MATHEMATICS

ALIEV R.M., GASIMOVA S.G.

BOUNDARY VALUE PROBLEMS AND OPTIMAL QUADRATURE FORMULAE

Abstract

Optimal quadrature formulae of S.M.Nokolski type are structured on the set of solutions of some boundary value problem. The nodes and coefficients of these quadrature formulae and their precise error have been found.

Consider the ordinary differential equation

$$y^{(2r)}(x)-\lambda y(x)=f(x) \tag{1}$$

under boundary conditions

$$y^{(\alpha_i)}(0)=y^{(\sigma_i)}(1)=0 \quad (i=0,1,...,r-1), \tag{2}$$

where $f(x)$ is a continuous function on the segment $[0,1]$, $\alpha_i<\alpha_{i+1}$, $\sigma_i<\sigma_{i+1}$ $(i=0,1,...,r-2)$ and $\alpha_i,\sigma_i\in\{0,1,...,2r-1\}$ $(i=0,1,...,r-1)$. Let $\lambda=0$ be not an eigen value of corresponding homogeneous problem.

Consider the differential equation (1) as functional on the space $L_p(0,1)$ functions $y(x)$, continuous on the segment $[0,1]$ with their derivatives up to order $2r$ includingly and satisfying the conditions (2).

By a class $C^{(2r)}L_p(0,1)$ we denote a set of solutions $y(x)$ of the boundary value problem (1), (2) for which

$$\|y(x)\|_{L_p(0,1)}=\left(\int_0^1|y(x)|^p dx\right)^{\frac{1}{p}}\le M, \quad (1\le p\le\infty). \tag{3}$$

Let for the totality on the segment $[0,1]$ continuous functions $f(x)$

$$\|f\|_{L_p(0,1)}=\left(\int_0^1|f(x)|^p dx\right)^{\frac{1}{p}}\le m, \quad (1\le p\le\infty).$$

In the present paper we find an optimal quadrature formula of the form

$$\int_0^1 y(x)dx=\sum_{k=1}^{N}\sum_{l=0}^{2r-2}A_n^{(l)}y^{(l)}(x_k)+R_N(f), \tag{4}$$

where $0<x_1<...<x_N<1$, for the class $C^{(2r)}L_p(0,1)$ $(1\le p\le\infty)$.

Theorem. *Among the quadrature formulae of the form (4) the unique formula determined by the coefficients*

$$A_k^{(2j+1)}=0, \quad (j=0,1,...,r-2)$$

$$A_k^{(2j)}=\frac{2h^{j+1}R_{q2r}^{(2r-2j+1)}(1)}{(2r)!}, \quad (j=0,1,...,r-1),$$

$$(-1)^l A_1^{(l)}=A_N^{(l)}=\frac{h^{l+1}}{(2r)!}\left\{\frac{(2r)!}{(l+1)!}\left[R_{q2r}(1)\right]^{\frac{l+1}{2r}}+(-1)^l R_{q2r}^{(2r-l+1)}(1)\right\},$$

$$\left(k=1,\ldots,N;\ l=0,1,\ldots,2r-2\right)$$

and nodes

$$x_k=\left(2k+\sqrt[2q]{R_{q2r}(1)}\right)\omega_N$$

is optimal for the class $C^{(2r)}L_p(0,1)$.

And the residual $R_N(y)$ *under indicated values* $A_k^{(l)}$ *and* x_k *in the class* $C^{(2r)}L_p(0,1)$ *has the estimate*

$$E_N\left[C^{(2r)}L_p\right]=\inf_{A_k^{(l)},x_k}\sup_{y\in C^{(2r)}L_p}\left|R_N(y)\right|=\frac{\left(|\lambda|M+m\right)R_{q2r}(1)}{(2r)!\sqrt[q]{2rq+1}}\omega_N,$$

where $R_{q2r}(t)$ *is a polynomial of degree* $2r$, *with a leading coefficient equal to zero, leastly deviating from zero on the segment* $[-1,1]$ *in the metric* L_q $\left(\frac{1}{p}+\frac{1}{q}=1\right)$,

