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MULTIPLE DIFFERENTIATION FORMULA OF SOME GENERALIZED FUNCTIONS

Abstract

Multiple differentiation formula of the generalized functions $x_\pm^\lambda$; $x_\pm^\lambda \ln^k x_\pm$; *are found.*

At the given paper the formula of multiple differentiation of generalized functions $x_\pm^\lambda; x_1^\lambda \ln^k x_\pm$ are given. These formulas whose proof is cited for the first time as it seems to us is of interest for all who has to work with such generalized functions. These differentiation formulas were applied successfully to the description of totality of fundamental system of solutions of some linear singular ordinary differential equations. The definitions and some other properties of these general functions were adopted from [2].

For the further statements let's denote by $[\lambda]_n$ the expression of the form:

$$[\lambda]_n = \lambda(\lambda-1)(\lambda-2)...(\lambda-n+1);\ n = 0,1,2,...;\ ([\lambda]_0 = 1).$$

The derivatives by the λ of the expression $[\lambda]_n$ we'll denote by $[\lambda]_n'$, $[\lambda]_n''$ and so on. It is evident that $[n]_n = n!$; $[\lambda]_n^{(n)} = n!$; $[n]_{n-1} = [n]_n$; $[\lambda]_n^{(q)} = 0$ when $q > n$, where n-is an integer. On the symbol $[n]_k^{(q)}$ we'll imply the value of the expression $[\lambda]_k^{(q)}$ when $\lambda = n$. We'll use the next lemma whose proof is given in [3].

Lemma. *The next correlations are true:*

$$[\lambda]_k^{(q)} = \sum_{j=0}^{r} q[\lambda]_{k-j-1}^{(q-1)}[\lambda-k+j]_j$$

$$[\lambda]_k^{(q+1)} = \sum_{j=0}^{r} q(q+1)[\lambda]_{k-j-1}^{(q-1)}[\lambda-k+j]_j';\ k = q+r.$$

P.1. The formulas of multiple differentiation of general functions x_+^λ and x_-^λ.

The general functions x_+^λ and x_-^λ are determined at $\operatorname{Re}\lambda > -1$ as functions:

$$x_+^\lambda = \begin{cases} x^\lambda & for\ x>0, \\ 0 & for\ x<0. \end{cases} \quad x_-^\lambda = \begin{cases} 0 & for\ x>0, \\ |x|^\lambda & for\ x<0. \end{cases}$$

And analytically continued on all the values $\lambda \neq -1,2,...,$ have the derivative, having the next form:

$$\left(x_\pm^\lambda\right)' = \pm\lambda x_\pm^{\lambda-1}. \tag{1.1}$$

It is evident that for $\lambda \neq m-1, m-2,\ldots$

$$\left(x_+^\lambda\right)^m = [\lambda]_m x_+^{\lambda-m}, \tag{1.2}$$

$$\left(x_-^\lambda\right)^m = (-1)^m[\lambda]_m x_-^{\lambda-m}. \tag{1.3}$$

For whole positives λ by the consecutive differentiation of the formula (1) we'll get:

$$\left(x_+^m\right)^{(m)} = m!1_+,\ \left(x_+^m\right)^{(m+1)} = m!\delta(x),\dots,\left(x_+^m\right)^{(m+q)} = m!\delta^{(q-1)}(x), \tag{1.4}$$

where q are positive integers, and 1_+ is a function which equals to 1 when $x>0$, and zero for $x<0$. Similarly

$$\left(x_-^m\right)^{(m)} = (-1)^m m!1_-,\ \left(x_-^m\right)^{(m+1)} = (-1)^{(m+1)} m!\delta(x),\dots\left(x_-^m\right)^{(m+q)} = (-1)^{m+1} m!\delta^{(q-1)}(x), \tag{1.5}$$

where 1_- is the function which equals o zero when $x>0$ and 1 for $x<0$.

The general function x_+^{-n}, defined as the value of the functional $F_{-n}(x_+,\lambda)$ when $\lambda=-n$ is differentiated by the formula:

$$\left(x_+^{-n}\right)' = -nx_+^{-n-1} + \frac{(-1)^n\delta^{(n)}(x)}{n!}.$$

Using this we'll get:

$$\left(x_+^{-n}\right)'' = (-n)\left(x_+^{-n-1}\right)' + \frac{(-1)^n\delta^{(n+1)}(x)}{n!} = -n\left[(-n-1)x_+^{-n-2} + \frac{(-1)^{n+1}\delta^{(n+1)}(x)}{(n+1)!}\right] +$$

$$+\frac{(-1)^n\delta^{(n+1)}(x)}{n!} = [-n]_2 x_+^{-n-2} + \frac{(-1)^{n+1}}{(n+1)!}[-n]_2'\delta^{(n+1)}(x).$$

By induction it is easy to prove, that for any whole positive q the next formula is true:

$$\left(x_+^{-n}\right)^q = [-n]_q x_+^{-n-q} + \frac{(-1)^{n+q-1}}{(n+q-1)!}[-n]_q'\delta_{(x)}^{(n+q-1)}. \tag{1.6}$$

