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ON THE NEGATIVE SPECTRUM OF GENERALIZED SCHRÖDINGER OPERATORS

Abstract

In the paper the negative spectrum of the Schrödinger operator which is obtained with substitution of Laplace operator on quasielliptic operator of higher order is studied. For some conditions the infinity of negative spectrum of this operator is established.

In the space R^n consider the differential operator

$$Lu = L_0 u + Q(x)u , \tag{1}$$

where

$$L_0 = \sum_{(\alpha,\lambda)=2} a_\alpha D^\alpha .$$

Here $x = (x_1, x_2, ..., x_n)$, $\alpha = (\alpha_1, ..., \alpha_n)$, $\lambda = \left(\frac{1}{p_1}, ..., \frac{1}{p_n}\right)$, $p_i \geq 1$ are integers $(\alpha, \lambda) = \sum_{i=1}^{n} \alpha_i \lambda_i$. As usually

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}} , \quad |\alpha| = \sum_{i=1}^{n} \alpha_i .$$

It is assumed that a_α are real constants such that

$$\gamma_2 \sum_{\alpha \in \mathcal{N}} \left|\xi^\alpha\right|^2 \geq \sum_{(\alpha,\lambda)=2} a_\alpha (i\xi)^\alpha > \gamma_1 \left(\sum_{\alpha \in \mathcal{N}} \left|\xi^\alpha\right|^2 \right) \tag{2}$$

$\xi \in R^n$, where $0 < \gamma_1 < \gamma_2$, $\mathcal{N}$ is some set of multiindices containing the set $\mathcal{M}$ consisting of all the elements as $(0,0,...,p_i,0,...,0)$, $i = 1,...,n$. The function $Q(x)$ is measurable and locally bounded. The operator L is determined as closure in $L_2(R^n)$ of the operator generated by the differential operator L on $\overset{\circ}{C}{}^\infty(R^n)$. It is assumed that L is a selfadjoint operator $L_2(R^n) \rightarrow L_2(R^n)$. It will be satisfied, if for example, $Q(x)$ is a bounded function.

Note that by virtue of the condition (2)

$$(L_0 u, u) > 0$$

if $u(x) \in \overset{\circ}{C}{}^\infty(R^n)$. In $L_2(R^4)$ a scalar product is denoted by $(\cdot,\cdot)$. In the case $Q(x) \equiv 0$, the operator L is positive and its spectrum is continuous coinciding with a positive semi-axis. The questions of discreteness and finiteness of a negative spectrum of the operator L are explicitly investigated in case, when L_0 is elliptic, i.e. $p_1 = p_2 = ... = p_n$ [1]. Particularly, many papers are devoted to studying of the case, when $L_0 = \Delta$, where Δ is a Laplace operator [2], [3]. Denote by

$$\rho(x) = \sum_{i=1}^{n} x_i^{2p_i} .$$

By investigating the spectrum of the operator L the Hardy inequality is essentially used [4]. One of such inequalities is assumed in the following lemma.

Lemma. *If* $u(x)\in \overset{\circ}{C}^{\infty}(R^n)$, $\sum_{i=1}^{n} p_i^{-1} > 2$, *then*

$$\int_{R^n} \rho^{-1}(x) u^2 dx \le \gamma_{\mathcal{N},n} \int_{R^n} \sum_{\alpha\in\mathcal{N}} \left|D^\alpha u\right|^2 dx, \tag{3}$$

where $\gamma_{\mathcal{N},n} = const$ *is independent of* u.

Theorem. *Let* $\mathcal{N} = \mathcal{M}\sum_{i=1}^{n} p_i^{-1} > 2$, $\gamma_{\mathcal{M},n}$ *be a lower bound of constants for which formula (3) is valid,*

$$\lim_{|x|\to\infty} \rho(x) Q-(x) > \gamma_{\mathcal{M},n}^{-1},$$

then the negative spectrum of the operator L *contains a finite number of points.*

Proof. Let $\mathcal{V}(x)\in \overset{\circ}{C}^{\infty}(R^n)$ be such that

$$\int_{R^n} \frac{\mathcal{V}^2}{\rho(x)} dx > \left(\gamma_{\mathcal{M},n} - \eta\right) \int_{R^n} \sum_{i=1}^{n} \left|\frac{\partial^{p_i}\mathcal{V}}{\partial x_i^{p_i}}\right|^2 dx, \tag{4}$$

where $\eta > 0$.