$$h=\omega_N=\left[2\left(N+\sqrt[2q]{R_{q2r}(1)}\right)\right]^{-1}.$$

Proof. Evidently, the equation $y^{(2r)}(x)=0$ under boundary conditions (2) has only a trivial solution $y(x)\equiv 0$. Let $G(x,t)$ be a Green function of the operator $\frac{d^{2r}}{dx^{2r}}$ with boundary conditions (2), and the boundary conditions (2) be self-adjoint. Then

$$y(x)=\int_0^1 G(x,t)y^{(2r)}(t)dt=\int_0^x G(x,t)y^{(2r)}(t)dt+$$

$$+\int_x^1 G(t,x)y^{(2r)}(t)dt=\int_0^1\left[G(x,t)E(t-x)+G(t,x)E(x-t)\right]y^{(2r)}(t)dt,$$

where $E(u)=\begin{cases}1 & for\quad u\ge 0,\\ 0 & for\quad u<0.\end{cases}$

Consequently, we can replace the boundary value problem (1), (2) by the equivalent integral equation

$$y(x)=\lambda\int_0^1\left[G(x,t)E(t-x)+G(t,x)E(x-t)\right]y(t)dt+\varphi(x), \qquad (5)$$

where

$$\varphi(x)=\int_0^1\left[G(x,t)E(t-x)+G(t,x)E(x-t)\right]f(t)dt.$$

We find from the formula (4) that

$$R_N(y)=\int_0^1 y(x)dx-\sum_{k=1}^{N}\sum_{l=0}^{2r-2}A_n^{(l)}y^{(l)}(x_k). \qquad (6)$$

Transform the expression (6) replacing $y(x)$ by the right hand side of the integral equation (5) [1] and get

$$R_N(y)=\int_0^1 K_{2r}(t)\left[\lambda y(t)+f(t)\right]dt, \qquad (7)$$

where

$$K_{2r}(t)=\int_0^t G(x,t)dx+\int_t^1 G(x,t)dx-$$
$$-\sum_{k=1}^{N}\sum_{l=0}^{2r-2}A_k^{(l)}\left[E(x_k-t)\frac{\partial^l G(t,x_k)}{\partial x^l}+E(t-x_k)\frac{\partial^l G(x_k,t)}{\partial x^l}\right]. \tag{8}$$

Applying Hölder equality to the integral (7) we get

$$|R_N(y)|\le\left(\int_0^1|K_{2r}(t)|^q dt\right)^{\frac{1}{q}}\left(\int_0^1|\lambda y(t)+f(t)|^p dt\right)^{\frac{1}{p}}\le$$
$$\le\left(\int_0^1|K_{2r}(t)|^q dt\right)^{\frac{1}{q}}\left[|\lambda|\left(\int_0^1|y(t)|^p dt\right)^{\frac{1}{p}}+\left(\int_0^1|f(t)|^p dt\right)^{\frac{1}{p}}\right]\le$$
$$\le[|\lambda|M+m]\left(\int_0^1|K_{2r}(t)|^q dt\right)^{\frac{1}{q}}. \tag{9}$$

Thus, the finding of optimal formula of the form (4) for the class $C^{(2r)}L_p(0,1)$ $(1\le p\le\infty)$ is reduced to the search of the least value integral

$$J_{2r}=\int_0^1|K_{2r}(t)|^q dt. \tag{10}$$

Study the function $K_{2r}(t)$ on segments

$$[0,x_1],\ [x_i,x_{i+1}]\quad (i=1,2,...,N-1),\quad [x_N,1].$$

One can easily note that Green's function $G(x,t)$ is a polynomial of degree not higher than $2r-1$ with respect to x or t for $x\le t$ and $x>t$.

Let $K_i(t)$ denote the function with which $K_{2r}(t)$ coincide on the segment $[x_i,x_{i+1}]\,(i=1,2,...,N-1)$.