Really, supposing the equality (1.6) fulfilled, differentiating it we'll get:

$$\left(x_+^{-n}\right)^{q+1} = [-n]_q\left(x_+^{-n-q}\right)' + \frac{(-1)^{n+q-1}}{(n+q-1)!}[-n]_q'\delta^{(n+q)}(x) =$$

$$= [-n]_q\left\{(-n-q)x_+^{-n-q-1} + \frac{(-1)^{n+q}}{(n+q)!}\delta^{(n+q)}(x)\right\} + \frac{(-1)^{n+q-1}}{(n+q-1)!}[-n]_q'\delta^{(n+q)}(x) =$$

$$= [-n]_{q+1}x_+^{-n-q-1} + \left\{[-n]_q - (n+q)[-n]_q'\right\}\frac{(-1)^{n+q}}{(n+q)!}\delta^{(n+q)}(x) =$$

$$= [-n]_{q+1}x_+^{-n-q-1} + [-n]_{q+1}'\frac{(-1)^{n+q}}{(n+q)!}\delta^{(n+q)}(x),$$

that means the truth of formula (1.6) for any q. The generalized function x_-^{-n}, defined as the value of functional $F_{-n}(x,\lambda)$ when $\lambda=-n$, is differentiated by the formula:

$$\left(x_-^{-n}\right)' = nx_-^{-n-1} - \frac{\delta^{(n)}(x)}{n!}.$$

Using this we'll get:

$$\left(x_-^{-n}\right)'' = n\left(x_-^{-n-1}\right)' - \frac{\delta^{(n+1)}(x)}{n!} = n\left[(n+1)x_-^{-n-2} - \frac{\delta^{(n+1)}(x)}{(n+1)!}\right] - \frac{\delta^{(n+1)}(x)}{n!} =$$

$$= (-1)^2[-n]_2 x_-^{-n-2} - \frac{1}{(n+1)!}[n+(n+1)]\delta^{(n+1)}(x) =$$

$$= (-1)^2[-n]_2 x_-^{-n-2} - \frac{1}{(n+1)!}[-n]_2'\delta^{(n+1)}(x).$$

By induction it is easy to prove that for any whole positive q the next formula is true:

$$\left(x_-^{-n}\right)^{(q)} = (-1)^q[-n]_q x_-^{-n-q} - \frac{(-1)^{q-1}}{(n+q-1)!}[-n]_q{}' \delta^{(n+q-1)}(x). \qquad (1.7)$$

P.2. The multiple differentiation formula of generalized functions $x_+^{\lambda} \ln^k x_+$.

The generalized functions $x_+^{\lambda} \ln^k x_+ (k = 1,2,...)$ are defined (see [2] p 114) as derivatives of k-th order by λ from the functional x_+^{λ} i.e.

$$x_+^{\lambda} \ln^k x_+ = \frac{\partial^k x_+^{\lambda}}{\partial \lambda^k} \qquad (k = 1,2,...).$$

Let's calculate the derivatives of these functions by x. The expansion of the function x_+^{λ} in the neighborhood of the regular point λ_0 in Taylor series has the form:

$$x_+^{\lambda} = x_+^{\lambda_0} + (\lambda - \lambda_0)\frac{\partial}{\partial \lambda} x_+^{\lambda_0} + \frac{1}{2!}(\lambda - \lambda_0)\frac{\partial^2 x_+^{\lambda_0}}{\partial \lambda^2} + ... + \frac{1}{k!}(\lambda - \lambda_0)\frac{\partial^k x_+^{\lambda_0}}{\partial \lambda^k} + ... =$$

$$= x_+^{\lambda_0} + (\lambda - \lambda_0) x_+^{\lambda_0} \ln x_+ + \frac{1}{2!}(\lambda - \lambda_0)^2 x_+^{\lambda_0} \ln^2 x_+ + ... + \frac{1}{k!}(\lambda - \lambda_0)^2 x_+^{\lambda_0} \ln^k x_+ + ...$$

Differentiating this equality we'll get:

$$\lambda x_+^{\lambda-1} = \left(x_+^{\lambda_0}\right)' + (\lambda - \lambda_0)\left(x_+^{\lambda_0} \ln x_+\right)' + \frac{1}{2!}(\lambda - \lambda_0)^2\left(x_-^{\lambda_0} \ln^2 x_-\right)' +$$

$$+ ... + \frac{1}{k!}(\lambda - \lambda_0)^k\left(x_+^{\lambda_0} \ln^k x_+\right)' + ...$$

Let's expand the $\lambda x_+^{\lambda-1}$ in the degrees $(\lambda - \lambda_0)$. For this it is enough to expand the function $x_+^{\lambda-1}$ at neighboring of the point λ_0 in Teylor series

$$\lambda x_+^{\lambda-1} = [(\lambda - \lambda_0) + \lambda_0]\left\{x_+^{\lambda_0-1} + (\lambda - \lambda_0) x_+^{\lambda_0-1} \ln x_+ + ... + \frac{1}{k!}(\lambda - \lambda_0)^k x_+^{\lambda_0-1} \ln^k x_+ + ...\right\}.$$

Comparing the degrees $(\lambda - \lambda_0)$ in the last two expansions we get the formula for the calculation of derivative function $x_+^{\lambda} \ln^k x_+$ at the point $\lambda = \lambda_0$:

$$\left(x_+^{\lambda_0} \ln^k x_+\right)' = \lambda x_+^{\lambda_0-1} \ln^k x_+ + k x_+^{\lambda_0-1} \ln^{k-1} x_+, \quad (k = 1,2,...). \qquad (2.1)$$

Using these formulas at the regular point λ we get:

$$\left(x_+^{\lambda} \ln^k x_+\right)'' = \lambda\left[(\lambda - 1) x_+^{\lambda-2} \ln^k x_+ + k x_+^{\lambda-2} \ln^{k-1} x_+\right] +$$

$$+ k\left[(\lambda - 1) x_+^{\lambda-2} \ln^{k-1} x_+ + (k-1) x_+^{\lambda-2} \ln^{k-2} x_+\right] =$$

$$= [\lambda]_2 x_+^{\lambda-2} \ln^k x_+ + k[\lambda]_2' x_+^{\lambda-2} \ln^{k-1} x_+ + [k]_2 x_+^{\lambda-2} \ln^{k-2} x_+ .$$

