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TWO-WEIGHTED INEQUALITIES FOR RIESZ POTENTIALS, GENERATED BY BESSEL DIFFERENTIAL OPERATORS

Abstract

In this we prove two-weight inequalities for the Riesz potentials, generated by Bessel differential operators $B=\frac{d^2}{dx^2}+\frac{\gamma}{x}\frac{d}{dx}$ *(B-Riesz potentials). In some special case we have found the necessary and sufficient conditions for pairs of weights ensuring the validity of strong type inequalities for the B-Riesz potentials.*

Let $R_+=]0,\infty[$, $\gamma>0$; $E_+(x,r)=\{y\in R_+ : |x-y|<r\}$, $E_+(0,r)=(0,r)$. We will denote by $L_p^\gamma(R_+)$ the space of measurable functions $f(x)$, $x\in R_+$ with the finite norm

$$\|f\|_{L_p^\gamma(R_+)}=\left(\int_{R_+}|f(x)|^p x^\gamma dx\right)^{1/p},\quad 1\le p<\infty .$$

We put $L_\infty^\gamma(R_+)=L_\infty(R_+)$, where $L_\infty(R_+)$ the class of all essential bounded functions f with the finite norm

$$\|f\|_{L_\infty^\gamma(R_+)}=\|f\|_{L_\infty(R_+)}=\operatorname*{ess\,sup}_{x\in R_+}|f(x)|.$$

Denote the T^y the B-shift operator acting according to the law

$$T^y f(x)=C_\gamma\int_0^\pi f\left(\sqrt{x^2+y^2-2xy\cos\alpha}\right)\sin^{\gamma-1}\alpha d\alpha ,$$

where $C_\gamma=\pi^{-\frac{1}{2}}\Gamma(\gamma+1/2)\Gamma^{-1}(\gamma)$.

We remark that T^y is closely connected with the $B=\frac{d^2}{dx^2}+\frac{\gamma}{x}\frac{d}{dx}$ (see [1] for details).

For the function $f:R_+\to R$ let us consider B-Riesz potentials

$$I_B^\alpha f(x)=\int_0^\infty T^y x^{\alpha-1-\gamma} f(y)y^\gamma dy,\quad 0<\alpha<1+\gamma .$$

The following theorem is valid.

Theorem 1. *Let* $0<\alpha<1+\gamma$, $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{1+\gamma}$, $1\le p<q<\infty$.

1) *If* $p=1$, $f\in L_1^\gamma(R_+)$, *then for all* $\lambda>0$

$$\int_{\{x\in R_+ : I_B^\alpha f(x)>\lambda\}} x^\gamma dx\le\left(\frac{C}{\lambda}\int_{R_+}|f(x)|x^\gamma dx\right)^q ,$$

where C does not depend on f.

2) *If* $1<p<\frac{1+\gamma}{\alpha}$, $f\in L_p^\gamma(R_+)$, *then* $I_B^\alpha f\in L_q^\gamma(R_+)$ *and*

$$\left(\int_{R_+}\left(I_B^\alpha f(x)\right)^q x^\gamma dx\right)^{1/q} \le C\left(\int f(x)|^p x^\gamma dx\right)^{1/p},$$

where C -depends only on p,γ .

Let ω be positive measurable function on R_+. Denote by $L^\gamma_{p,\omega}(R_+)$ the set of measurable functions $f(x)$, $x\in R_+$, with the finite norm

$$\|f\|_{L^\gamma_{p,\omega}(R_+)} = \left(\int_{R_+}|f(x)|^p\omega(x)x^\gamma dx\right)^{\frac{1}{p}},\quad 1\le p<\infty .$$

Definition 1. *The weight function* ω *belongs to the class* $A_p^\gamma(R_+)$ *for* $1<p<\infty$ *if*

$$\sup_{x,r\in R_+}|E_+(x,r)|_\gamma^{-1}\int_{E_+(x,r)}\omega(y)y^\gamma dy\left(|E_+(x,r)|_\gamma^{-1}\int_{E_+(x,r)}\omega^{-\frac{1}{p-1}}(y)y^\gamma dy\right)^{p-1}<\infty .$$

and ω *belongs to* $A_1^\gamma(R_+)$ *if there exists a positive constant C such that for any* $x\in R_+$ *and* $r>0$

$$|E_+(x,r)|_\gamma^{-1}\int_{E_+}\omega^{-\frac{1}{p-1}}(y)y^\gamma dy\le C\operatorname*{ess\,inf}_{y\in E_+(x,r)}\omega(y).$$