It follows from the equality (8) that on the segment $[x_i,x_{i+1}]$ $(i=1,2,...,N-1)$ the function $K_{2r}(t)$ is a polynomial and it is determined by the equality

$$K_i(t)=\int_0^t G(x,t)dx+\int_t^1 G(t,x)dx-$$
$$-\sum_{k=i}^{N}\sum_{l=0}^{2r-2}A_k^{(l)}G_k^{(l)}(t,x_k)-\sum_{k=0}^{i-1}\sum_{l=0}^{2r-2}A_k^{(l)}G_k^{(l)}(x_k,t). \tag{11}$$

If we differentiate the formula (11) $2r$ times on t we get

$$K_i^{(2r)}(t)=\int_0^t\frac{\partial^{2r}G(x,t)}{\partial t^{2r}}dx+\left.\frac{\partial^{2r-1}G(x,t)}{\partial t^{2r-1}}\right|_{x=t-0}+$$
$$+\int_t^1\frac{\partial^{2r}G(t,x)}{\partial t^{2r}}dx-\left.\frac{\partial^{2r-1}G(t,x)}{\partial t^{2r-1}}\right|_{x=t+0}-\frac{\partial^{2r}}{\partial t^{2r}}\left[\sum_{k=i}^{N}\sum_{l=0}^{2r-2}A_k^{(l)}G_x^{(l)}(t,x_k)+\right. \tag{12}$$
$$\left.+\sum_{k=0}^{i-1}\sum_{l=0}^{2r-2}A_k^{(l)}G_x^{(l)}(x_k,t)\right]=\left.\frac{\partial^{2r-1}G(x,t)}{\partial t^{2r-1}}\right|_{x=t-0}-\left.\frac{\partial^{2r-1}G(t,x)}{\partial t^{2r-1}}\right|_{x=t+0}=1\ .$$

Hence, it follows that $K_i(t)$ $(i=1,2,...,N-1)$ is a polynomial of degree $2r$ with a leading coefficient that is equal to $\frac{1}{2r}$.

Consequently on the basis of (8) and (12) we can deduce that $K_i(t)$ on the segment $[x_i, x_{i+1}]$ $(i=1,2,..,N-1)$ has the following form

$$K_i(t)=\frac{t^{2r}}{(2r)!}+\sum_{j=0}^{2r-1}c_j t^j,$$

where c_j is a constant number.

Now study the function $K_{2r}(t)$ on the segment $[0,x_0]$. For $t\in[0,x_0]$ we find from the formula (7) that

$$K_{2r}(t)=\int_0^t G(x,t)dx+\int_t^1 G(t,x)dx-\sum_{k=1}^{N}\sum_{l=0}^{2r-2}A_k^{(l)}G_x^{(l)}(t,x_k). \tag{13}$$

By virtue of the fact that optimal quadrature formula of the form (3) is precise for the polynomials of the degree no higher than $2r-1$, then

$$\int_0^1 G(t,x)dx=\int_0^t G(t,x)dx+\int_t^1 G(t,x)dx=\sum_{k=1}^{N}\sum_{l=0}^{2r-2}A_k^{(l)}G_x^{(l)}(t,x_k). \tag{14}$$

It follows from the expressions (13) and (14) that

$$K_{2r}(t)=\int_0^t G(x,t)dx+\int_t^1 G(t,x)dx-\int_0^t G(t,x)dx-\int_t^1 G(t,x)dx=$$

$$=\int_0^t G(x,t)dx-\int_0^t G(t,x)dx=\int_0^t[G(x,t)-G(t,x)]dx. \tag{15}$$

Expanding the difference $G(x,t)-G(t,x)$ in Tayler polynomial at the point $x=t$, and taking into account the continuity of $G_x^{(j)}(x,t)$ $(j=0,1,2,...,2r-l)$ we get

$$G(x,t)-G(t,x)=\sum_{j=0}^{2r-2}\frac{G_x^{(j)}(x,t)\big|_{t=x-0}-G_x^{(j)}(t,x)\big|_{t=x+0}}{j!}(t-x)^j+$$

$$+\frac{G_x^{(2r-1)}(x,x-0)-G_x^{(2r-1)}(x+0,x)}{(2r-1)!}(t-x)^{2r-1}=\frac{1}{(2r-1)!}(t-x)^{2r-1}. \tag{16}$$

Considering the formula (16), from the formula (15) for $t\in[0,x_0]$ we find

$$K_{2r}(t)=\frac{1}{(2r-1)!}\int_0^t (t-x)^{2r-1}dx=\frac{t^{2r}}{(2r)!}.$$

We prove analogously that on the segment $[x_N,1]$ the function $K_{2r}(t)$ equals to $\frac{(t-1)^{2r}}{(2r)!}$.