By the mathematical induction method it is easy to prove that the next formula is true:

$$\left(x_+^{\lambda} \ln k x_+\right)^{(q)} = \sum_{j=0}^{q} \frac{[k]_j [\lambda]_q^{(j)} x_+^{\lambda-q} \ln^{k-j} x_+}{j!}. \qquad (2.2)$$

For the values $q \le k$, λ isn't integers. If λ is whole then $q \le \lambda$. Really, for values $q = 1,2$. We believe in the truth of (2) by simply calculation, let suppose that (2.2) is true at $q = n - 1$, i.e.

$$\left(x_+^{\lambda} \ln^k x_+\right)^{(n-1)} = \sum_{j=0}^{n-1} \frac{[k]_j [\lambda]_{n-1}^{(j)} x_+^{\lambda-n+1} \ln^{k-j} x_+}{j!}.$$

Differentiating once more we get:

$$\left(x_+^\lambda \ln^k x_+\right)^{(n)} = \sum_{j=0}^{n-1}\frac{[k]_j[\lambda]_{n-1}^{(j)}}{j!}\left[(\lambda-n+1)x_+^{\lambda-n}\ln^{k-j}x_+ + (k-j)x_+^{\lambda-n}\ln^{k-j-1}x_+\right]=$$

$$=[\lambda]_{n-1}(\lambda-n+1)x_+^{\lambda-n}\ln^k x_+ + \sum_{j=1}^{n-1}\frac{[k]_j[\lambda]_{n-1}^{(j)}}{j!}(\lambda-n+1)x_+^{\lambda-n}\ln^{k-j}x_+ +$$

$$+\sum_{j=0}^{n-2}\frac{[k]_j[\lambda]_{n-1}^{(j)}}{j!}(k-j)x_+^{\lambda-n}\ln^{k-j-1}x_+ + \frac{[k]_{n-1}[\lambda]_{n-1}^{(n-1)}}{(n-1)!}(k-n+1)x_+^{\lambda-n}\ln^{k-n}x_+ =$$

$$=[\lambda]_n x_+^{\lambda-n}\ln^k x_+ + \sum_{j=1}^{n-1}\left[\frac{[k]_j[\lambda]_{n-1}^{(j)}(\lambda-n+1)}{j!} + \frac{[k]_{j-1}[\lambda]_{n-1}^{(j-1)}}{(j-1)!}(k-j+1)\right]x_+^{\lambda-n}\ln^{k-j}x_+ +$$

$$+[k]_n x_+^{\lambda-n}\ln^{k-n}x_+ .$$

We can transform the expression standing inside of brackets A in the next from:

$$\frac{[k]_j}{j!}[\lambda]_{n-1}^{(j)}(\lambda-n+1) + \frac{[k]_{j-1}[\lambda]_{n-1}^{(j-1)}}{(j-1)!}(k-j+1) = \frac{[k]_j}{j!}\left\{[\lambda]_{n-1}^{(j)}(\lambda-n+1)+\right.$$

$$\left.+j[\lambda]_{n-1}^{(j-1)}\right\} = \frac{[k]_j}{j!}\left\{[\lambda]_{n-1}^{(j)}(\lambda-n+1)\right\}^{(j)} = \frac{[k]_j}{j!}[\lambda]_n^{(j)}.$$

So, we have:

$$\left(x_+^\lambda \ln^k x_+\right)^{(n)} = [\lambda]_n x_+^{\lambda-n}\ln^k x_+ + \sum_{j=1}^{n-1}\frac{[k]_j[\lambda]_n^{(j)}}{j!}x_+^{\lambda-n}\ln^{k-j}x_+ +$$

$$+[k]_n x_+^{\lambda-n}\ln^{k-n}x_+ = \sum_{j=0}^{n}\frac{[k]_j[\lambda]_n^{(j)}}{j!}x_+^{\lambda-n}\ln^{k-j}x_+ ,$$

which finishes the proof by induction. Let's consider the case when $q>k$ and λ isn't whole. At $q=k$ from (2) we have:

$$\left(x_+^\lambda \ln^k x_+\right)^k = \sum_{j=0}^{k-1}\frac{[k]_j[\lambda]_k^{(j)}}{j!}x_+^{\lambda-k}\ln^{k-j}x_+ + [k]_k x_+^{\lambda-k}.$$

Differentiating by x we get:

$$\left(x_+^\lambda \ln^k x_+\right)^{(k+1)} = \sum_{j=0}^{k-1}\frac{[k]_j[\lambda]_k^{(j)}}{j!}\left[(\lambda-k)x_+^{\lambda-k-1}\ln^{k-j}x_+ + (k-j)x_+^{\lambda-k-1}\ln^{k-j-1}x_+\right]+$$

$$+[k]_k\left(x_+^{\lambda-k}\right)' = \frac{[k]_0[\lambda]_k^{(0)}}{0!}(\lambda-k)x_+^{\lambda-k-1}\ln^k x_+ + \sum_{j=1}^{k-1}\frac{[k]_j[\lambda]_k^{(j)}}{j!}(\lambda-k)x_+^{\lambda-k-1}\ln^{k-j}x_+ +$$

$$+\sum_{j=1}^{k-2}\frac{[k]_j[\lambda]_k^{(j)}}{j!}(k-j)x_+^{\lambda-k-1}\ln^{k-j-1}x_+ + \frac{[k]_{k-1}[\lambda]_k^{(k-1)}}{(k-1)!}(\lambda-k+1)x_+^{\lambda-k-1} + [k]_k\left(x_+^{\lambda-k}\right)' =$$

