GULIEV V.S., MUSTAFAEV R.Ch.

TWO-WEIGHT INEQUALITIES FOR POTENTIALS DEFINED ON SPACES OF HOMOGENEOUS TYPE

Abstract

The sufficient conditions for pairs of monotone weights ensuring the validity of two-weight inequalities for the integral of the fractional order on homogeneous spaces, are found. In some cases these conditions are as well necessary for the corresponding inequalities to be fulfilled.

Several different definitions of homogeneous spaces can be found in the literature. Here we refer particularly to [2].

Definition 1. *Let X be a non empty set. A function $d: X \times X \to [0,\infty]$ is called quasidistance if it satisfies following conditions:*

i) for every x and y in X $d(x,y)=0$ if and only if $x=y$,

ii) for every x and y in X $d(x,y)=d(y,x)$,

iii) there exists a positive constant C_0 such that for every $x,y,z \in X$

$$d(x,y) \le C_0[d(x,z)+d(z,y)].$$

The non empty set x equipped with the quasi-distance d is a space of homogeneous type (X,d,μ) if there exists a positive measure μ defined on a σ-algebra of subsets of X which contains the open set of X and the balls $B_r(x)$ with the following property: there exists a positive constant A such that for every x in X and every $r>0$

$$0<\mu(B_{2r}(x)) \le A\mu(B_r(x)).$$

Definition 2. *Almost everywhere positive, locally summable function $\omega: X \to R$ is called the weight.*

Definition 3. *The function $g: X \to R_+^1$ is said to be radial if $g(x)=g(d(x,a))$.*

Denote by $L_{p,\omega}(X,\mu)$ the space of functions $f(x)$, $x \in X$ for which

$$\|f\|_{L_{p,\omega}(X,\mu)} = \left(\int_X |f(x)|^p \omega(x) d\mu \right)^{1/p} < \infty, \quad 1 \le p \le \infty.$$

Let a be an arbitrary fixed point from X; $d=\sup\{d(x,a): x \in X\}$.

The function

$$T_\gamma f(x) = \int_X (d(x,y))^{\gamma-1} f(y) d\mu, \quad 0<\gamma<1$$

is called the integral of the fractional order on X (see, e.g., [7]).

Theorem 1. (see [7]) *Let $1<p<\gamma^{-1}$, $q^{-1}=p^{-1}-\gamma$. The following two conditions are equivalent:*

(1) T_γ maps continuously $L_p(X,\mu)$ into $L_q(X,\mu)$.

(2) There exists a constant $C>0$ such that $\mu B(x,r) \le Cr$ for any $x \in X$ and $r>0$.

In what follows it will be assumed that

$$\exists C>0,\ \forall x \in X,\ r>0,\ \mu B(x,r) \le Cr. \qquad (*)$$

Further we will need the following weighted variants of Hardy-Littlewood theorem.

Theorem 2 (see [1,8,9]). *Suppose* $1 \le p \le q < \infty$, $p' = p/(p-1)$ *and that* $u(t)$ *and* $v(t)$ *are positive measurable functions on* $(0,d)$.

1) The inequality

$$\left(\int_0^d u(t) \left| \int_0^t \varphi(\tau) d\tau \right|^q dt \right)^{1/q} \le C \left(\int_0^d |\varphi(t)|^p v(t) dt \right)^{1/p} \qquad (1)$$

holds with C *independent of* φ *if and only if the weights* u, v *satisfy the conditions*

$$\sup_{0<t<d} \left(\int_t^d u(\tau) d\tau \right)^{p/q} \left(\int_0^t v(\tau)^{1-p'} d\tau \right)^{p-1} < \infty .$$

2) The inequality

$$\left(\int_0^d u(t) \left| \int_t^d \varphi(\tau) d\tau \right|^q dt \right)^{1/q} \le C \left(\int_0^d |\varphi(t)|^p v(t) dt \right)^{1/p} \qquad (2)$$

holds with C *independent of* φ *if and only if the weights* u, v *satisfy the condition*

$$\sup_{0<t<d} \left(\int_0^t u(\tau) d\tau \right)^{p/q} \left(\int_t^d v(\tau)^{1-p'} d\tau \right)^{p-1} < \infty .$$

We consider monotone radial weights ω, ω_1. Note that, in our case, ω, ω_1 vanish nowhere except for, maybe, at 0 and d (if $d = \infty$).

As far as we study two weighted inequalities, we deal with the pair (ω, ω_1).

Formulate requirements on the pair (ω, ω_1).

Definition 4. *Let* $\omega(t)$, $\omega_1(t)$ *be a positive functions on* $(0,d)$, $1 < p \le q < \infty$.

The set of pair of weights (ω, ω_1), satisfying conditions

$$\sup_{0<t<d} \left(\int_t^d \omega_1(\tau) \tau^{-1-q/p'} d\tau \right)^{p/q} \left(\int_0^t \omega(\tau)^{1-p'} d\tau \right)^{p-1} < \infty , \qquad (3)$$

$$\sup_{0<t<d} \left(\int_0^t \omega_1(\tau) d\tau \right)^{p/q} \left(\int_t^d \omega(\tau)^{1-p'} \tau^{-1-p'/q} d\tau \right)^{p-1} < \infty , \qquad (4)$$

$$d < \infty \Rightarrow \omega_1(d-0) < \infty,\ \omega(d-0) < \infty$$

is denoted by S_{pq}^{-}.

X-radial weights $(\omega(d(x,a)), \omega_1(d(x,a)))$, constructed by $(\omega, \omega_1) \in S_{pq}^{-}$ are denoted by $S_{pq}^{-}(X)$.

Let us emphasize following feature of weighted pair $(\omega(t), \omega_1(t)) \in S_{pq}^{-}$, at the monotone case.

Lemma 1. *Let* $\omega(t)$, $\omega_1(t)$ *be monotone positive functions on* $(0,d)$, $1 < p \le q < \infty$. *If* $\omega(t)$, $0 < t < d$ *is nondecreasing function, then the condition (3) implies (4), and if* $\omega(t)$, $0 < t < d$ *is nonincreasing function, then (4) follows (3).*

Proof. Let $\omega(t), \omega_1(t)$ be positive nondecreasing functions on $(0,d)$, and (3) is valid. Then the following relation is valid:

$$\exists C>0,\quad \forall t>0,\ \omega_1(t)^{p/q}\le C\omega(t/2). \tag{5}$$

Actually

$$\left(\int_t^d \omega_1(\tau)\tau^{-1-q/p'}d\tau\right)^{p/q}\ge C\omega_1(t)^{p/q}\left(t^{-q/p'}-d^{-q/p'}\right)^{p/q},$$

and for $0<t\le d/2$

$$\left(\int_t^d \omega_1(\tau)\tau^{-1-q/p'}d\tau\right)^{p/q}\ge C\omega_1(t)^{p/q}t^{1-p}.$$

And

$$\left(\int_0^t \omega(\tau)^{1-p'}d\tau\right)^{p-1}\ge\left(\int_0^{t/2}\omega(\tau)^{1-p'}d\tau\right)^{p-1}\ge C\omega(t/2)^{-1}t^{p-1}.$$

Multiplying these inequalities and using the condition (3), we get (5) for $t\le d/2$. For $d>t>d/2$ we have

$$\omega_1^{p/q}(t)\le\omega_1^{p/q}(d-0)\le\frac{\omega_1^{p/q}(d-0)}{\omega_1^{p/q}(d/2)}C\omega(d/4)\le C\frac{\omega_1^{p/q}(d-0)}{\omega_1^{p/q}(d/2)}\omega(t/2).$$

So we obtain $\omega_1(t)^{p/q}\le C\omega(t/2)$ with C independent of $t\in(0,d)$.