The properties of the class $A_1^\gamma(R_+)$ are analogous to those of the B.Muckenhoupt classes. In particular, if $\omega\in A_p^\gamma(R_+)$, then $\omega\in A_{p-\varepsilon}^\gamma(R_+)$ for a certain sufficiently small $\varepsilon>0$ and $\omega\in A_{p_1}^\gamma(R_+)$ for any $p_1>p$.

Note that, $x^\alpha\in A_p^\gamma(R_+)$, $1<p<\infty$, if and only if $-(1+\gamma)<\alpha<(1+\gamma)(p-1)$ and $x^\alpha\in A_1^\gamma(R_+)$, if and only if $-(1+\gamma)<\alpha\le 0$.

Theorem 2. *Let* $1<p<\dfrac{1+\gamma}{\alpha}$, $\dfrac{1}{q}=\dfrac{1}{p}-\dfrac{a}{1+\gamma}$. *Then the following two condition are equivalent:*

(i) There is a constant $C>0$ *such that for any* $f\in L^\gamma_{p,\omega}$ *(*R_+ *the inequality*

$$\left(\int_{R_+}\left(I_B^\alpha\left(f\omega^\alpha\right)(x)\right)^q\omega(x)x^\gamma dx\right)^{\frac{1}{q}}\le C\left(\int_{R_+}|f(x)|^p\omega(x)x^\gamma dx\right)^{\frac{1}{p}}$$

holds.

(ii) $\omega\in A^\gamma_{1+\frac{q}{p'}}(R_+)$, $p'=\dfrac{p}{p-1}$.

For Riesz potentials, Theorem 2 is due to B.Muckenhoupt and R.L.Wheeden [3]. The classes $A_p^\gamma(R_+)$ are the analogies of the well-known Muckenhoup weight classes.

In the sequel, we shall need the following weighted version of Hardy's inequality.

Theorem 3. *Let* $1\le p\le q\le\infty$ *and let* $u(t),v(t)$ *be positive functions on* $(0,\infty)$.

(i) For the validity of the inequality

$$\left(\int_0^\infty u(t)\left|\int_0^t \varphi(\tau)d\tau\right|^q dt\right)^{1/q} \le K_1\left(\int_0^\infty |\varphi(t)|^p v(t)dt\right)^{1/p}$$

with a constant K_1, *not depending on* φ, *it is necessary and sufficient that*

$$\sup_{t>0}\left(\int_t^\infty u(\tau)d\tau\right)^{p/q}\left(\int_0^t v(\tau)^{1-p'}d\tau\right)^{p-1} < \infty .$$

(ii) For the validity of the inequality

$$\left(\int_0^\infty u(t)\left|\int_t^\infty \varphi(\tau)d\tau\right|^q dt\right)^{1/q} \le K_2\left(\int_0^\infty |\varphi(t)|^p v(t)dt\right)^{1/p}$$

with a constant K_2, *not depending on* φ, *it is necessary and sufficient that*

$$\sup_{t>0}\left(\int_0^t u(\tau)d\tau\right)^{p/q}\left(\int_t^\infty v(\tau)^{1-p'}d\tau\right)^{p-1} < \infty .$$

We note that Theorem 3 was established by Muckenhoupt for $1 \le p = q \le \infty$ and Kokilashvili, Mazja for $p < q$ (see [4,5,6).