$$=\frac{[k]_0[\lambda]_k^{(0)}}{0!}(\lambda-k)x_+^{\lambda-k-1}\ln^k x_+ + \sum_{j=1}^{k-1}\left\{\frac{[k]_j[\lambda]_k^{(j)}(\lambda-k)}{j!}+\right.$$

$$\left.+\frac{[k]_{j-1}[\lambda]_k^{(j-1)}(k-j+1)}{(j-1)!}\right\}x_+^{\lambda-k-1}\ln^{k-j}x_+ \frac{[k]_{k-1}[\lambda]_k^{(k-1)}}{(k-1)!}x_+^{\lambda-k-1} + [k]_k\left(x_+^{\lambda-k}\right)' =$$

$$=\frac{[k]_0[\lambda]_k^{(0)}(\lambda-k)x_+^{\lambda-k-1}\ln^k x_+}{0!}+\sum_{j=1}^{k-1}\frac{[k]_j}{j!}\left\{[\lambda]_k^{(j)}(\lambda-k)+j[\lambda]_k^{(j-1)}\right\}x_+^{\lambda-k-1}\ln^{k-j}x_+ +$$

$$+\frac{[k]_{k-1}[\lambda]_k^{(k-1)}x_+^{\lambda-k-1}}{(k-1)!}+[k]_k\left(x_+^{\lambda-k}\right)'=\frac{[k]_0[\lambda]_{k+1}^{(0)}x_+^{\lambda-k-1}\ln^k x_+}{0!}+$$

$$\sum_{j=1}^{k-1}\frac{[k]_j[\lambda]_{k+1}^{(j)}}{j!}x_+^{\lambda-k-1}\ln^{k-j}x_+ +k[\lambda]_k^{(k-1)}x_+^{\lambda-k-1}+[k]_k\left(x_+^{\lambda-k}\right)'=$$

$$\sum_{j=1}^{k-1}\frac{[k]_j}{j!}[\lambda]_{k+1}^{(j)}x_+^{\lambda-k-1}\ln^{k-j}x_+ +\sum_{j=0}^{1}k[\lambda]_{k-j}^{(k-1)}[\lambda-k-1+j]_j x_+^{\lambda-k-1}.$$

Using lemma 1 we can write, the second sum in the form:

$$\sum_{j=0}^{1}k[\lambda]_{k-j}^{(k-1)}[\lambda-k+j]_j=[\lambda]_{k+1}^{(k)}.$$

So, we have:

$$\left(x_+^{\lambda}\ln^k x_+\right)^{(k+1)}=\sum_{j=0}^{k}\frac{[k]_j}{j!}[\lambda]_{k+1}^{(j)}x_+^{\lambda-k-1}\ln^{k-j}x_+.$$

Let's psrove again by induction that the next formula is true:

$$\left(x_+^{\lambda}\ln^k x_+\right)^{(k+p)}=\sum_{j=0}^{k}\frac{[k]_j[\lambda]_{k+p}^{(j)}}{j!}x_+^{\lambda-k-p}\ln^{k-j}x_+ \tag{2.3}$$

for any $P=1,2,\dots$.

We see the truth of formula (2.2) for $p=1$ with simply calculations. Let suppose that it is true for $p=n-1$, i.e.

$$\left(x_+^{\lambda}\ln x_+\right)^{(k+n-1)}=\sum_{j=0}^{k}\frac{[k]_j[\lambda]_{k+m-1}^{(j)}}{j!}x_+^{\lambda-k-n+1}\ln^{k-j}x_+.$$

Differentiating this equality we'll get:

$$\left(x_+^{\lambda}\ln^k x_+\right)^{(k+n)}=\sum_{j=0}^{k}\frac{[k]_j}{j!}[\lambda]_{k+n-1}^{(j)}\left[(\lambda-k-n+1)x_+^{\lambda-k-n}\ln^{k-j}x_+ +(k-j)x_+^{\lambda-k-n}\ln^{k-j-1}x_+\right]=$$

$$=\frac{[k]_0[\lambda]_{k+n-1}^{(0)}}{0!}(\lambda-k-n+1)x_+^{\lambda-k-n}\ln^k x_+ +\sum_{j=1}^{k-1}\frac{[k]_j}{j!}[\lambda]_{k+n-1}^{(j)}(\lambda-k-n+1)x_+^{\lambda-k-n}\ln^{k-j}x_+ +$$

$$+\frac{[k]_k}{k!}[\lambda]_{k+n-1}^{(k)}(\lambda-k-n+1)x_+^{\lambda-k-n}+\sum_{j=1}^{k-2}\frac{[k]_j[\lambda]_{k+n-1}^{(j)}(k-j)}{j!}x_+^{\lambda-k-n}\ln^{k-j-1}x_+ +$$

$$+\frac{[k]_{k-1}[\lambda]_{k+n-1}^{(k-1)}}{(k-1)!}x_+^{\lambda-k-n}=\frac{[k]_0[\lambda]_{k+n}^{(0)}x_+^{\lambda-k-n}\ln^k x_+}{0!}+\sum_{j=1}^{k-1}\left\{\frac{[k]_j[\lambda]_{k+n-1}^{(j)}(\lambda-k-n+1)}{j!}+\right.$$

$$\left.\frac{[k]_{j-1}[\lambda]_{k+n-1}^{(j-1)}(k-j+1)}{(j-1)!}\right\}x_+^{\lambda-k-n}\ln^{k-j}x_+ +\left\{[\lambda]_{k+n-1}^{(k-1)}(\lambda-k-n+1)+k[\lambda]_{k+n-1}^{(k-1)}\right\}x_+^{\lambda-k-n}=$$

$$=\frac{[k]_0[\lambda]_{k+n}^{(0)}}{0!}x_+^{\lambda-k-n}\ln^k x_+ +\sum_{j=1}^{k-1}\frac{[k]_j[\lambda]_{k+n}^{(j)}}{j!}x_+^{\lambda-k-n}\ln^{k-j}x_+ +[\lambda]_{k+n}^{(k)}x_+^{\lambda-k-n}=$$

$$=\sum_{j=0}^{k}\frac{[k]_j[\lambda]_{k+n}^{(j)}}{j!}x_+^{\lambda-k-n}\ln^{k-j}x_+,$$

which finishes the proof.