On the other hand, we have

$$K_2=\sup_{0<t<d}\left(\int_0^t\omega_1(\tau)d\tau\right)^{p/q}\left(\int_t^d\omega(\tau)^{1-p'}\tau^{-1-p'/q}d\tau\right)^{p-1}\le$$

$$\le C\sup_{0<t<d}\omega_1(t)^{p/q}t^{p/q}\omega(t)^{-1}\left(t^{-p'/q}-d^{-p'/q}\right)^{p-1}=$$

$$=C\sup_{0<t<d}\frac{\omega_1(t)^{p/q}}{\omega(t)}\left(1-\left(\frac{t}{d}\right)^{p'/q}\right)^{p-1}\le C\sup_{0<t<d}\frac{\omega_1(t)^{p/q}}{\omega(t)}.$$

Since $\omega_1(t)^{p/q}\le C\omega(t)$, we have $K_2<\infty$.

Let $\omega(t)$ be positive nondecreasing, $\omega_1(t)$ positive nonincreasing functions on $(0,d)$, and (3) is valid. Then (5) is true.

Actually

$$\infty>K_1=\sup_{0<t<d}\left(\int_t^d\omega_1(\tau)\tau^{-1-q/p'}d\tau\right)^{p/q}\left(\int_0^t\omega(\tau)^{1-p'}d\tau\right)^{p-1}\ge$$

$$\ge\sup_{0<t<d/2}\left(\int_t^{2t}\omega_1(\tau)\tau^{-1-q/p'}d\tau\right)^{p/q}\left(\int_0^t\omega(\tau)^{1-p'}d\tau\right)^{p-1}\ge$$

$$\ge C\sup_{0<t<d/2}\omega_1(2t)^{p/q}t^{-p/p'}\omega(t)^{-1}t^{p-1}=C\sup_{0<t<d/2}\frac{\omega_1(2t)^{p/q}}{\omega(t)}.$$

Thus $\exists C>0,\ \forall t:0<t<d,\ \omega_1(t)^{p/q}\le\omega(t/2)\le C\omega(t)$.

Taking into account (5), we have

$$K_2=\sup_{0<t<d}\left(\int_0^t\omega_1(\tau)d\tau\right)^{p/q}\left(\int_t^d\omega(\tau)^{1-p'}\tau^{-1-p'/q}d\tau\right)^{p-1}\le$$

$$\geq C\sup_{0<t<d}\left(\int_0^t \omega^{q/p}(\tau)d\tau\right)^{p/q}\left(\int_t^d \omega(\tau)^{1-p'}\tau^{-1-p'/q}d\tau\right)^{p-1}\leq$$

$$\leq C\sup_{0<t<d}\omega(t)t^{p/q}\omega(t)^{-1}\left(t^{-p'/q}\right)^{p-1}=C<\infty .$$

The fact, that for nonincreasing $\omega(t), 0<t<d$ (4) follows (3), are proved analogously. ■

From the proof of lemma 1 immediately follows

Lemma 2. *Let $\omega(t), \omega_1(t)$ be monotone positive functions on $(0,d), 1<p\leq q<\infty$. If $\omega(t), 0<t<d$ is nondecreasing function, then the condition (3) implies*

$$\exists C>0,\ \forall 0<t<d,\ \omega_1(t)^{p/q}\leq C\omega(t/2).$$

Lemma 3. *Let $\omega(t), \omega_1(t)$ be monotone positive functions on $(0,d), 1<p\leq q<\infty$. If $\omega(t), 0<t<d$ is nonincreasing function, then the condition (4) implies*

$$\exists C>0,\ \forall 0<t<d,\ \omega_1(t/2)^{p/q}\leq C\omega(t).$$

The lemma below is proved analogously.

Lemma 4. *Let $\omega(t), \omega_1(t)$ be nondecreasing positive functions on $(0,d), 1<p\leq q<\infty$. Then the condition (3) implies*

$$\exists C>0,\ \forall 0<t<\frac{d}{2C_0},\ \omega_1(2C_0t)^{p/q}\leq C\omega(t).$$

The following theorem is true

Theorem 3. *Let μ satisfies the condition (*) $0<\gamma<1$, $1<p<1/\gamma$, $1/p-1/q=\gamma$ and $\omega(t), \omega_1(t)$ be monotone functions.*

If $(\omega,\omega_1)\in S_{pq}^{-}(X)$, then the inequality

$$\left(\int_X |T_\gamma f(x)|^q\omega_1(d(x,a))d\mu(x)\right)^{1/q}\leq C\left(\int_X |f(x)|^p\omega(d(x,a))d\mu(x)\right)^{1/p} \qquad (6)$$

holds for $f\in L_{p,\omega(d(x,a))}(X,\mu)$.

Proof. Let $f\in L_{p,\omega(d(x,a))}(X,\mu)$.

At first consider the case: $\omega(t), \omega_1(t)$ are positive increasing functions on $(0,d)$.

In order to prove that $T_\gamma f(x)$ exists for almost all $x\in X$ we take a arbitrary fixed $\tau>0$ and represent f as the sum f_1+f_2, where

$$f_1(x)=\begin{cases} f(x), & \text{if} \quad d(x,a)>\tau \\ 0, & \text{if} \quad d(x,a)\leq\tau \end{cases},\quad f_2(x)=f(x)-f_1(x) .$$

Since ω is positive increasing function on $(0,d)$ then $f_1\in L_p(X,\mu)$. Therefore, by theorem 1 $T_\gamma f_1(x)$ absolutely converges for almost all $x\in X$. Show that $T_\gamma f_2(x)$ is finite for any $x: d(x,a)\geq 2C_0\tau$.