Theorem 4. *Let* $0 < \alpha < 1+\gamma$, $1 < p < \frac{1+\gamma}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{1+\gamma}$ *and* $\omega(t), \omega_1(t)$ *be the positive, increasing function on* $(0,\infty)$. *If for* (ω,ω_1) *the condition*

$$\sup_{t>0}\left(\int_t^\infty \omega_1(\tau)\tau^{-1-\frac{(1+\gamma)q}{p'}}d\tau\right)^{p/q}\left(\int_0^{t/2}\omega(\tau)^{1-p'}\tau^\gamma d\tau\right)^{p-1} < \infty \tag{1}$$

is fulfilled, then there exists a constant $c > 0$ *such that for an arbitrary* $f \in L^\gamma_{p,w}(R_+)$ *the inequality*

$$\left(\int_{R_+}\left|I_B^\alpha f(x)\right|^q \omega_1(x)x^\gamma dx\right)^{1/q} \le C\left(\int_{R_+}|f(x)|^p \omega(x)x^\gamma dx\right)^{1/p} \tag{2}$$

is valid.

Proof. Without restriction of generality we may assume that the function ω_1 has the form

$$\overline{\omega}_1(t) = \overline{\omega}_1(0) + \int_0^t \varphi(u)du ,$$

where $\overline{\omega}_1(0) = \lim_{t\to 0}\omega_1(t)$, and $\varphi(t) \ge 0$ on the interval $(0,\infty)$.

Note that the condition (1) implies the following relations:

$$\exists C_1 > 0,\ \forall t > 0,\ \ \omega_1(t)^{p/q} \le C_1\omega\left(\frac{t}{2}\right), \tag{3}$$

$$\exists C_2 > 0,\ \forall t > 0,\ \left(\int_t^\infty \varphi(\tau)\tau^{-\frac{(1+\gamma)q}{p'}}d\tau\right)^{p/q} \times$$

$$\times\left(\int_0^{t/2}\varphi(\tau)\tau^{1-p'}\tau^\gamma d\tau\right)^{p-1} \le C_2 . \tag{4}$$

The relation (3) follows from the fact that

$$\left(\int_t^\infty \omega_1(\tau)\tau^{-1-\frac{(1+\gamma)q}{p'}}d\tau\right)^{p/q} \geq C\omega_1(t)^{p/q}t^{-\frac{(1+\gamma)p}{p'}},$$

$$\left(\int_0^{t/2}\omega(\tau)\tau^{1-p'}\tau^\gamma d\tau\right)^{p-1} \leq C\omega\left(\frac{t}{2}\right)^{-1}t^{\frac{(1+\gamma)p}{p'}}.$$

and (4) follows from the inequalities

$$\int_t^\infty \varphi(\tau)\tau^{-\frac{(1+\gamma)q}{p'}}d\tau = \frac{(1+\gamma)q}{p'}\int_t^\infty \varphi(t)tdt\int_t^\infty \lambda^{-1-\frac{(1+\gamma)q}{p'}}d\lambda =$$

$$= \frac{(1+\gamma)q}{p'}\int_t^\infty \lambda^{-1-\frac{Qq}{p'}}d\lambda\int_t^\lambda \varphi(\tau)d\tau \leq \frac{(1+\gamma)q}{p'}\int_t^\infty \omega_1(\tau)\tau^{-1-\frac{(1+\gamma)q}{p'}}d\tau.$$

Clearly,

$$\left\|I_B^\alpha f\right\|_{L^\gamma_{q,\omega_1}(R_+)} \leq \left(\int_{R_+}\left|I_B^\alpha f(x)\right|^q x^\gamma dx\int_0^x \varphi(t)dt\right).$$

If $\omega(0+)>0$, then $L^\gamma_{p,\omega}(R_+)\subset L^\gamma_p(R_+)$, and if $\omega(0+)=0$, then it follows from $\overline{\omega}_1(t)\leq\omega_1(t)\leq\omega(t/2)^{2/p}$ that $\overline{\omega}_1=0$. Consequently, if $\omega(0+)=0$, then $A_2=0$. If $\omega(0+)>0$, then $f\in L^\gamma_p(R_+)$, and hence by Theorem 1 we have

$$A_2 \leq C\overline{\omega_1}(0)^{1/q}\left(\int_{R_+}|f(x)|^p x^\gamma dx\right)^{1/p} \leq C\left(\int_{R_+}|f(x)|^p \omega_1(2x)^{p/q}x^\gamma dx\right)^{1/p} \leq$$

$$\leq C\left(\int_{R_+}|f(x)|^p\omega(x)x^\gamma dx\right)^{1/p} = C\|f\|_{L^\gamma_{p,\omega}(R_+)}.$$