Comparing (2.2) and (2.3) we finally get:

$$\left(x_+^\lambda \ln^k x_+\right)^{(q)} = \sum_{j=0}^{q} \frac{[k]_j [\lambda]_q^{(j)} x_+^{\lambda-q} \ln^{k-j} x_+}{j!} \tag{2.4}$$

λ isn't whole, $q = 1,2,\ldots,$ at $j > k$, $[k]_j = 0$

j has the value 0,1,2,…,k and we get the formulas (3).

P. 3. The differentiation of generalized functions $x_+^{-n} \ln^k x_+$.

The functionals $x_+^{-n} \ln^k x_+ (k = 1,2...)$ are defined (see [2] p.117) as derivatives by λ of the function $F_{-n}(x_+, \lambda)$ at $\lambda = -n$ i.e.

$$\frac{\partial E_{-n}(x_+,\lambda)}{\partial \lambda}\Big|_{\lambda=-n} = x_+^{-n} \ln x_+, \frac{\partial^2 E_{-n}(x_+,\lambda)}{\partial \lambda^2}\Big|_{\lambda=-n} = x_+^{-n} \ln^2 x_+ \text{ and so on.}$$

Let's calculate the derivative of functional $x_+^{-n} \ln^k x_+$, $(k = 1,2,\ldots)$.

The generalized function x_+^λ at the neighborhood $\lambda = -n$ is expanded in next Loran series (see [2]. P.117).

$$x_+^\lambda = \frac{(-1)^{n-1}\delta^{(n-1)}(x)}{(n-1)!(\lambda+n)} + x_+^{-n} + (\lambda+n)x_+^{-n} \ln^k x_+ + \ldots + \frac{(\lambda+n)^k}{k!} x_+^{-n} \ln^k x_+ + \ldots.$$

Differentiating this equality by x, which is always possible for the generalized functions we get:

$$\lambda x_+^{\lambda-1} = \frac{(-1)^{n-1}\delta^{(n)}(x)}{(n-1)!(\lambda+n)} + \left(x_+^{-n}\right)' + (\lambda+n)\left(x_+^{-n} \ln x_+\right)' + \ldots.$$

On other hand the functional $\lambda x_+^{\lambda-1}$ at the neighborhood of the point $\lambda = -n$ is expanded to the next Loran series:

$$\lambda x_+^{\lambda-1} = [(\lambda+n)-n]\left[\frac{(-1)^n\delta^{(n)}(x)}{n!(\lambda+n)} + x_+^{-(n+1)} + (\lambda+n)x_+^{-(n+1)} \ln x_+ \ldots + \right.$$

$$\left. + \frac{(\lambda+n)^k}{k!} x_+^{-(n+1)} \ln x_+ + \ldots\right]$$

equating the coefficients at the same degrees $(\lambda+n)$ we get:

$$\left(x_+^{-n}\right)' = -nx_+^{-(n+1)} + \frac{(-1)^n\delta^{(n)}(x)}{n!}, \quad \left(x_+^{-n} \ln x_+\right)' = -nx_+^{-(n+1)} \ln x_+ + x_+^{-(n+1)},$$

$$\left(x_+^{-n} \ln^2 x_+\right)' = -nx_+^{-(n+1)} \ln^2 x_+ + 2x_+^{-(n+1)} \ln x_+, \ldots$$

$$\ldots, \left(x_+^{-n} \ln^k x_+\right)' = -nx_+^{-(n+1)} \ln^k x_+ + kx_+^{-(n+1)} \ln^{k-1} x_+.$$

Using the last formula let's calculate the higher derivatives:

$$\left(x_+^{-n} \ln^k x_+\right)'' = -n\left[(-n-1)x_+^{-n-2} \ln^k x_+ + kx_+^{-n-2} \ln^{k-1} x_+\right] + k\left[(-n-1)x_+^{-n-2} \ln^{k-1} x_+ + \right.$$

$$\left. + (k-1)x_+^{-n-2} \ln^{k-2} x_+\right] = [-n]_2 x_+^{-n-2} \ln^k x_+ + k[-n]_2' x_+^{-n-2} \ln^{k-1} x_+ + [k]_2 x_+^{-n-2} \ln^{k-2} x_+.$$

$$\left(x_+^{-n} \ln^k x_+\right)''' = [-n]_2\left[(-n-2)x_+^{-n-3} \ln^k x_+ + kx_+^{-n-3} \ln^{k-1} x_+\right] +$$

$$k[-n]_2'\left[(-n-2)x_+^{-n-3} \ln^{k-1} x_+ + (k-1)x_+^{-n-3} \ln^{k-2} x_+\right] + [k]_2\left[(-n-2)x_+^{-n-3} \ln^{k-2} x_+ + \right.$$

$$\left. (k-2)x_+^{-n-3} \ln^{k-3} x_+\right] = [-n]_3 x_+^{-n-3} \ln^k x_+ + k\left\{[-n]_2 + [-n]_2'(-n-2)\right\} x_+^{-n-3} \ln^{k-1} x_+ +$$