Taking into account that $d(x,a)\geq C_0\tau$, $d(y,a)\leq\tau$ implies $d(x,y)\geq 1/C_0 d(x,a)-$ $-d(y,a)\geq\tau$ we have

$$\left|T_\gamma f_2(x)\right| \le \int_{d(y,a)\le\tau} (d(x,y))^{\gamma-1}|f(y)|d\mu \le \tau^{\gamma-1}\int_{d(y,a)\le\tau}\frac{|f(y)|\omega(d(y,a))^{1/p}}{\omega(d(y,a))^{1/p}}d\mu .$$

Applying Hölder inequality we obtain

$$\left|T_\gamma f_2(x)\right| \le C\tau^{\gamma-1}\|f\|_{L_{p,\omega(d(x,a))}(X,\mu)}\left(\int_{d(y,a)\le\tau}\omega(d(y,a))^{1-p'}d\mu\right)^{1/p'} .$$

Then

$$\int_{d(y,a)\le\tau}\omega(d(y,a))^{1-p'}d\mu = \sum_{n=0}^{\infty}\int_{B(a,2^{-n}\tau)\setminus B(a,2^{-n-1}\tau)}\omega(d(y,a))^{1-p'}d\mu \le$$

$$\le \sum_{n=0}^{\infty}\omega\left(2^{-n-1}\tau\right)^{1-p'}\int_{B(a,2^{-n}\tau)\setminus B(a,2^{-n-1}\tau)}d\mu \le C\sum_{n=0}^{\infty}\omega\left(2^{-n-1}\tau\right)^{1-p'}2^{-n-1}\tau \le C\int_0^{\tau}\omega(t)^{1-p'}dt .$$

Last integral, by assumption of the theorem, converges, hence $T_\gamma f_2(x)$ exists for almost any $x\in X\setminus B(a,\tau)$. Since $X\setminus\{a\}=\bigcup_{\tau>0}(X\setminus B(a,\tau))$ then the existence of $T_\gamma f(x)$ for almost any $x\in X$ follows from these reasonings.

Let $\overline{\omega}_1$ be an arbitrary continuous increasing function on $(0,d)$ such that $\overline{\omega}_1(t)\le\omega_1(t)$, $\overline{\omega}_1(0)=\omega_1(0+)$ and $\overline{\omega}_1(t)=\int_0^t\varphi(\tau)d\tau+\overline{\omega}_1(0)$, $t\in(0,d)$ (the existence of such $\overline{\omega}_1$ is obvious, for example, $\overline{\omega}_1(t)=\int_0^t\omega_1'(\tau)d\tau+\omega_1(0+)\le\omega_1(t)$).

Note that the following relation are valid:

$$\exists C>0,\ \forall t>0,\ \left(\int_t^d\varphi(\tau)\tau^{-q/p'}d\tau\right)^{p/q}\left(\int_0^t\omega(\tau)^{1-p'}d\tau\right)^{p-1}\le C . \qquad (7)$$

The inequality (7) follows from the inequalities

$$\int_t^d\varphi(\tau)\tau^{-q/p'}d\tau=\frac{q}{p'}\int_t^d\varphi(\tau)d\tau\int_\tau^d\lambda^{-1-q/p'}d\lambda+d^{-q/p'}\int_t^d\varphi(\tau)d\tau\le$$

$$\le\frac{q}{p'}\int_t^d\lambda^{-1-q/p'}d\lambda\int_t^\lambda\varphi(\tau)d\tau+d^{-q/p'}\omega_1(d-0)\le$$

$$\le\frac{q}{p'}\int_t^d\lambda^{-1-q/p'}\omega_1(\lambda)d\lambda+d^{-q/p'}\omega_1(d-0),$$

and

$$\int_0^t\omega(\tau)^{1-p'}d\tau\le 2\int_0^{t/2}\omega(\tau)^{1-p'}d\tau\le 2\int_0^{d/2}\omega(\tau)^{1-p'}d\tau<\infty ,$$

with provision for the inequality (3).

We have

$$\left\|T_\gamma f\right\|_{L_{q,\overline{\omega}_1(d(x,a))}(X,\mu)}\le\left(\int_X\left|T_\gamma f(x)\right|^q d\mu\int_0^{d(x,a)}\varphi(t)dt\right)^{1/q}+$$

$$+\left(\overline{\omega}_1(0)\int_X\left|T_\gamma f(x)\right|^q d\mu\right)^{1/q}=A_1+A_2 .$$

$$\left|T_\gamma f_2(x)\right| \le \int_{d(y,a)\le\tau} (d(x,y))^{\gamma-1}|f(y)|d\mu \le \tau^{\gamma-1} \int_{d(y,a)\le\tau} \frac{|f(y)|\omega(d(y,a))^{1/p}}{\omega(d(y,a))^{1/p}} d\mu .$$

Applying Hölder inequality we obtain

$$\left|T_\gamma f_2(x)\right| \le C\tau^{\gamma-1}\|f\|_{L_{p,\omega(d(x,a))}(X,\mu)} \left(\int_{d(y,a)\le\tau} \omega(d(y,a))^{1-p'} d\mu \right)^{1/p'} .$$

Then

$$\int_{d(y,a)\le\tau} \omega(d(y,a))^{1-p'} d\mu = \sum_{n=0}^{\infty} \int_{B(a,2^{-n}\tau)\setminus B(a,2^{-n-1}\tau)} \omega(d(y,a))^{1-p'} d\mu \le$$

$$\le \sum_{n=0}^{\infty} \omega\left(2^{-n-1}\tau\right)^{1-p'} \int_{B(a,2^{-n}\tau)\setminus B(a,2^{-n-1}\tau)} d\mu \le C\sum_{n=0}^{\infty} \omega\left(2^{-n-1}\tau\right)^{1-p'} 2^{-n-1}\tau \le C\int_0^\tau \omega(t)^{1-p'} dt .$$

Last integral, by assumption of the theorem, converges, hence $T_\gamma f_2(x)$ exists for almost any $x \in X \setminus B(a,\tau)$. Since $X \setminus \{a\} = \bigcup_{\tau>0} (X \setminus B(a,\tau))$ then the existence of $T_\gamma f(x)$ for almost any $x \in X$ follows from these reasonings.

Let $\overline{\omega}_1$ be an arbitrary continuous increasing function on $(0,d)$ such that $\overline{\omega}_1(t) \le \omega_1(t)$, $\overline{\omega}_1(0) = \omega_1(0+)$ and $\overline{\omega}_1(t) = \int_0^t \varphi(\tau)d\tau + \overline{\omega}_1(0)$, $t \in (0,d)$ (the existence of such $\overline{\omega}_1$ is obvious, for example, $\overline{\omega}_1(t) = \int_0^t \omega_1'(\tau)d\tau + \omega_1(0+) \le \omega_1(t)$).

Note that the following relation are valid:

$$\exists C>0,\ \ \forall t>0,\ \ \left(\int_t^d \varphi(\tau)\tau^{-q/p'}d\tau\right)^{p/q} \left(\int_0^t \omega(\tau)^{1-p'}d\tau\right)^{p-1} \le C . \qquad (7)$$

The inequality (7) follows from the inequalities

$$\int_t^d \varphi(\tau)\tau^{-q/p'}d\tau = \frac{q}{p'}\int_t^d \varphi(\tau)d\tau \int_\tau^d \lambda^{-1-q/p'}d\lambda + d^{-q/p'}\int_t^d \varphi(\tau)d\tau \le$$

$$\le \frac{q}{p'}\int_t^d \lambda^{-1-q/p'}d\lambda \int_t^\lambda \varphi(\tau)d\tau + d^{-q/p'}\omega_1(d-0) \le$$

$$\le \frac{q}{p'}\int_t^d \lambda^{-1-q/p'}\omega_1(\lambda)d\lambda + d^{-q/p'}\omega_1(d-0),$$

and

$$\int_0^t \omega(\tau)^{1-p'}d\tau \le 2\int_0^{t/2} \omega(\tau)^{1-p'}d\tau \le 2\int_0^{d/2} \omega(\tau)^{1-p'}d\tau < \infty ,$$

with provision for the inequality (3).

