}L_{21}(s)F_1^*(s)ds\right], \qquad (15)$$

where $F_i^*(t)$ is a bounded function expressed by given functions, and Φ_1 is a function denoted in theorem 1.

From the inequalities (15) follows the statement of the theorem.

Theorem 4. *Let the operators* $K_i(t,s;x,y,\lambda)$ *for any fixed* $t,s\in[-1,1+a]$ *act in* $X\times Y\times E$, *has derivatives by* x,y *and* λ *which we'll denote by* K'_{ix},K'_{iy} *and* $K'_{i\lambda}$ $(i=1,2)$.

Let operators K'_{i1},K'_{ix},K'_{iy} *and* $K'_{i\lambda}$ $(i=1,2)$ *be continuous and satisfy the inequalities*

$$\left\|K'_{ix}(t,s;x,y)h\right\|\le L_{i1}(s)\|h\|,\ \ h=\overline{x}-x\in X,\ (i=1,2),$$

$$\left\|K'_{iy}(t,s;x,y)\overline{h}\right\|\le L_{i2}(s)\left\|\overline{h}\right\|,\ \ \overline{h}=\overline{y}-y\in Y,\ (i=1,2),$$

$$\left\|K'_{i\lambda}(t,s;x,y)h_1\right\|\le\overline{\psi}_i(t,s)\|h_1\|,\ \ h_1=\overline{\lambda}-\lambda\in E,\ (i=1,2),$$

where $L_{ij}(s),\psi_i(t,s)$ $(i,j=1,2,-1\le t,\ s\le 1+a)$ *fulfilling the conditions of theorem 1.*

Then the solution of the system (12) has continuous derivative by λ *coinciding with the solution of the system*

$$\begin{aligned}U(t)h_1&=\int_{a-t}^{t}\{K'_{1x}[t,s;x(s),y(s),\lambda]U(s)h_1+\\&+K'_{1y}[t,s;x(s),y(s),\lambda]V(s)h_1+K'_{1\lambda}[t,s;x(s),y(s),\lambda]\}ds,\\V(t)h_1&=\int_{-1}^{1+a}\{K'_{2x}[t,s;x(s),y(s),\lambda]U(s)h_1+\\&+K'_{2y}[t,s;x(s),y(s),\lambda]V(s)h_1+K'_{2\lambda}[t,s;x(s),y(s),\lambda]\}ds.\end{aligned} \qquad (16)$$

Proof. Denoting the solution of the system (12) according to the value $(\overline{x}(t),\overline{y}(t))$ by $\lambda=\overline{\lambda}$, we have

$$\left\|x(t)-\overline{x}(t)\right\|\le sign\left(t-\frac{a}{2}\right)\int_{a-t}^{t}\{\left\|K'_{1x}[t,s;x(s),y(s),\lambda]\right\|\left\|x(s)-\overline{x}(s)\right\|+$$

$$+\left\|K'_{1y}[t,s;x(s),y(s),\lambda]\right\|\left\|y(s)-\overline{y}(s)\right\|+\left\|K'_{1\lambda}[t,s;x(s),y(s),\lambda]\right\|\left\|\overline{\lambda}-\lambda\right\|+$$

$$+\left\|\omega_1\left[t,s;x,y,x-\bar{x}\right]\right\|+\left\|\theta_1\left(t,s;x,y,\lambda,y-\bar{y}\right)\right\|+\left\|\gamma_1\left(t,s;x,y,\lambda,\lambda-\bar{\lambda}\right)\right\|\Big\}ds,$$

$$\left\|\bar{y}(t)-\bar{y}(t)\right\|\le\int_{-1}^{1+a}\Big\{\left\|K'_{2x}\left[t,s;x(s),y(s),\lambda\right]\right\|\left\|x(s)-\bar{x}(s)\right\|+$$

$$+\left\|K'_{2\lambda}\left[t,s;x(s),y(s),\lambda\right]\right\|\left\|y(s)-\bar{y}(s)\right\|+\left\|K'_{2\lambda}\left[t,s;x(s),y(s),\lambda\right]\right\|\left\|\bar{\lambda}-\lambda\right\|+$$