$$+[k]_2\{[-n]_2'+(n-2)\}x_+^{-n-3}\ln^{k-2}x_+ +[k]_3x_+^{-n-3}\ln^{k-3}x_+ =[-n]_3x_+^{-n-3}\ln^k x_+ +$$
$$+k[-n]_3'x_+^{-n-3}\ln^{k-1}x_+ +\frac{[k]_2}{2!}[-n]_3'x_+^{-n-3}\ln^{k-2}x_+ +[k]_3x_+^{-n-3}\ln^{k-3}x_+,...,$$

$$\left(x_+^{-n}\ln^k x_+\right)^{(q)}=\sum_{j=0}^{q}\frac{[k]_j[-n]_q^{(j)}}{j!}x_+^{-n-q}\ln^{k-j}x_+,\quad q\le k \tag{3.1}$$

at $q=k$ we have:

$$\left(x_+^{-n}\ln^k x_+\right)^{(k)}=[-n]_k x_+^{-n-k}\ln^k x_+ +k[-n]_k' x_+^{-n-k}\ln^{k-1}x_+$$
$$+\frac{[k]_2}{2!}[-n]_k'' x_+^{-n-k}\ln^{k-2}x_+ +...++\frac{[k]_{k-1}}{(k-1)!}[-n]_k^{(k-1)}x_+^{-n-k}\ln x_+ +[k]_k x_+^{-n-k}.$$

Differentiating once more we have:

$$\left(x_+^{-n}\ln^k x_+\right)^{k+1}=[-n]_k\left[(-n-k)x_+^{-n-k-1}\ln^k x_+ +kx_+^{-n-k-1}\ln^{k-1}x_+\right]+$$
$$+k[-n]_k'\left[(-n-k)x_+^{-n-k-1}\ln^{k-1}x_+ +(k-1)x_+^{-n-k-1}\ln^{k-2}x_+\right]+...+$$
$$\frac{[k]_{k-1}}{(k-1)!}[-n]_k^{(k-1)}\left[(-n-k)x_+^{-n-k-1}\ln x_+ +x_+^{-n-k-1}\right]+[k]_k\left(x_+^{-n-k}\right)'=$$
$$=[-n]_{k+1}x_+^{-n-k-1}\ln^k x_+ +\frac{[k]_1}{1!}[-n]_{k+1}'x_+^{-n-k-1}\ln^{k-1}x_+ +...+$$
$$+\frac{[k]_{k-1}}{(k-1)!}[-n]_{k+1}^{(k-1)}x_+^{-n-k-1}\ln x_+ +\frac{[k]_{k-1}}{(k-1)!}[-n]_k^{(k-1)}x_+^{-n-k-1}+[k]_k\left(x_+^{-n-k}\right)',$$

....

$$\left(x_+^{-n}\ln^k x_+\right)^{(k+q)}=[-n]_{k+q}x_+^{-n-k-q}\ln^k x_+ +\frac{[k]_1}{1!}[-n]_{k+q}'x_+^{-n-k-q}\ln^{k-1}x_+ ...+$$
$$+\frac{[k]_{k-1}}{(k-1)!}[-n]_{k+q}^{(k-1)}x_+^{-n-k-q}\ln x_+ +\frac{[k]_{k-1}}{(k-1)!}\left\{[-n]_{k+q-2}^{(k-1)}x_+^{-n-k-q}-[-n]_{k+q-2}^{(k-1)}(x_+^{-n-k-q+1})'+\right.$$
$$\left.+[-n]_{k+q-3}^{(k-1)}\left(x_+^{-n-k-q+2}\right)''+...+[-n]_k^{(k-1)}(x_+^{-n-k-1})^{(q-1)}\right\}+[k]_k\left(x_+^{-n-k}\right)^{(q)}.$$

So we have:

$$\left(x_+^{-n}\ln^k x_+\right)^{(k+q)}=\sum_{j=0}^{k-1}\frac{[k]_j}{j!}[-n]_{k+q}^{(j)}x^{-n-k-q}\ln^{k-j}x_+ +\sum_{j=0}^{q}\frac{[k]_{k-1}}{(k-1)!}[-n]_{k+q-1-j}^{(k-1)}(x_+^{-n-k-q+j})^{(j)}$$

Applying the formula (1.6) from p.1. we get:

$$\left(x_+^{-n}\ln^k x_+\right)^{(k+q)}=\sum_{j=0}^{k-1}\frac{[k]_j}{j!}[-n]_{k+q}^{(j)}x_+^{-n-k-q}\ln^{k-j}x_+ +$$
$$+\sum_{j=0}^{q}k[-n]_{k+q-1+j}^{(k-1)}\left\{[-n-k-q+j]x_+^{-n-k-q}+\frac{(-1)^{n+k+q-1}}{(n+k+q-1)!}[-n-k-q+j]_j'\times\right.$$
$$\left.\times\delta^{(n+k+q-1)}(x)\right\}=\sum_{j=0}^{k-1}\frac{[k]_j}{j!}[-n]_{k+q}^{(j)}x_+^{-n-k-q}\ln^{k-j}x_+ +[-n]_{k+q}^{(k)}x_+^{-n-k-q}+$$
$$+\frac{(-1)^{n+k+q-1}[-n]_{k+q}^{(k+1)}}{(n+k+q-1)!(k+1)}\delta_{(x)}^{(n+k+q-1)}=\sum_{j=0}^{k}\frac{[k]_j[-n]_{k+q}^{(j)}x_+^{-n-k-q}\ln^{k-j}x_+}{j!}+$$

$$+\frac{(-1)^{n+k+q-1}[-n]_{k+q}^{(k+1)}}{(n+k+q-1)!(k+1)}\delta_{(x)}^{(n+k+q-1)}.$$

Thus, we receive the formula:

$$\left(x_+^{-n}\ln^k x_+\right)^{(k+q)}=\sum_{j=0}^{k}\frac{[k]_j[-n]_{k+q}^{(j)}x_+^{-n-k-q}\ln^{k-j}x_+}{j!}+$$
$$+\frac{(-1)^{n+k+q-1}[-n]_{k+q}^{(k+1)}}{(n+k+q-1)!(k+1)}\delta^{(n+k+q-1)}(x). \quad (3.2)$$

Connecting (3.1) and (3.2) we'll get:

$$\left(x_+^{-n}\ln^k x_+\right)^{(q)}=\sum_{j=0}^{k}\frac{[k]_j[-n]_q^{(j)}x_+^{-n-q}\ln^{k-j}x_+}{j!}+\frac{(-1)^{n+q-1}[-n]_q^{(k+1)}}{(n+q-1)!(k+1)}\delta^{(n+q-1)}(x). \quad (3.3)$$

From this formula at $q\le k$ we have (3.1) and at $q>k$ we get (3.2).