We have

$$\|T_\gamma f\|_{L_{q,\overline{\omega}_1(d(x,a))}(X,\mu)} \le \left(\int_X |T_\gamma f(x)|^q d\mu \int_0^{d(x,a)} \varphi(t)dt \right)^{1/q} +$$

$$+\left(\overline{\omega}_1(0) \int_X |T_\gamma f(x)|^q d\mu \right)^{1/q} = A_1 + A_2 .$$

$$+C\frac{\omega_1(d-0)^{1/q}}{\omega\left(\frac{d}{2C_0}\right)^{1/p}}\left\{\int_{X\setminus B\left(a,\frac{d}{2C_0}\right)}|f(x)|^p\,\omega(d(x,a))d\mu(x)\right\}^{1/p}\le C\|f\|_{L_{p,\omega(d(x,a))}}(X,\mu).$$

Let us estimate A_{12}. From $d(x,a)>t$, $d(y,a)<t/(2C_0)$ implies

$$\frac{1}{2C_0}d(x,a)\le d(x,y)\le C_0(d(x,a)+d(y,a))\le C_0\left(1+\frac{1}{2C_0}\right)d(x,a).$$

Then

$$\int_{d(x,a)>t}\left|\int_{d(y,a)<t/(2C_0)}(d(x,y))^{\gamma-1}f(y)d\mu(y)\right|^q d\mu(x)\le$$

$$\le C\int_{d(x,a)>t}d(x,a)^{(\gamma-1)q}\,d\mu(x)\left(\int_{d(y,a)<t/(2C_0)}|f(y)|d\mu(y)\right)^q.$$

Now we will estimate $\int_{d(x,a)>t}d(x,a)^{(\gamma-1)q}\,d\mu(x)$.

$$\int_{d(x,a)>t}d(x,a)^{(\gamma-1)q}\,d\mu(x)=\sum_{n=0}^{\infty}\int_{B(a,2^{n+1}t)\setminus B(a,2^n t)}d(x,a)^{(\gamma-1)q}\,d\mu(x)\le$$

$$\le\sum_{n=0}^{\infty}(2^n t)^{(\gamma-1)q}\,\mu B(a,2^{n+1}t)\le C\sum_{n=0}^{\infty}(2^n t)^{(\gamma-1)q}\,2^{n+1}t\le$$

$$\le Ct^{1+(\gamma-1)q}\sum_{n=0}^{\infty}2^{n(1+(\gamma-1)q)}=Ct^{1+(\gamma-1)q}.$$

Then

$$\int_{d(y,a)<t/(2C_0)}|f(y)|d\mu(y)=\sum_{n=0}^{\infty}\int_{B(a,2^{-n}t/(2C_0))\setminus B(a,2^{-n-1}t/(2C_0))}|f(y)|d\mu(y)\le$$

$$\le C\sum_{n=0}^{\infty}\left(\int_{B(a,2^{-n}t/(2C_0))\setminus B(a,2^{-n-1}t/(2C_0))}(d(y,a))^{p'}\,d\mu(y)\right)^{1/p'}\times$$

$$\times\left(\int_{B(a,2^{-n}t/(2C_0))\setminus B(a,2^{-n-1}t/(2C_0))}|f(y)|^p(d(y,a))^{-p}\,d\mu(y)\right)^{1/p}\le$$

$$\le C\sum_{n=0}^{\infty}2^{-n}t/(2C_0)\left(\mu B(a,2^{-n}t/(2C_0))\right)^{1/p'}\times$$

$$\times\left(\int_{B(a,2^{-n}t/(2C_0))\setminus B(a,2^{-n-1}t/(2C_0))}|f(y)|^p(d(y,a))^{-p}\,d\mu(y)\right)^{1/p}\le$$

$$\le C\sum_{n=0}^{\infty}\left(2^{-n}t/(2C_0)\right)^{1+1/p'}\left(\int_{X\setminus B(a,2^{-n-1}t/(2C_0))}|f(y)|^p(d(y,a))^{-p}\,d\mu(y)\right)^{1/p}\le$$

$$\leq C\int_0^{t/(4C_0)}\tau^{1/p'}\left(\int_{X\setminus B(a,\tau)}|f(y)|^p(d(y,a))^{-p}d\mu(y)\right)^{1/p}.$$

Therefore

$$\int_{d(x,a)>t}\left|\int_{d(y,a)<t/2C_0}(d(x,y))^{\gamma-1}f(y)d\mu(y)\right|^q d\mu(x)\leq$$

$$\leq Ct^{-q/p'}\left(\int_0^{t/(4C_0)}\tau^{1/p'}\left(\int_{X\setminus B(a,\tau)}|f(y)|^p(d(y,a))^{-p}d\mu(y)\right)^{1/p}\right)^q.$$

Hence

$$A_{12}\leq C\left(\int_0^d\varphi(t)t^{-q/p'}\left(\int_0^t\tau^{1/p'}\left(\int_{X\setminus B(a,\tau)}|f(y)|^p(d(y,a))^{-p}d\mu(y)\right)^{1/p}d\tau\right)^q dt\right)^{1/q}.$$

By (7) and theorem 2, we have

$$A_{12}\leq C\left\{\int_0^d\tau^{p-1}\left(\int_{X\setminus B(a,\tau)}|f(y)|^p(d(y,a))^{-p}d\mu(y)\right)\omega(\tau)d\tau\right\}^{1/p}=$$

$$=C\left(\int_X|f(y)|^p(d(y,a))^{-p}d\mu(y)\int_0^{d(y,a)}\omega(\tau)\tau^{p-1}\right)^{1/p}\leq$$

$$\leq C\left(\int_X|f(y)|^p\omega(d(y,a))d\mu(y)\right)^{1/p}=C\|f\|_{L_{p,\omega(d(x,a))}(X,\mu)}.$$

Combining the estimates for A_1 and A_2, we obtain (6) for $\omega_1=\overline{\omega}_1$. If the function $\omega_1\geq 0$ $(0\leq t<d)$ is increasing then there is the sequence of differentiable functions $\varphi_n(t)$ such that $\lim\limits_{n\to\infty}\varphi_n(t)=\omega_1(t)$ a.e., $\varphi_n(t)\leq\omega_1(t)$, $\varphi_n(0)=\omega_1(0)$, $\varphi_n(t)$ is nondecreasing and $\varphi_n(t)=\int_0^\tau\varphi_n'(\tau)d\tau+\varphi_n(0)$. By Faton's theorem on passing to the limit under the integral sign, this gives (6).

We now consider the case when ω, ω_1 be a positive decreasing function on $(0,d)$.