$$+\left\|\omega_2\left[t,s;x,y,x-\bar{x}\right]\right\|+\left\|\theta_2\left(t,s;x,y,\lambda,y-\bar{y}\right)\right\|+\left\|\gamma_1\left(t,s;x,y,\lambda,\lambda-\bar{\lambda}\right)\right\|\Big\}ds,$$

where $\omega_i,\theta_i,\gamma_i,\ (i,j=1,2,...)$ are infinitismall quantities. Hence fixing the values λ and $\bar{\lambda}$, $\lambda\ne\bar{\lambda}$ we obtain

$$\frac{\left\|\bar{x}(t)-x(t)\right\|}{\left\|\bar{\lambda}-\lambda\right\|}\le sign\left(t-\frac{a}{2}\right)\times$$

$$\times\int_{a-t}^{t}\left\{L_{11}(s)\frac{\left\|\bar{x}(s)-x(s)\right\|}{\left\|\bar{\lambda}-\lambda\right\|}+L_{12}(s)\frac{\left\|\bar{y}(s)-y(s)\right\|}{\left\|\bar{\lambda}-\lambda\right\|}+\psi_1(t,s)\right\}ds,$$

$$\frac{\left\|\bar{y}(s)-y(s)\right\|}{\left\|\bar{\lambda}-\lambda\right\|}\le\int_{-1}^{1+a}\left\{L_{21}(s)\frac{\left\|\bar{x}(s)-x(s)\right\|}{\left\|\bar{\lambda}-\lambda\right\|}+L_{22}(s)\frac{\left\|\bar{y}(s)-y(s)\right\|}{\left\|\bar{\lambda}-\lambda\right\|}+\psi_2(t,s)\right\}ds,$$

where

$$\psi_i(t,s)=\bar{\psi}_i(t,s)+\frac{\left\|\omega_i\right\|+\left\|\theta_i\right\|+\left\|\gamma_i\right\|}{\left\|\lambda-\bar{\lambda}\right\|},\quad (i=1,2).$$

Obviously, that $\psi_i(t,s)\ (i=1,2;\ -1\le t,\ s\le 1+a)$ satisfy the conditions of theorem 1 and besides as λ and $\bar{\lambda}$ are fixed, then the equalities $\dfrac{\left\|x(t)-\bar{x}(t)\right\|}{\left\|\bar{\lambda}-\lambda\right\|}$ and $\dfrac{\left\|y(t)-\bar{y}(t)\right\|}{\left\|\bar{\lambda}-\lambda\right\|}$ are continuous functions on $[-1,1+a]$.

Then applying theorem 1, we have

$$\frac{\left\|x(t)-\bar{x}(t)\right\|}{\left\|\lambda-\bar{\lambda}\right\|}\le F(t),\qquad \frac{\left\|y(t)-\bar{y}(t)\right\|}{\left\|\lambda-\bar{\lambda}\right\|}\le\Phi(t),$$

where $F(t),\Phi(t)$ are bounded functions. Hence it implies that the limits

$$\lim_{\bar{\lambda}\to\lambda}\frac{\left\|x(t)-\bar{x}(t)\right\|}{\left\|\lambda-\bar{\lambda}\right\|}=x'_\lambda(t),\quad \lim_{\bar{\lambda}\to\lambda}\frac{\left\|y(t)-\bar{y}(t)\right\|}{\left\|\bar{\lambda}-\lambda\right\|}=y'_\lambda(t)$$

exist.

Let's show, that these derivatives by λ coincide with the solution of the system of equations (16).