Thus

$$K_{2r}(t)=\begin{cases}\frac{t^{2r}}{(2r)!} & for \quad t\in[0,x_0],\\ \frac{t^{2r}}{(2r)!}+\sum_{j=0}^{2r-1}c_j t^j & for \quad t\in[x_i,x_{i+1}] \ (i=1,2,...,N),\\ \frac{(t-1)^{2r}}{(2r)!} & for \quad t\in[x_N,1]\end{cases} \tag{17}$$

Consequently, the finding of the optimal quadrature formula of the form (4) was reduced to the finding of the least value of the integral (10) under the condition (17).

One can easily note that the function $K_{2r}(t)$ on the segments $[x_i, x_{i+1}]$ $(i = 1,2,...,N-1)$ coincides with a polynomial that leastly deviates from zero on the metric L_q.

To this aid, if we integrate the integral (10) with respect to $A_i^{(l)}$ and x_i and equal them to zero we get

$$\frac{\partial J_{2r}}{\partial A_i^{(l)}} = q\int_0^1 |K_{2r}(t)|^{q-1} signK_{2r}(t)\left[E(x_i - t)\frac{\partial^l G(t,x_i)}{\partial x^l} + E(t - x_i)\frac{\partial^l G(x_i,t)}{\partial x^l}\right]dt = 0, \quad (18)$$

$$\frac{\partial J_{2r}}{\partial x_i} = q\int_0^1 |K_{2r}(t)|^{q-1} signK_{2r}(t)\sum_{l=0}^{2r-2} A_i^{(l)}\left[E(x_i - t)\frac{\partial^{l+1} G(t,x_i)}{\partial x^{l+1}} + E(t - x_i)\frac{\partial^{l+1} G(x_i,t)}{\partial x^{l+1}}\right]dt = 0. \quad (19)$$

Taking into account the equality (18), it follows from the equality (19) that

$$\frac{\partial J_{2r}}{\partial x_i} = q\int_0^1 |K_{2r}(t)|^{q-1} signK_{2r}(t)\left[E(x_i - t)\frac{\partial^{2r-1} G(t,x_i)}{\partial x^{2r-1}} + E(t - x_i)\frac{\partial^{2r-1} G(x_i,t)}{\partial x^{2r-1}}\right]dt = 0. \quad (20)$$

The relations between (18) and (20) we can rewrite in the form

$$\frac{\partial J_{2r}}{\partial A_i^{(l)}} = \int_0^{x_i} |K_{2r}(t)|^{q-1} signK_{2r}(t)\frac{\partial^l G(t,x_i)}{\partial x^l}dt + \int_{x_i}^1 |K_{2r}(t)|^{q-1} signK_{2r}(t)\frac{\partial^l G(x_i,t)}{\partial x^l}dt = 0, \quad (21)$$

$$\frac{\partial J_{2r}}{\partial x_i} = \int_0^{x_i} |K_{2r}(t)|^{q-1} signK_{2r}(t)\frac{\partial^{2r-1} G(t,x_i)}{\partial x^{2r-1}}dt + \int_{x_i}^1 |K_{2r}(t)|^{q-1} signK_{2r}(t)\frac{\partial^{2r-1} G(x_i,t)}{\partial x^{2r-1}}dt = 0. \quad (22)$$

By virtue of the equality (16) we'll have from the relations (21) and (22)

$$\frac{\partial J_{2r}}{\partial A_i^{(l)}} = \int_0^{x_i} |K_{2r}(t)|^{q-1} signK_{2r}(t)\left[\frac{\partial^l G(t,x_i)}{\partial x^l} - \frac{(-1)^l (t - x_i)^{2r-l-1}}{(2r-l-1)!}\right]dt + \int_{x_i}^1 |K_{2r}(t)|^{q-1} signK_{2r}(t)\frac{\partial^l G(x_i,t)}{\partial x^l}dt = 0, \quad (23)$$

$$\frac{\partial J_{2r}}{\partial x_i} = \int_0^{x_i} |K_{2r}(t)|^{q-1} signK_{2r}(t)\left[\frac{\partial^{2r-1} G(x_i,t)}{\partial x^{2r-1}} + 1\right]dt + \int_{x_i}^1 |K_{2r}(t)|^{q-1} signK_{2r}(t)\frac{\partial^{2r-1} G(x_i,t)}{\partial x^{2r-1}}dt = 0. \quad (24)$$