Let $\overline{\overline{\omega}}_1$ be an arbitrary continuous decreasing function on $(0,d)$ such that $\overline{\overline{\omega}}_1(t)\leq\omega_1(t)$, $\overline{\overline{\omega}}_1(d)=\omega_1(d-0)$ and $\overline{\overline{\omega}}_1(t)=\int_t^d\psi(\tau)d\tau+\overline{\overline{\omega}}_1(d)$, $\psi(t)\geq 0$, $t\in(0,d)$ (the existence of such $\overline{\overline{\omega}}_1$ is obvious, for example, $\overline{\overline{\omega}}_1(t)=\int_t^d(-\omega_1'(\tau))d\tau+\omega_1(d-0)\leq\omega_1(t)$).

Note that the condition (4) implies the following relation:

$$\exists C_4>0,\ \forall t>0,\ \left(\int_0^t\psi(\tau)\tau d\tau\right)^{p/q}\left(\int_t^d\omega(\tau)^{1-p'}\tau^{-1-p'/q}d\tau\right)^{p-1}\leq C_4. \qquad (8)$$

The relation (8) follows from the inequalities

$$\int_0^t \psi(\tau)\tau d\tau = \int_0^t \psi(\tau)d\tau \int_0^\tau ds = \int_0^t ds \int_s^t \psi(\tau)d\tau \le \int_0^t \omega_1(s)ds .$$

We have

$$\left\|T_\gamma f\right\|_{L_{q,\bar{\bar{\omega}}_1(d(d,a))}(X,\mu)} \le C\left(\int_X \left|T_\gamma f(x)\right|^q d\mu \int_{d(x,a)}^d \psi(t)dt \right)^{1/q} +$$

$$+\left(\bar{\bar{\omega}}_1(d-0)\int_X \left|T_\gamma f(x)\right|^q d\mu\right)^{1/q} = B_1 + B_2 .$$

Since $\omega(d-0)=\lim\limits_{t\to d-0}\omega(t)>0$, then $L_{p,\omega(d(x,a))}(X,\mu)\subset L_p(X,\mu)$, and by theorem 1

$$B_2 \le C\bar{\bar{\omega}}_1(d-0)^{1/q}\left(\int_X |f(x)|^p d\mu\right)^{1/p} \le C\left(\int_X |f(x)|^p \omega_1(d(x,a)/2)^{p/q} d\mu\right)^{1/p} \le$$

$$\le C\left(\int_X |f(x)|^p \omega(d(x,a))d\mu\right)^{1/p} = C\|f\|_{L_{p,\omega(d(x,a))}(X,\mu)} .$$

We now estimate B_1:

$$B_1 \le \left(\int_0^d \psi(t)dt \int_{d(x,a)<t} \left|T_\gamma f(x)\right|^q d\mu\right)^{1/q} \le B_{11} + B_{12},$$

where

$$B_{11} = \left(\int_0^d \psi(t)dt \int_{d(x,a)<t} \left| \int_{d(y,a)<2C_0t} (d(x,y))^{\gamma-1} f(y)d\mu(y)\right|^q d\mu\right)^{1/q},$$

$$B_{12} = \left(\int_0^d \psi(t)dt \int_{d(x,a)<t} \left| \int_{d(y,a)>2C_0t} (d(x,y))^{\gamma-1} f(y)d\mu(y)\right|^q d\mu\right)^{1/q}.$$

Further from the relation

$$\int_{d(y,a)\le t} |f(y)|^p d\mu(y) \le \frac{1}{\omega(t)} \int_{d(y,a)\le t} |f(y)|\omega(d(y,a))d\mu(y)$$

we get that $f\in L_p(B(a,t))$ for any $t>0$.

Thus, taking into account lemma 3, by the theorem 1 and the Minkovsky inequality with the exponent $q/p\ge 1$ we have

$$B_{11} \le C\left\{ \int_0^d \psi(t)\left(\int_{d(x,a)<2C_0t} |f(x)|^p d\mu(x)\right)^{q/p} dt\right\}^{1/q} \le C\left\{ \int_X |f(x)|^p \left(\int_{\frac{d(x,a)}{2C_0}}^d \psi(t)dt\right)^{p/q} d\mu(x)\right\}^{1/p} \le$$

$$\le C\left\{ \int_X |f(x)|^p \omega_1\left(\frac{1}{2C_0}d(x,a)\right)^{p/q} d\mu(x)\right\}^{1/p} \le C\|f\|_{L_{p,\omega(d(x,a))}(X,\mu)} .$$

Let us estimate B_{12}. From $d(x,a)<t,\ d(y,a)>2C_0t$ implies

$$\frac{1}{2C_0}d(y,a)\le d(x,y)\le C_0(d(x,a)+d(y,a))\le C_0\left(1+\frac{1}{2C_0}\right)d(y,a).$$

Then

$$\int_{d(x,a)<t}\left|\int_{d(y,a)>2C_0t}(d(x,y))^{\gamma-1}f(y)d\mu(y)\right|^q d\mu(x)\le$$

$$\le C\int_{d(x,a)<t}d\mu(x)\left(\int_{d(y,a)>2C_0t}|f(y)|(d(y,a))^{\gamma-1}d\mu(y)\right)^q.$$

We have $\int_{d(x,a)<t}d\mu(x)\le \mu B(a,t)\le Ct$.

Now we will estimate $\left(\int_{d(y,a)>2C_0t}|f(y)|(d(y,a))^{\gamma-1}d\mu(y)\right)^q$. Let $d=\infty$. Then

$$\int_{d(y,a)>2C_0t}|f(y)|(d(y,a))^{\gamma-1}d\mu(y)=\sum_{n=0}^{\infty}\int_{B(a,2^{n+1}2C_0t)\setminus B(a,2^n2C_0t)}|f(y)|(d(y,a))^{\gamma-1}d\mu(y)\le$$

$$\le\sum_{n=0}^{\infty}\left(2^n2C_0t\right)^{\gamma-2}\int_{B(a,2^{n+1}2C_0t)}|f(y)|\,d(y,a)d\mu(y)\le$$

$$\le\sum_{n=0}^{\infty}\left(2^n2C_0t\right)^{\gamma-2}\left(\mu B(a,2^{n+1}2C_0t)\right)^{1/p'}\left(\int_{B(a,2^{n+1}2C_0t)}|f(y)|^p(d(y,a))^p d\mu(y)\right)^{1/p}\le$$

$$\le\sum_{n=0}^{\infty}\left(2^n2C_0t\right)^{\gamma-2}\left(2^{n+1}2C_0t\right)^{1/p'}\left(\int_{B(a,2^{n+1}2C_0t)}|f(y)|^p(d(y,a))^p d\mu(y)\right)^{1/p}\le$$

$$\le\sum_{n=0}^{\infty}\left(2^n2C_0t\right)^{-1/q-1}\left(\int_{B(a,2^{n+1}2C_0t)}|f(y)|^p(d(y,a))^p d\mu(y)\right)^{1/p}\le$$

$$\le C\int_{4C_0t}^{d}\tau^{-1/q-2}\left(\int_{B(a,\tau)}|f(y)|^p(d(y,a))^p d\mu(y)\right)^{1/p}.$$