P. 4. The differentiating of the generalized functions $x_+^n\ln^k x_+$.

By formula (2.4) p.2 at $\lambda=n$, $q>k$ we have:

$$\left(x_+^n\ln^k x_+\right)^{(n)}=\sum_{j=0}^{k}\frac{[k]_j[n]_n^{(j)}x_+^0\ln^{k-j}x_+}{j!}=\sum_{j=0}^{k-1}\frac{[k]_j[n]_n^{(j)}x_+^0\ln^{k-j}x_+}{j!}+\frac{[k]_k[n]_n^{(k)}x_+^0}{k!}.$$

Differentiating once more we get:

$$\left(x_+^n\ln^k x_+\right)^{(n+1)}=\sum_{j=0}^{k-1}\frac{[k]_j[n]_n^{(j)}(k-j)x_+^{-1}\ln^{k-j-1}x_+}{j!}+\frac{[k]_k[n]_n^{(k)}\left(x_+^0\right)'}{k!}.$$

Accepting the notation

$$B_j=\frac{[k]_j[n]_n^{(j)}(k-j)}{j!}\quad (j=0,1,..,k-1),\quad B_k=\frac{[k]_k[n]_n^{(k)}}{k!},$$

the last equality let's write in the form:

$$\left(x_+^n\ln^k x_+\right)^{(n+1)}=\sum_{j=0}^{k-1}B_jx_+^{-1}\ln^{k-j-1}x_+ + B_k\left(x_+^0\right)'$$

Applying the formula (2.3) p.2 we'll get:

$$\left(x_+^n\ln^k x_+\right)^{(n+q)}=\sum_{j=0}^{k-1}B_j\left\{\sum_{i=0}^{k-j-1}\frac{[k-j-1]_i[-1]_{q-1}^{(i)}}{i!}x_+^{-q}\ln^{k-j-1-i}x_+ +\right.$$
$$\left.+\frac{(-1)^{q-1}[-1]_{q-1}^{(k-j)}\delta_{(x)}^{(q-1)}}{(q-1)!(k-j)}\right\}+B_k\left(x_+^0\right)^{(q)}=\sum_{j=0}^{k-1}B_j\sum_{i=0}^{k-j-1}\frac{[k-j-1]_i[-1]_{q-1}^{(i)}}{i!}x_+^{-q}\ln^{k-j-1-i}x_+ +$$
$$+\left\{\sum_{j=0}^{k-1}\frac{B_j(-1)^{q-1}[-1]_{q-1}^{(k-j)}}{(j-1)(k-j)}+B_k\right\}\delta^{(q-1)}(x).$$

The expression under the sign of first sum let's arrange in degrees $\ln x_+$:

$$\sum_{j=0}^{k-1} B_j \sum_{i=0}^{k-j-1} \frac{[k-j-1]_i [-1]_{q-1}^{(i)}}{i!} x_+^{-q} \ln^{k-j-1-i} x_+ = B_0 \sum_{i=0}^{k-2} \frac{[k-1]_i [-1]_{q-1}^{(i)}}{i!} x_+^{-q} \ln^{k-1-i} x_+ +$$

$$+ B_0 \frac{[k-1]_{k-1} [-1]_{q-1}^{(k-1)}}{(k-1)!} x_+^{-q} + B_1 \sum_{i=0}^{k-3} \frac{[k-2]_i [-1]_{q-1}^{(i)}}{i!} x_+^{-q} \ln^{k-2-i} x_+ +$$

$$+ B_1 \frac{[k-2]_{k-2} [-1]_{q-1}^{(k-2)}}{(k-2)!} x_+^{-q} + \ldots + B_{k-2} \frac{[1]_0 [-1]_{q-1}^{(0)} x^{-q} \ln x_+}{0!} + B_{k-2} \frac{[1]_1 [-1]_{q-1}^{(1)} x^{-q}}{1!} +$$

$$+ B_{k-1} \frac{[0]_0 [-1]_{q-1}^{(0)} x^{-q}}{0!} = \left\{ B_{k-2} \frac{[1]_0 [-1]_{q-1}^{(0)}}{0!} + B_{k-3} \frac{[2]_1 [-1]_{q-1}^{(2)}}{1!} + \ldots + \right.$$

$$\left. + B_1 \frac{[k-2]_{k-3} [-1]_{q-1}^{(k-3)}}{(k-3)!} + B_0 \frac{[k-1]_{k-2} [-1]_{q-1}^{(k-2)}}{(k-2)!} \right\} x_+^{-q} \ln x_+ + \left\{ B_{k-3} \frac{[2]_0 [-1]_{q-1}^{0}}{0!} + \right.$$

$$\left. + \ldots + B_1 \frac{[k-2]_{k-4} [-1]_{q-1}^{(k-4)}}{(k-4)!} + B_0 \frac{[k-1]_{k-3} [-1]_{q-1}^{(k-3)}}{(k-3)!} \right\} x_+^{-q} \ln x_+ + \ldots +$$