Let us now estimate A_1.

$$A_1 \leq \left(\int_0^\infty \varphi(t)dt\int_t^\infty \left|I_B^\alpha f(x)\right|^q x^\gamma dx\right)^{1/q} \leq A_{11}+A_{12},$$

where

$$A_{11} = \left(\int_0^t \varphi(t)dt\int_t^\infty\left|\int_{t/2}^\infty T^y x^{\alpha-1-\gamma}f(y)y^\gamma dy\right|^q x^\gamma dx\right)^{1/q},$$

$$A_{12} = \left(\int_0^t \varphi(t)dt\int_t^\infty\left|\int_0^{t/2} T^y x^{\alpha-1-\gamma}f(y)y^\gamma dy\right|^q x^\gamma dx\right)^{1/q}.$$

Further, it follows from the relation

$$\int_t^\infty |f(y)|y^\gamma dy \leq \frac{1}{\omega(t)}\int_t^\infty |f(y)|^p\omega(y)y^\gamma dy,$$

that $f\in L^\gamma_p(t,\infty)$ for any $t>0$.

By virtue of Theorem 1 and owing to the Minkowsky's inequality with the exponent $\frac{q}{p} \geq 1$, we have

$$A_{11} \leq C\left\{\int_0^\infty \varphi(t)dt\left(\int_{t/2}^\infty |f(x)|^p x^\gamma dx\right)^{q/p}\right\}^{1/q} \leq$$

$$\leq C\left\{\int_{R_+} |f(x)|^p \left(\int_0^{2x} \varphi(t)dt\right)^{p/q} x^\gamma dx\right\}^{1/p} \leq$$

$$\leq C\left\{\int_{R_1} |f(x)|^p (\omega_1(2x))^{p/q} x^\gamma dx\right\}^{1/p} \leq C\|f\|_{L^\gamma_{p,\omega}(R_1)}.$$

Estimate now A_{12}.

For $x > t,\ 0 < y < t/2$ we have the inequality $\frac{1}{2}x \leq |x-y| \leq x+y \leq \frac{3}{2}x$. Then

$$\int_t^\infty \left|\int_0^{t/2} T^y x^{\alpha-1-\gamma} f(y) y^\gamma dy\right|^q x^\gamma dx \leq \int_t^\infty x^{(\alpha-1-\gamma)q} x^\gamma dx \left(\int_0^{t/2} |f(y)| y^\gamma dy\right)^q =$$

$$= Ct^{1+\gamma-(\alpha-1-\gamma)q}\left(\int_0^{t/2} |f(y)| y^\gamma dy\right)^q = Ct^{-\frac{(1+\gamma)q}{p'}}\left(\int_0^{t/2} |f(y)| y^\gamma dy\right)^q.$$

Choosing $\beta > \frac{1+\gamma}{p'} + 1 + \gamma$ and using the Hölder inequality, we have

$$\int_0^{t/2} |f(y)| y^\gamma dy = \beta \int_0^{t/2} y^{\gamma-\beta} |f(y)| dy \int_0^y \tau^{\beta-1} d\tau =$$

$$= \beta \int_0^{t/2} \tau^{\beta-1} d\tau \int_\tau^{t/2} |f(y)| y^{-\beta} y^\gamma dy \leq$$

$$\leq \beta \int_0^{t/2} \tau^{\beta-1}\left(\int_\tau^{t/2} |f(y)|^p y^{-(1+\gamma)p} dy\right)^{1/p}\left(\int_\tau^{t/2} y^{(1+\gamma-\beta)p'} y^\gamma dy\right)^{1/p'} d\tau \leq$$

$$\leq C\int_0^{t/2} \tau^{\gamma+\frac{1+\gamma}{p'}}\left(\int_\tau^{t/2} |f(y)| y^{-(1+\gamma)p} y^\gamma dy\right)^{1/p} d\tau.$$