Let $d<\infty$. Define $m:=\max\{n:2^n2C_0t\le d\}$. Then

$$\int_{d(y,a)>2C_0t}|f(y)|(d(y,a))^{\gamma-1}d\mu(y)=\sum_{n=0}^{m}\int_{B(a,2^{n+1}2C_0t)\setminus B(a,2^n2C_0t)}|f(y)|(d(y,a))^{\gamma-1}d\mu(y)\le$$

$$\le\sum_{n=0}^{m}\left(2^n2C_0t\right)^{\gamma-2}\int_{B(a,2^{n+1}2C_0t)}|f(y)|d(y,a)d\mu(y)\le$$

$$\le \sum_{n=0}^{m}\left(2^n 2C_0 t\right)^{\gamma-2}\left(\mu B\left(a,2^{n+1}2C_0 t\right)\right)^{1/p'}\left(\int_{B\left(a,2^{n+1}2C_0 t\right)}|f(y)|^p\,(d(y,a))^p\,d\mu(y)\right)^{1/p} \le$$

$$\le \sum_{n=0}^{m}\left(2^n 2C_0 t\right)^{\gamma-2}\left(2^{n+1}2C_0 t\right)^{1/p'}\left(\int_{B\left(a,2^{n+1}2C_0 t\right)}|f(y)|^p\,(d(y,a))^p\,d\mu(y)\right)^{1/p} \le$$

$$\le \sum_{n=0}^{m}\left(2^n 2C_0 t\right)^{-1/q-1}\left(\int_{B\left(a,2^{n+1}2C_0 t\right)}|f(y)|^p\,(d(y,a))^p\,d\mu(y)\right)^{1/p} \le$$

$$\le C\int_{4C_0 t}^{d}\tau^{-1/q-2}\left(\int_{B(a,\tau)}|f(y)|^p\,(d(y,a))^p\,d\mu(y)\right)^{1/p}d\tau + Cd^{-1/q}\|f\|_{L_p(X,\mu)}.$$

Consequently

$$\int_{d(x,a)<t}\left|\int_{d(y,a)>2C_0 t}(d(x,y))^{\gamma-1}f(y)\,d\mu(y)\right|^q d\mu(x)\le$$

$$\le Ct\left(\int_{t}^{d}\tau^{-1/q-2}\left(\int_{B(a,\tau)}|f(y)|^p\,(d(y,a))^p\,d\mu(y)\right)^{1/p}d\tau + d^{-1/q}\|f\|_{L_p(X,\mu)}\right)^q.$$

Thus

$$B_{12}\le C\left(\int_0^d \psi(t)t\left(\int_t^d \tau^{-1/q-2}\left(\int_{B(a,\tau)}|f(y)|^p\,(d(y,a))^p\,d\mu(y)\right)^{1/p}d\tau\right)^q dt\right)^{1/q}+$$

$$+Cd^{-1/q}\|f\|_{L_p(X,\mu)}\left(\int_0^d\psi(t)t\,dt\right)^{1/q}.$$

In view of theorem 2 and (8) we have

$$B_{12}\le C\left(\int_0^d \tau^{-p/q-2p}\left(\int_{B(a,\tau)}|f(y)|^p\,(d(y,a))^p\,d\mu(y)\right)\omega(\tau)\tau^{p/q+p-1}d\tau\right)^{1/p}+$$

$$+Cd^{-1/q}\|f\|_{L_{p,\omega(d(x,a))}(X,\mu)}=C\left(\int_0^d\tau^{-1-p}\left(\int_{B(a,\tau)}|f(y)|^p\,(d(y,a))^p\,d\mu(y)\right)\omega(\tau)d\tau\right)^{1/p}+$$

$$+Cd^{-1/q}\|f\|_{L_{p,\omega(d(x,a))}(X,\mu)}=C\left(\int_X|f(y)|^p\,(d(y,a))^p\,d\mu(y)\int_{d(y,a)}^{d}\tau^{-1-p}\omega(\tau)d\tau\right)^{1/p}+$$

$$+Cd^{-1/q}\|f\|_{L_{p,\omega(d(x,a))}(X,\mu)}=C\left(\int_X|f(y)|^p\,\omega(d(y,a))d\mu(y)\right)^{1/p}+$$

$$+Cd^{-1/q}\|f\|_{L_{p,\omega(d(x,a))}(X,\mu)}\le C\|f\|_{L_{p,\omega(d(x,a))}(X,\mu)}.$$

Combining the estimates of B_1 and B_2, we get (6) for $\omega_1=\overline{\overline{\omega}}_1$. By Fatou's theorem on passing to the limit under the integral sign, this gives (6).

Let us also consider the case when ω is increasing, ω_1 is decreasing function on $(0,d)$.

Denote by $M = \inf_{0<t<d} \omega(t)$. We have

$$\omega_1(t) \le \omega_1(t/n) \le C\omega(t/(2n))^{q/p} \to M^{q/p}C \quad \text{for} \quad n \to \infty .$$

Therefore, in view of theorem 1, we obtain

$$\|T_\gamma f(x)\|_{L_{q,\omega_1(d(x,a))}(X,\mu)} = \left(\int_X |T_\gamma f(x)|^q \omega_1(d(x,a)) d\mu(x) \right)^{1/q} \le$$

$$\le M^{1/p} C^{1/q} \left(\int_X |T_\gamma f(x)|^q d\mu(x) \right)^{1/q} \le M^{1/p} C^{1/q} C_1 \left(\int_X |f(x)|^p d\mu(x) \right)^{1/p} \le$$

$$\le C_1 C^{1/q} \left(\int_X |f(x)|^p \omega(d(x,a)) d\mu(x) \right)^{1/p} = C_1 C^{1/q} \|f(x)\|_{L_{p,\omega(d(x,a))}(X,\mu)} .$$

Finally, in the case of decreasing function ω and increasing function ω_1 the proof is carried out along the same lines.

The following reverse theorem holds.

Theorem 4. *Let* $d = \infty \quad (\equiv \mu X = \infty)$,

$$\exists C > 0 \quad \mu B(x,r) = Cr, \quad \forall x \in X, \ r > 0 \qquad (9)$$

and $0 < \gamma < 1$, $1 < p < 1/\gamma$, $1/p - 1/q = \gamma$ *. Suppose that for the pair of monotone weights* (ω,ω_1) *the inequality (6) is valid. Then* $(\omega,\omega_1) \in S^{\sim}_{pq}(X)$.