We get from the relations (23) and (24)

$$\frac{(-1)^{l+1}}{(2r-l-1)!}\int_0^{x_i} |K_{2r}(t)|^{q-1} signK_{2r}(t)(t - x_i)^{2r-l-1}dt =$$

$$= -\int_0^1 |K_{2r}(t)|^{q-1} signK_{2r}(t) \frac{\partial^l G(x_i,t)}{\partial x^l} dt, \qquad (25)$$

$$(l = 0,1,...,2r-2, \ i = 1,2,...,N),$$

$$\int_0^{x_i} |K_{2r}(t)|^{q-1} signK_{2r}(t) dt = -\int_0^1 |K_{2r}(t)|^{q-1} signK_{2r}(t) \frac{\partial^{2r-1} G(x_i,t)}{\partial x^{2r-1}} dt, \quad (i = 1,2,...,N). \qquad (26)$$

It follows from the equality (26) that the integral

$$\int_0^{x_i} |K_{2r}(t)|^{q-1} signK_{2r}(t) dt$$

for all the values $x_i \, (i = 1,2,...,N)$ equal to the constant number.

Consequently,

$$\int_0^{x_i} |K_{2r}(t)|^{q-1} signK_{2r}(t) dt = 0.$$

Then it follows from the relation (24) that

$$\int_{x_i}^1 |K_{2r}(t)|^{q-1} signK_{2r}(t) dt = 0. \qquad (27)$$

Now taking into account the relation (27) we find from the relation (25)

$$\int_0^{x_i} |K_{2r}(t)|^{q-1} signK_{2r}(t) t^j dt = \int_{x_i}^1 |K_{2r}(t)|^{q-1} signK_{2r}(t) t^j dt \qquad (28)$$

$$(j = 0,1,...,2r-2, \ i = 1,2,...,N).$$

If in the equality we replace x_i by x_{i-1}, we get

$$\int_0^{x_{i-1}} |K_{2r}(t)|^{q-1} signK_{2r}(t) t^j dt = \int_{x_{i-1}}^1 |K_{2r}(t)|^{q-1} signK_{2r}(t) t^j dt. \qquad (29)$$

Further, subtracting term by term (29) from (28), we find

$$\int_{x_{i-1}}^{x_i} |K_{2r}(t)|^{q-1} signK_{2r}(t) t^j dt = -\int_{x_{i-1}}^{x_i} |K_{2r}(t)|^{q-1} signK_{2r}(t) t^j dt. \qquad (30)$$

Hence

$$\int_{x_{i-1}}^{x_i} |K_{2r}(t)|^{q-1} signK_{2r}(t) t^j dt = 0 \qquad (31)$$

$$(j = 0,1,...,2r-2, \ i = 1,2,...,N).$$

Consequently the function $K_{2r}(t)$ on the segments $[x_{i-1}, x_i]$ $(i = 1,2,...,N)$ coincides with a polynomial that leastly deviates from zero on the metric L_q ([3], p.3). Then a polynomial of degree $2r$ with leading coefficient for t^{2r}, equaling to $1/2r$ leastly deviating from zero on the segment $[x_i, x_{i+1}]$ $(i = 1,2,...,N)$ in the metric L_q has the form

$$P_i(t) = \frac{h_i^{2r}}{(2r)!} R_{q2r}\left(\frac{t - t_i}{h_i}\right), \quad where \quad h_i = \frac{x_i - x_{i-1}}{2}, \ t_i = \frac{x_i + x_{i-1}}{2},$$

$R_{q2r}(t)$ is an algebraic polynomial of degree $2r$ with a coefficient that equals to zero for t^{2r} leastly deviating from zero on the segment $[-1,1]$, in metric L_q. Since $K_{2r}(t)$ is a continuous function on segments $[x_{i-1}, x_i]$ $(i = 1,2,...,N)$, the polynomials $P_i(t)$ and

$$P_{i+1}(t)=\frac{h_{i+1}}{(2r)!}R_{q2r}\left(\frac{t-t_{i+1}}{h_i}\right)$$

have to coincide at the point $x_i=t_i+h_i=t_{i+1}-h_{i+1}$, i.e. $P_i(x_i)=P_{i+1}(x_i)$.