Indeed,

$$h(t)-U(t)h_1=\int_{a-t}^{t}\Big\{K'_{1x}\left[t,s;x,\bar{y},\bar{\lambda}\right]\left[h-U(s)h_1\right]+K'_{1y}\left(t,s;x,y,\bar{\lambda}\right)\times$$

$$\times\left[h(s)-V(s)h_1\right]+\left[K'_{1x}\left(t,s;x,\bar{y},\bar{\lambda}\right)-K'_{1x}\left(t,s;x,y,\bar{\lambda}\right)\right]U(s)h_1+$$

$$+\left[K'_{1y}\left(t,s;x,y,\bar{\lambda}\right)-K'_{1y}\left(t,s;x,y,\lambda\right)\right]V(s)h_1+\omega_1\left(t,s;x,y,\lambda,\bar{x}-x\right)+$$

$$+\theta_1\left(t,s;x,y,\lambda,\bar{y}-y\right)+\gamma_1\left(t,s;x,y,\lambda,\bar{\lambda}-\lambda\right)\Big\}ds,$$

$$h - V(t)h_1 = \int_{-1}^{1+a}\{K'_{2x}[t,s;x,\bar{y},\bar{\lambda}][h(s)-U(s)h_1] + K'_{2y}(t,s;x,y,\bar{\lambda})\times$$
$$\times[\bar{h}(s)-V(s)h_1] + [K'_{2x}(t,s;x,\bar{y},\bar{\lambda}) - K'_{2x}(t,s;x,y,\bar{\lambda})]U(s)h_1 +$$
$$+[K'_{2y}(t,s;x,y,\bar{\lambda}) - K'_{2y}(t,s;x,y,\lambda)]V(s)h_1 + \omega_2(t,s;x,y,\lambda,\bar{x}-x) +$$
$$+\theta_2(t,s;x,y,\lambda,\bar{y}-y) + \gamma_1(t,s;x,y,\lambda,\bar{\lambda}-\lambda)\}ds.$$

Hence, fixing the value $\|h_1\| \neq 0$ and allowing for conditions of the theorem we have

$$\frac{\|h(t)-U(t)h_1\|}{\|h_1\|} \leq sign\left(t-\frac{a}{2}\right)\int_{a-t}^{t}\{L_{11}(s)\frac{\|h(s)-U(s)h_1\|}{\|h_1\|} +$$
$$+ L_{12}(s)\frac{\|\bar{h}(s)-V(s)h_1\|}{\|h_1\|} + \|K'_{1x}(t,s;x,\bar{y},\bar{\lambda}) - K'_{1x}(t,s;x,y,\bar{\lambda})\|\|U(s)\| + \tag{17}$$
$$+\|K'_{1y}(t,s;x,y,\bar{\lambda}) - K'_{1y}(t,s;x,y,\lambda)\|\|V(s)\| + \frac{\|\omega_1\|}{\|h_1\|} + \frac{\|\theta_1\|}{\|h_1\|} + \frac{\|\gamma_1\|}{\|h_1\|},$$

$$\frac{\|\bar{h}(t)-V(t)h_1\|}{\|h_1\|} \leq \int_{-1}^{1+a}\{L_{21}(s)\frac{\|h(s)-U(s)h_1\|}{\|h_1\|} + L_{22}(s)\frac{\|\bar{h}(s)-V(s)h_1\|}{\|h_1\|} +$$
$$+\|K'_{2x}(t,s;x,\bar{y},\bar{\lambda}) - K'_{2x}(t,s;x,y,\lambda)\|\|U(s)\| + \tag{18}$$
$$+\|K'_{2y}(t,s;x,y,\bar{\lambda}) - K'_{2y}(t,s;x,y,\lambda)\|\|V(s)\| + \frac{\|\omega_2\|}{\|h_1\|} + \frac{\|\theta_2\|}{\|h_1\|} + \frac{\|\gamma_2\|}{\|h_1\|}\}ds,$$

where $h(t) = \bar{x}(t) - x(t) \in X$, $\bar{h}(t) = \bar{y}(t) - y(t) \in Y$, $h_1 = \bar{\lambda} - \lambda \in E$.

Under the conditions of theorem one can affirm the following:
Let $\varepsilon > 0$ be an arbitrary number. Then there exists such $\delta_1(\varepsilon) > 0$, that for $\|h\| < \delta_1$ are fulfilled $\frac{\|\omega_i\|}{\|h\|} < \varepsilon$ $(i=1,2)$. On number δ_1 there'll be found such $\delta_2 > 0$, that from $\|h_1\| < \delta_2$ follows $\|h\| < \delta_1$. Consequently for $\|h_1\| < \delta_2$ holds $\frac{\|\omega_i\|}{\|h\|} < \varepsilon$ $(i=1,2)$. Further for $\forall\varepsilon > 0$ thee exists $\delta_3 > 0$ such that for $\|h_1\| < \delta_3$ holds $\frac{\|\gamma_i\|}{\|h_1\|} < \varepsilon$ $(i=1,2)$.