Consequently,

$$A_{12} < C\left\{\int_0^\infty \varphi(2t) t^{-\frac{(1+\gamma)q}{p'}}\left[\int_0^t \tau^{\gamma+\frac{1+\gamma}{p'}}\left(\int_\tau^\infty |f(y)| y^{-(1+\gamma)p} y^\gamma dy\right)^{1/p} d\tau\right]^q dt\right\}^{1/q}.$$

By (3) and theorem 1,

$$A_{12} \leq C\left\{\int_0^\infty \tau^{(1+\gamma)p\left(1+\frac{1}{p'}\right)-p}\left(\int_0^\tau |f(y)| y^{-(1+\gamma)p} y^\gamma dy\right)\omega(\tau)\tau^{-\gamma(p-1)} d\tau\right\}^{1/p} =$$

$$=C\left(\int_0^\infty \tau^{(1+\gamma)p-1}\omega(\tau)d\tau\int_0^\tau |f(y)|y^{-(1+\gamma)p}y^\gamma dy\right)^{1/p}=$$

$$=C\left(\int_{R_+}|f(y)|y^{-(1+\gamma)p}y^\gamma dy\int_0^y \tau^{(1+\gamma)p-1}\omega(\tau)d\tau\right)^{1/p}\leq$$

$$\leq C\left(\int_{R_+}|f(x)|^p\omega(x)x^\gamma dx\right)^{1/p}=C\|f\|_{L^\gamma_{p,\omega}}.$$

Combining the estimates for A_1 and A_2, we obtain (2) for $\omega_1=\omega_1$. By Fatou's theorem on passing to the limit under the integral sign, this gives (2).

Theorem 5. *Let* $0<\alpha<1+\gamma$, $1<p<\frac{1+\gamma}{\alpha}$, $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{1+\gamma}$ *and* $\omega(t),\omega_1(t)$ *be the positive, decreasing functions on* $(0,\infty)$. *If for* (ω,ω_1) *the condition*

$$\sup_{t>0}\left(\int_0^{t/2}\omega_1(\tau)\tau^{1+\gamma}d\tau\right)^{p/q}\left(\int_t^\infty \omega(\tau)^{1-p'}\tau^{-1-\frac{(1+\gamma)p'}{q}}d\tau\right)^{p-1}<\infty \qquad (5)$$

is fulfilled, then the inequality (2) holds.

In the case of positive decreasing functions ω and ω the proof is carried out along the same lines, with the decrease of the weight functions taken into account.

The sufficient conditions for general weights ensuring the validity of the two-weight strong inequality for the operator I_B^α is given in the following theorem.

Theorem 6. *Let* $0<\alpha<Q$, $1<p<\frac{Q}{\alpha}$, $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{Q}$ *and let* ω *and* ω *be the positive functions on* $(0,\infty)$ *satisfy the conditions.*

1) *there exists a constant* $b>0$ *such that for an arbitrary* $t>0$ *the inequality*

$$\left(\sup_{t<\tau\leq 8t}v(\tau)\right)^{\frac{p}{q}}\leq b\inf_{t<\tau\leq 8t}w(\tau)$$

holds;

2) $$\sup_{t>0}\left(\int_t^\infty \omega_1(\tau)\tau^{-1-\frac{(1+\gamma)q}{p'}}d\tau\right)^{p/q}\left(\int_0^t \omega(\tau)^{1-p'}\tau^\gamma d\tau\right)^{p-1}<\infty;$$

3) $$\sup_{t>0}\left(\int_0^t \omega_1(\tau)\tau^{1+\gamma}d\tau\right)^{p/q}\left(\int_t^\infty \omega(\tau)^{1-p'}\tau^{-1-\frac{(1+\gamma)p'}{q}}d\tau\right)^{p-1}<\infty.$$

Then the inequality (2) holds.