Proof. Let ω be a positive increasing function on $(0,\infty)$ and $t > 0$. Choose the function f by the following way

$$f(y) = \begin{cases} \omega(d(y,a))^{1-p'}, & \text{if} \quad d(y,a) < t/(2C_0), \\ 0, & \text{if} \quad d(y,a) \ge t/(2C_0). \end{cases}$$

Estimate $\left(\int_X |T_\gamma f(x)|^q \omega_1(d(x,a)) d\mu(x) \right)^{1/q}$.

$$\left(\int_X |T_\gamma f(x)|^q \omega_1(d(x,a)) d\mu(x) \right)^{1/q} =$$

$$= \left(\int_X \left| \int_{d(y,a)<t/(2C_0)} d(x,y)^{\gamma-1} \omega(d(y,a))^{1-p'} d\mu(y) \right|^q \omega_1(d(x,a)) d\mu(x) \right)^{1/q} \ge$$

$$\ge \left(\int_{d(x,a)>t} \left| \int_{d(y,a)<t/(2C_0)} d(x,y)^{\gamma-1} \omega(d(y,a))^{1-p'} d\mu(y) \right|^q \omega_1(d(x,a)) d\mu(x) \right)^{1/q} \ge$$

$$\ge \int_{d(y,a)<t/(2C_0)} \omega(d(y,a))^{1-p'} d\mu(y) \cdot \left(\int_{d(y,a)>t} d(x,y)^{(\gamma-1)q} \omega_1(d(x,a)) d\mu(x) \right)^{1/q} .$$

On the other hand, we have

$$\left(\int_X |f(x)|^p \omega(d(x,a))d\mu(x)\right)^{1/p} = \left(\int_{d(x,a)<t/(2C_0)} \omega(d(x,a))^{1-p'} d\mu(x)\right)^{1/p}.$$

Combining last two inequalities, we get from (6):

$$\left(\int_{d(x,a)<t/(2C_0)} \omega(d(x,a))^{1-p'} d\mu(x)\right)^{1/p'} \left(\int_{d(x,a)>t} d(x,a)^{(\gamma-1)q} \omega_1(d(x,a))d\mu(x)\right)^{1/q} \le C.$$

We have

$$\int_{d(y,a)<t/(2C_0)} \omega(d(y,a))^{1-p'} d\mu(y) = \sum_{n=0}^{\infty} \int_{B(a,2^{-n}t/(2C_0))\setminus B(a,2^{-n-1}t/(2C_0))} \omega(d(y,a))^{1-p'} d\mu(y) \ge$$

$$\ge \sum_{n=0}^{\infty} \omega(2^{-n}t/(2C_0))^{1-p'} \left(\mu B(a,2^{-n}t/(2C_0)) - \mu B(a,2^{-n-1}t/(2C_0))\right) =$$

$$= C\sum_{n=0}^{\infty} \omega(2^{-n}t/(2C_0))^{1-p'} (2^{-n}t/(2C_0)) \ge C\sum_{0}^{\infty} \int_{2^{-n}t/(2C_0)}^{2^{-n+1}t/(2C_0)} \omega(\tau)^{1-p'} d\tau =$$

$$= C\int_0^{t/C_0} \omega(\tau)^{1-p'} d\tau \ge C\int_0^t \omega(\tau)^{1-p'} d\tau .$$

Indeed

$$\int_0^t \omega(\tau)^{1-p'} d\tau = \int_0^{t/C_0} \omega(\tau)^{1-p'} d\tau + \int_{t/C_0}^t \omega(\tau)^{1-p'} d\tau \le$$

$$\le \int_0^{t/C_0} \omega(\tau)^{1-p'} d\tau + (1-1/C_0)t \cdot \omega(t/C_0)^{1-p'} \le$$

$$\le \int_0^{t/C_0} \omega(\tau)^{1-p'} d\tau + (C_0-1)\int_0^{t/C_0} \omega(\tau)^{1-p'} d\tau = C_0 \int_0^{t/C_0} \omega(\tau)^{1-p'} d\tau .$$

Now estimate $\left(\int_{d(x,a)>t} d(x,a)^{(\gamma-1)q} \omega_1(d(x,a))d\mu(x)\right)^{1/q}$. Let $\omega_1 \uparrow (0,\infty)$. Then

$$\int_{d(x,a)>t} d(x,a)^{(\gamma-1)q} \omega_1(d(x,a))d\mu(x) =$$

$$= \sum_{n=0}^{\infty} \int_{B(a,2^{n+1}t)\setminus B(a,2^n t)} d(x,a)^{(\gamma-1)q} \omega_1(d(x,a))d\mu(x) \ge$$

$$\ge C\sum_{n=0}^{\infty} (2^{n+1}t)^{(\gamma-1)q+1} \omega_1(2^n t) \ge C\sum_{n=0}^{\infty} \int_{2^{n-1}t}^{2^n t} \tau^{-q/p'-1} \omega_1(\tau)d\tau =$$

$$= C\int_{t/2}^{\infty} \tau^{-q/p'-1} \omega_1(\tau)d\tau \ge C\int_{t/2}^{\infty} \tau^{-q/p'-1} \omega_1(\tau)d\tau .$$

Thus

$$\sup_{0<t<\infty} \left(\int_t^{\infty} \omega_1(\tau)\tau^{-1-q/p'} d\tau\right)^{p/q} \left(\int_0^t \omega(\tau)^{1-p'} d\tau\right)^{p-1} < \infty .$$

If $\omega_1 \downarrow (0,\infty)$, then

$$\int\limits_{d(x,a)>t} d(x,a)^{(\gamma-1)q}\omega_1(d(x,a))d\mu(x) \geq C\sum_{n=0}^{\infty}\left(2^{n+1}t\right)^{-q/p'}\omega_1\left(2^{n+1}t\right) \geq$$

$$\geq C\sum_{n=0}^{\infty}\int\limits_{2^{n+1}t}^{2^{n+2}t}\tau^{-q/p'-1}\omega_1(\tau)d\tau = C\int\limits_{2t}^{\infty}\tau^{-q/p'-1}\omega_1(\tau)d\tau .$$

Since

$$\int\limits_0^{2t}\omega(\tau)^{1-p'}d\tau = \int\limits_0^{t}\omega(\tau)^{1-p'}d\tau + \int\limits_t^{2t}\omega(\tau)^{1-p'}d\tau \leq$$

$$\leq \int\limits_0^{t}\omega(\tau)^{1-p'}d\tau + t\omega(t)^{1-p'} \leq 2\int\limits_0^{t}\omega(\tau)^{1-p'}d\tau ,$$

we have

$$\sup_{0<t<\infty}\left(\int\limits_{2t}^{\infty}\omega_1(\tau)\tau^{-1-q/p'}d\tau\right)^{p/q}\left(\int\limits_0^{2t}\omega(\tau)^{1-p'}d\tau\right)^{p-1} < \infty .$$

Now assume that ω is a positive decreasing function on $(0,\infty)$ and $t>0$, and define the function f by the following way:

$$f(y) = \begin{cases} \left(\omega(d(y,a))d(y,a)^{1-\gamma}\right)^{1-p'}, & if \quad d(y,a)>t \\ 0, & if \quad d(y,a)\leq t \end{cases}$$

Then

$$\|f\|_{L_{p,\omega(d(x,a))}(X,\mu)} = \left(\int\limits_{d(x,a)>t}\omega(d(x,a))^{1-p'}(d(x,a))^{-p'/q-1}d\mu(x)\right)^{1/p} .$$