Analogously, for this $\varepsilon > 0$ there'll be found such $\delta_1^*(\varepsilon) > 0$, that for $\|\bar{h}\| \leq \delta_1^*$ will be $\frac{\|\theta_i\|}{\|\bar{h}\|} < \varepsilon$ $(i=1,2)$. Also on number $\delta_1^* > 0$ one can choose such $\delta_2^* > 0$, that from $\|h_1\| < \delta_2^*$ follows $\|\bar{h}\| < \delta_1^*$. Consequently for $\varepsilon > 0$ there exists such $\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_1^*, \delta_2^*\}$, that for $\|h_1\| < \delta$

$$\frac{\|\omega_i\|}{\|h\|} < \varepsilon, \quad \frac{\|\theta_i\|}{\|h\|} < \varepsilon, \quad \frac{\|\gamma_i\|}{\|h\|} < \varepsilon,$$
$$\|K'_{ix}(t,s;\bar{x},\bar{y},\bar{\lambda}) - K'_{ix}(t,s;x,y,\lambda)\| < \varepsilon,$$

$$\left\|K'_{iy}\left(t,s;x,y,\bar{\lambda}\right)-K'_{iy}\left(t,s;x,y,\lambda\right)\right\|<\varepsilon \quad (i=1,2)$$

are fulfilled.

Obviously,

$$\frac{\|\omega_i\|}{\|h_1\|}=\frac{\|\omega_i\|}{\|h\|}\cdot\frac{\|h\|}{\|h_1\|}<\varepsilon\frac{\|h\|}{\|h_1\|}<\varepsilon F(t), \quad (i=1,2)$$

$$\frac{\|\theta_i\|}{\|h_1\|}=\frac{\|\theta_i\|}{\|\bar{h}\|}\cdot\frac{\|\bar{h}\|}{\|h_1\|}<\varepsilon\frac{\|\bar{h}\|}{\|h_1\|}<\varepsilon\Phi(t), \quad (i=1,2).$$

Then we will rewrite the inequalities (17) and (18) in the form

$$\frac{\|h-U(t)h_1\|}{\|h_1\|}\le sign\left(t-\frac{a}{2}\right)\int_{a-t}^{t}\left\{L_{11}(s)\frac{\|h(s)-U(s)h_1\|}{\|h_1\|}+\right.$$
$$\left.+L_{12}(s)\frac{\|\bar{h}(s)-V(s)h_1\|}{\|h_1\|}+\varepsilon\psi_1(t,s)\right\}ds, \qquad (19)$$

$$\frac{\|\bar{h}-V(t)h_1\|}{\|h_1\|}\le\int_{-1}^{1+a}\left\{L_{21}(s)\frac{\|h(s)-U(s)h_1\|}{\|h_1\|}+L_{22}(s)\frac{\|\bar{h}(s)-V(s)h_1\|}{\|h_1\|}+\varepsilon\psi_2(t,s)\right\}ds,$$

where $\psi_i(t,s)=2+2\Phi(t)+F(t)$ $(i=1,2)$ which satisfy the conditions of theorem 1. Then applying theorem 1 to inequalities (19) we have

$$\frac{\|h-U(t)h_1\|}{\|h_1\|}<\varepsilon F_1(t),\ \frac{\|\bar{h}(s)-V(s)h_1\|}{\|h_1\|}+\varepsilon F_2(t),$$

where $F_i(t)$ $(i=1,2)$ are bounded functions expressed by the given function.

Hence

$$\lim_{\|h_1\|\to 0}\frac{\|h(t)-U(t)h_1\|}{\|h_1\|}=0, \qquad \lim_{\|h_1\|\to 0}\frac{\|h(t)-V(t)h_1\|}{\|h_1\|}=0.$$

Q.E.D.

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