Proof. Represent the left-hand side of inequality (2) as follows:

$$\left(\int_{R_+}|I_B^\alpha f(x)|^q\omega_1(x)x^\gamma dx\right)^{1/q}=\left(\sum_{k\in Z}\int_{2^k}^{2^{k+1}}|I_B^\alpha f(x)|^q\omega_1(x)x^\gamma dx\right)^{1/q}\leq$$

$$\leq\left(\sum_{k\in Z}\int_{2^k}^{2^{k+1}}\left|I_B^\alpha\left(f\chi_{\{0<x\leq 2^{k-1}\}}\right)(x)\right|^q\omega_1(x)x^\gamma dx\right)^{1/q}+$$

$$+\left(\sum_{k\in Z}\int_{2^k}^{2^{k+1}}\left|I_B^\alpha\left(f\chi_{\{2^{k-1}<x\le 2^{k+2}\}}\right)(x)\right|^q\omega_1(x)x^\gamma dx\right)^{1/q}+$$

$$+\left(\sum_{k\in Z}\int_{2^k}^{2^{k+1}}\left|I_B^\alpha\left(f\chi_{\{x>2^{k+2}\}}\right)(x)\right|^q\omega_1(x)x^\gamma dx\right)^{1/q}=A_1+A_2+A_3.$$

Estimate A_1. For $2^k<x\le 2^{k+1},\ y\le 2^{k-1}$ the inequality are valid. Using Theorem 3, we obtain

$$A_1\le C\left(\sum_{k\in Z}\int_{2^k}^{2^{k+1}}\omega_1(x)^{(\alpha-1-\gamma)q}\left(\int_0^x|f(y)|y^\gamma dy\right)^q x^\gamma dx\right)^{1/q}=$$

$$=C\left(\int_{2^k}^{2^{k-1}}\omega_1(x)^{(\alpha-1-\gamma)q}\left(\int_0^x|f(y)|y^\gamma dy\right)^q x^\gamma dx\right)^{1/q}=$$

$$=C\left(\int_0^\infty\omega_1(x)x^{-1-\frac{(1+\gamma)q}{p'}}\left(\int_0^x|f(y)|y^\gamma dy\right)^q dx\right)^{1/q}\le C\left(\int_0^\infty|f(x)|^p\omega(x)x^\gamma dx\right)^{1/p}.$$

Let us now estimate A_3. For $2^k<x\le 2^{k+1},\ y\ge 2^{k+2}$ the inequalities $x\le y,\ |x-y|\ge\frac{1}{2}y$ are valid. Using again Theorem 3, we get

$$A_3\le C\left(\sum_{k\in Z}\int_{2^k}^{2^{k+1}}\omega_1(x)\left(\int_x^\infty T^y x^{\alpha-1-\gamma}|f(y)|y^\gamma dy\right)^q x^\gamma dx\right)^{1/q}\le$$

$$\le\left(\int_{R_+}\omega_1(x)x^\gamma\left(\int_x^\infty y^{\alpha-1}|f(y)|\right)^q dx\right)\le C\left(\int_{R_+}|f(x)|\omega(x)x^\gamma dx\right)^{1/p}.$$

Due to the strong type inequality (p,q) for the operator I_B^α we have

$$A_2\le C\left(\sum_{k\in Z}\left(\sup_{2^{k-1}<x\le 2^{k+2}}\omega_1(x)\right)\int_{R_+}\left|I_B^\alpha\left(f\chi_{\{2^{k-1}<x\le 2^{k+2}\}}\right)(x)\right|^q x^\gamma dx\right)^{1/q}\le$$

$$\le C\left(\sum_{k\in Z}\left(\inf_{2^{k-1}<x\le 2^{k+2}}\omega(x)\right)\left(\int_{R_+}\left|f\chi_{\{2^{k-1}<x\le 2^{k+2}\}}(x)\right|^p x^\gamma dx\right)^{q/p}\right)^{1/q}\le$$

$$\le C\left(\sum_{k\in Z}\left(\int_{2^{k-1}}^{2^{k+2}}|f(x)|^p\omega(x)x^\gamma dx\right)^{q/p}\right)^{1/q}\le C\left(\int_{R_+}|f(x)|^p\omega(x)x^\gamma dx\right)^{1/p}.$$

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