On the other hand

$$\|T_\gamma f\|_{L_{q,\omega_1(d(x,a))}(X,\mu)} \geq$$

$$\geq \left(\int\limits_{d(x,a)<t/(2C_0)}\omega_1(d(x,a))\left(\int\limits_{d(y,a)>t}(d(x,y))^{\gamma-1}\omega(d(y,a))^{1-p'}d(y,a)^{(1-\gamma)(1-p')}d\mu(y)\right)^q d\mu(x)\right)^{1/q} \geq$$

$$\geq C\left(\int\limits_{d(x,a)<t/(2C_0)}\omega_1(d(x,a))d\mu(x)\right)^{1/q}\left(\int\limits_{d(y,a)>t}\omega(d(y,a))^{1-p'}(d(y,a))^{-p'/q-1}d\mu(y)\right).$$

Therefore

$$\left(\int\limits_{d(x,a)<t/(2C_0)}\omega_1(d(x,a))d\mu(x)\right)^{p/q}\left(\int\limits_{d(y,a)>t}\omega(d(y,a))^{1-p'}(d(y,a))^{-p'/q-1}d\mu(y)\right)^{p-1} \leq C.$$

Estimate $\int\limits_{d(y,a)>t}\omega(d(y,a))^{1-p'}(d(y,a))^{-p'/q-1}d\mu(y)$

$$\int\limits_{d(y,a)>t}\omega(d(y,a))^{1-p'}(d(y,a))^{-p'/q-1}d\mu(y) =$$

$$= \sum_{n=0}^{\infty}\int\limits_{B(a,2^{n+1}t)\setminus B(a,2^n t)}\omega(d(y,a))^{1-p'}(d(y,a))^{-p'/q-1}d\mu(y) \geq$$

$$\geq C\sum_{n=0}^{\infty}\omega\left(2^n t\right)^{1-p'}\left(2^{n+1}t\right)^{-p'/q}\geq C\sum_{n=0}^{\infty}\int_{2^{n-1}t}^{2^n t}\omega(\tau)^{1-p'}\tau^{-p'/q-1}d\tau=$$

$$=C\int_{t/2}^{\infty}\omega(\tau)^{1-p'}\tau^{-p'/q-1}d\tau\geq\int_{t}^{\infty}\omega(\tau)^{1-p'}\tau^{-p'/q-1}d\tau\ .$$

Let $\omega_1\downarrow(0,\infty)$, then

$$\int_{d(y,a)<t/(2C_0)}\omega_1(d(x,a))d\mu(x)=$$

$$=\sum_{n=0}^{\infty}\int_{B(a,2^{-n}t/(2C_0))\setminus B(a,2^{-n-1}t/(2C_0))}\omega_1(d(x,a))d\mu(x)\geq$$

$$\geq C\sum_{n=0}^{\infty}2^{-n}t/(2C_0)\omega_1\left(2^{-n}t/(2C_0)\right)\geq C\sum_{n=0}^{\infty}\int_{2^{-n}t/(2C_0)}^{2^{-n+1}t/(2C_0)}\omega_1(\tau)d\tau=$$

$$=C\int_0^{t/C_0}\omega_1(\tau)d\tau\geq C\int_0^{t}\omega_1(\tau)d\tau\ .$$

If $\omega_1\uparrow(0,\infty)$, then

$$\int_{d(x,a)<t/(2C_0)}\omega_1(d(x,a))d\mu(x)\geq C\sum_{n=0}^{\infty}2^{-n}t/(2C_0)\omega_1\left(2^{-n-1}t/(2C_0)\right)\geq$$

$$\geq C\sum_{n=0}^{\infty}\int_{2^{-n-1}t/(2C_0)}^{2^{-n-2}t/(2C_0)}\omega_1(\tau)d\tau=C\int_0^{t/(4C_0)}\omega_1(\tau)d\tau\ .$$

Since

$$\int_{t/(4C_0)}^{\infty}\omega(\tau)^{1-p'}\tau^{-p'/q-1}d\tau\leq C\int_{t}^{\infty}\omega(\tau)^{1-p'}\tau^{-p'/q-1}d\tau\ ,$$

we get

$$\sup_{0<t<\infty}\left(\int_0^{t/(4C_0)}\omega_1(\tau)d\tau\right)^{p/q}\left(\int_{t/(4C_0)}^{\infty}\omega(\tau)^{1-p'}\tau^{-p'/q-1}d\tau\right)^{p-1}<\infty\ .$$

Note that in [3] the criteria for weak and strong two-weighted inequalities are obtained for integral transforms with positive kernels, which for T_γ take the following form:

Theorem 5. *Let* $0<\gamma<1,\ 1<p<1/\gamma,\ 1/p-1/q=\gamma,$

$$B(x,R)\setminus B(x,r)\neq\emptyset,\quad\forall r,R:0<r<R<\infty,\tag{10}$$

and $\omega(t),\ \omega_1(t)$ *be monotone functions. Then for the inequality*

$$\left(\int_X\left|T_\gamma f(x)\right|^q\omega_1(x)d\mu\right)^{1/q}\leq C\left(\int_X\left|f(x)\right|^p\omega(x)d\mu\right)^{1/p}$$

to hold, where the constant C *does not depend on* f, *it is necessary and sufficient that the following two conditions be fulfilled simultaneously:*

$$\sup_{\substack{x\in X\\ r>0}}\left(\int_{B(x,6C_0r)}\omega_1(x)d\mu(x)\right)^{\frac{1}{q}}\left(\int_{X\setminus B(x,r)}d(x,y)^{(\gamma-1)p'}\omega^{1-p'}(y)d\mu(y)\right)^{\frac{1}{p'}}<\infty\ ,\tag{11}$$

$$\sup_{\substack{x\in X\\ r>0}}\left(\int_{B(x,6C_0r)}\omega^{1-p'}(x)d\mu(x)\right)^{\frac{1}{p'}}\left(\int_{X\setminus B(x,r)}d(x,y)^{(\gamma-1)q}\omega_1(y)d\mu(y)\right)^{\frac{1}{q}}<\infty. \qquad (12)$$

Remark 1. Note, that the condition (10) implies that $\mu X=+\infty$. But our results include the case of $\mu X<+\infty$.

The set of pairs satisfying (11) and (12) we denote by $S_{pq}(X)$.

Remark 2. From theorems 3,4 and 5 it follows that:

Theorem 6. Let $0<\gamma<1$, $1<p<1/\gamma$, $1/p-1/q=\gamma$, the conditions (9) and (10) hold, and $\omega(t)$, $\omega_1(t)$ be monotone functions. Then

$$(\omega,\omega_1)\in S^{-}_{pq}(X)\Leftrightarrow(\omega,\omega_1)\in S_{pq}(X).$$

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Guliev V.S., Mustafaev R.Ch.
Institute of Mathematics and Mechanics of AS Azerbaijan.
9, F.Agayev str., 370141, Baku, Azerbaijan.
Tel.: 39-47-20.

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