$$+ \left\{ B_1 \frac{[k-2]_0 [-1]_{q-1}^{0}}{0!} + B_0 \frac{[k-1]_1 [-1]_{q-1}^{(1)}}{(1)!} \right\} x_+^{-q} \ln^{k-2} x_+ + B_0 \frac{[k-1]_0 [-1]_{q-1}^{0}}{0!} x_+^{-q} \ln^{k-1} x_+ +$$

$$+ \left\{ B_0 [-1]_{q-1}^{(k-1)} + B_1 [-1]_{q-1}^{(k-2)} + \ldots + B_{k-1} [-1]_{q-1}^{(0)} \right\} x_+^{-q} = \sum_{j=1}^{k-1} \omega_j x_+^{-q} \ln^{k-j} x_+ + \omega_k x_+^{-q} =$$

$$= \sum_{j=1}^{k-1} \omega_j x_+^{-q} \ln^{k-j} x_+ .$$

So:

$$\sum_{j=0}^{k-1} B_j \sum_{i=0}^{k-j-1} \frac{[k-j-1]_i [-1]_{q-1}^{(i)}}{i!} x_+^{-q} \ln^{k-j-1-i} x_+ = \sum_{j=1}^{k} \omega_j x_+^{-q} \ln^{k-j} x_+ ,$$

where by ω_j we denote the expression:

$$\omega_j = \sum_{i=0}^{j-1} B_{j-1-i} \frac{[k-j+i]_i [-1]_{q-1}^{(i)}}{i!} \quad (j = 1,2,\ldots,k)$$

thus we get the formula:

$$\left(x_+^n \ln^k x_+\right)^{(n+q)} = \sum_{j=1}^{k} \omega_j x_+^{-q} \ln^{k-j} x_+ + \omega_0 \delta^{(q-1)}(x), \tag{4.1}$$

where,

$$\omega_j = \sum_{i=0}^{j-1} B_{j-1-i} \frac{[k-j+i]_i [-1]_{q-1}^{(i)}}{i!} \qquad (j = 1,2,\ldots,k).$$

P. 5. The multiple differentiation of the generalized functions $x_-^\lambda \ln^k x$.

The generalization of the functions $x_-^\lambda \ln^k x_-$ $(k = 1,2,\ldots)$ are defined as derivatives of k-th order by λ from the functional x_-^λ, i.e.

$$x^\lambda \ln^k x_- = \frac{\partial^k x_-^\lambda}{\partial \lambda^k} \quad (k = 1,2,\ldots).$$

The derivatives by x of these functionals are calculated similarly, as the derivatives of the functionals $x_+^\lambda \ln^k x_+$. But, taking into account the equality $(x^\lambda, \varphi(x)) = (x_+^\lambda, \varphi(-x))$, the formulas of differentiation for the generalized functions $x_-^\lambda \ln^k x_-$ we can get from corresponding differentiation formulas for $x_+^\lambda \ln^k x_+$, substituting $\varphi(x)$ by $\varphi(-x)$ the expression of the form $\varphi^q(x)$ by $(-1)^q\ \varphi(0)$. For example form formula (2.4) p.2. get for the derivative by q-th order of the functional $(x_+^\lambda \ln^k x_+)$, it is easy to find the formula for the derivative of the same order of the functional $x_-^\lambda \ln^k x_-$. Really,

$$\left(\left(x_-^\lambda \ln^k x_-\right)^{(q)},\ \varphi(x)\right) = (-1)^q (x_-^\lambda \ln^k x_-,\ \varphi^{(q)}(x)) =$$
$$= (-1)^q \left(x_+^\lambda \ln^k x_+,\ \ (-1)^q \varphi^{(q)}(-x)\right) = (-1)^q \left((x_+^\lambda \ln^k x_+)^{(q)},\ \varphi(-x)\right) =$$
$$= (-1)^q \left(\sum_{j=0}^{q} \frac{[k]_j [\lambda]_q^{(j)}}{j!} x_+^{\lambda-q} \ln^{k-j} x_+,\ \ \varphi(-x)\right) = (-1)^q \left(\sum_{j=0}^{q} \frac{[k]_j [\lambda]_q^{(j)} x_-^{\lambda-q} \ln^{k-j} x_-}{j!}, \varphi(x)\right).$$

Whence

$$\left(x_-^\lambda \ln^k x_-\right)^{(q)} = \sum_{j=0}^{q} \frac{(-1)^q [k]_j [\lambda]_q^{(j)}}{j!} x_-^{\lambda-q} \ln^{k-j} x_-, \tag{5.1}$$

that is true for all q non-integer λ and at $q \le \lambda$ for integer λ. Similarly, at different combinations λ and q we'll get the next differentiation formula for the generalized functions $x_-^\lambda \ln^k x_-$. To the formula (3.3) p.3. corresponds the formula

$$\left(x_-^\lambda \ln^k x_-\right)^{(q)} = \sum_{j=0}^{k} \frac{(-1)^q [k]_j [-n]_q^{(j)}}{j!} x_-^{-n-q} \ln^{k-j} x_- + \frac{(-1)^q [-n]_q^{(k+1)}}{(n+q-1)!(k+1)} \delta^{(n+q-1)}(x) \tag{5.2}$$

and the formula (1) p.4. corresponds to the formula:

$$\left(x_-^\lambda \ln^k x_-\right)^{n+q} = \sum_{j=1}^{k} (-1)^{(n+q)} \omega_j x_-^{-q} \ln^{k-j} x_- + \omega_0 (-1)^{n-1} \delta^{(q-1)}(x), \tag{5.3}$$

where ω_j are the same, as in p.3.

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