Hence we conclude that $h_i=h_{i+1}=h \quad (i=1,2,..,N)$.

Then

$$P_1(x_1)=\frac{h^{2r}}{(2r)!}R_{q2r}(1)\ , \tag{32}$$

$$P_N(x_N)=\frac{h^{2r}}{(2r)!}R_{q2r}(1)\ . \tag{33}$$

On the other hand

$$P_1(x_1)=K_{2r}(x_1)=\frac{x_1^{2r}}{(2r)!}, \tag{34}$$

$$P_N(x_N)=K_{2r}(x_N)=\frac{(1-x_N)^{2r}}{(2r)!}\ . \tag{35}$$

Comparing the relations (32) and (34) and also the relations (33) and (35), we find

$$x_1=1-x_{N-1},\ h=\frac{x_1}{\sqrt[2r]{R_{q2r}(1)}}\ . \tag{36}$$

Taking into account (36), we find from the equality $x_N=x_1+N2h$

$$x_1=\frac{\sqrt[2r]{R_{q2r}(1)}}{2\left(N+\sqrt[2r]{R_{q2r}(1)}\right)}. \tag{37}$$

Then

$$h=\frac{1}{2\left(N+\sqrt[2r]{R_{q2r}(1)}\right)}. \tag{38}$$

Substituting into the equality $x_i=x_i+2ih$ instead of x_i and h the right hand sides of the equality (37)-(38) we get

$$x_i=\left(2i+\sqrt[2r]{R_{q2r}(1)}\right)\omega_N\ , \qquad (i=1,2,...,N), \tag{39}$$

where $\omega_N=h$.

Then we determine

$$E_N\left[C^{(2r)}L_p\right]=\inf_{x_k,A_i^{(i)}}\ \sup_{y\in C^{(2r)}L_p}|R_N(y)|=$$

$$=(|\lambda|M+m)\left(\int_0^{x_1}|K_{2r}(t)|^q\,dt+\sum_{i=1}^{N-1}\int_{x_i}^{x_{i+1}}|K_{2r}(t)|^q\,dt+\int_{x_N}^{1}|K_{2r}(t)|^q\,dt\right)^{1/q}. \tag{40}$$

According to the relation (17) at each of the segments $[x_{i-1},x_i]\,(i=1,2,...,N)$ the function $K_{2r}(t)$ is a polynomial of the order $2r$ with a higher term $t^{2r}/(2r)!$. Then by substituting in the right hand side of the equality (40) the integrand $K_{2r}(t)$ with a polynomial of the degree $2r$ with a higher $\frac{1}{(2r)!}t^{2r}$ term leastly deviating from zero on the segment $[x_{i-1},x_i]\,(i=1,2,...,N)$ in the metric L_q and taking into account the equality ([2], p.139)

$$\int_{-1}^{1}\left|R_{q2r}(t)\right|^{q-1}dt=\frac{2\left[R_{2rq}(1)\right]}{2rq+1},$$

we get

$$E_N\left[C^{(2r)}L_p\right]=\frac{\left(|\lambda|M+m\right)R_{2rq}(1)}{(2r)!\sqrt[2r]{2rq+1}}\omega_N^{2r}.$$

The coefficients $A_k^{(i)}$ that correspond to the minimum of the integral $\int_0^1\left|K_{2r}(t)\right|^q dt$ are determined by S.M.Nikolskii's scheme [2].

References

[1]. Алиев Р.М. *Оптимальные квадратурные формулы на множествах решений краевых задач.* Диф.урав., 1989, т.25, №7, с.1161-1171.
[2]. Никольский С.М. *Квадратурные формулы.* М., Наука, 1988, 227 с.
[3]. Тиман А.Ф. *Теория приближения функций действительного переменного.* М., Физматгиз, 1960, 624 с.

Aliev R.M., Gasimova S.G.
Azerbaijan University of Architecture and Construction.
5, A. A.Sultanova str., 370073, Baku, Azerbaijan.

Received November 5, 2000; Revised February 22, 2001.
Translated by Mirzoyeva K.S.