Then

$$\int_{R^n} \sum_{i=1}^{n} \left|\frac{\partial^{p_i}\mathcal{V}}{\partial x_i^{p_i}}\right|^2 dx < \left(\gamma_{\mathcal{M},n}^{-1} + \varepsilon\right) \int_{R^n} \frac{\mathcal{V}^2}{\rho(x)} dx.$$

Let $\delta > 0$, $v_\delta = v\sigma\left(\delta^{-\frac{1}{p_1}} x_1,\dots,\delta^{-\frac{1}{p_1}} x_n\right)$ and

$$\sigma(x) = \begin{cases} 0, & \sum_{i=1}^{n} x_i^{2p_i} < \frac{1}{2} \\ 1, & \sum_{i=1}^{n} x_i^{2p_i} > 1 \ . \end{cases}$$

Since $|\mathcal{V}_\delta| \le \mathcal{V}$

$$\left|\frac{\mathcal{V}_\delta^2}{\rho(x)}\right| \le \frac{\mathcal{V}^2}{\rho(x)} \quad \text{then} \quad \int_{R^n} \frac{\mathcal{V}_\delta^2}{\rho(x)} dx \to \int_{R^n} \frac{\mathcal{V}^2}{\rho(x)} dx \quad \text{for} \quad \delta\to 0.$$

Consider $\int_{R^n} \left|\frac{\partial^{p_i}(\mathcal{V}-\mathcal{V}_\delta)^2}{\partial x_i^{p_i}}\right| dx$

$$\int_{R^n} \left|\frac{\partial^{p_i}(\mathcal{V}-\mathcal{V}_\delta)^2}{\partial x_i^{p_i}}\right| dx = \int_{R^n} \left|\frac{\partial^{p_i}\left[\mathcal{V}\left(1+\sigma\left(\delta^{-1}x\right)\right)\right]}{\partial x_i^{p_i}}\right|^2 dx \le$$

$$\le \sum C_{p_i'} \int \left|\mathcal{V}^{(p_i')}\right|^2 \left|\sigma^{p_i''}\left(\delta^{-1}x\right)\right|^2 \delta^{-\frac{2p_i''}{p_i}} dx +$$

$$+ \int_{R^n} \left|\mathcal{V}^{(p_i)}\right|^2 \left[1 - \sigma\left(\delta^{-1}x\right)\right]^2 dx .$$

For $\delta \to 0$ the last integral approaches to zero

$$\int_{R^n} \frac{\left|\mathcal{V}^{(p_i)}\sigma^{(p_i)}\left(\delta^{-1}x\right)\right|^2}{\delta^{\frac{2p''}{p_i}}} dx < \int_{R^n} \frac{\varphi^2(x)}{\sum_{i=1}^{n} x_i^{2p_i}} dx,$$

where $\varphi(x) = \mathcal{V}^{(p_i)}\sigma^{(p_i)}\left(\delta^{-1}\right)$.

It is obvious that

$$\int_{R^n} \frac{\varphi^2(x)}{\sum_{i=1}^{n} x_i^{2p_i}} = \int_{\frac{\delta^2}{2} < \sum_{i=1}^{n} x_i^{2p_i} < \delta^2} \frac{\varphi^2(x)}{\sum_{i=1}^{n} x_i^{2p_i}} dx.$$

From the Hardy inequality (3) (remind that $\varphi(x) \in \overset{\circ}{C}{}^{\infty}\left(R^N\right), \sum_{i=1}^{n} p_i^{-1} > 2$)

$$\int_{R^n} \frac{\varphi^2(x)}{\sum_{i=1}^{n} x_i^{2p_i}} dx < +\infty.$$

Therefore

$$\int_{\frac{\delta^2}{2} < \sum_{i=1}^{n} x_i^{2p_i} < \delta^2} \frac{\varphi^2(x)}{\sum_{i=1}^{n} x_i^{2p_i}} dx \to 0 \ \text{ for } \delta \to 0.$$

Thus

$$\int_{R^n} \left|\frac{\partial^{p_i}(\mathcal{V} - \mathcal{V}_\delta)}{\partial x_i^{p_i}}\right|^2 dx \to 0 \ \text{ for } \delta \to 0.$$

Hence we obtain that the inequality (4) is also valid for v_δ

$$\int_{R^n} \frac{\mathcal{V}_\delta^2}{\rho(x)} > \left(\gamma_{\mathcal{M},n}^{-1} + \varepsilon\right)^{-1} \int_{R^n} \sum_{i=1}^{n} \left|\frac{\partial^{p_i}\mathcal{V}_\delta}{\partial x_i^{p_i}}\right|^2 dx.$$

Now we consider $\mathcal{V}_N(x) = \mathcal{V}_\delta\left(N^{-\frac{1}{p_1}}x_1, \dots, N^{-\frac{1}{p_n}}x_n\right)$ for any $N \neq 0$. We substitute

$$x_i N^{-\frac{1}{p_i}} = x_i', \quad x' = \left(x_1', \dots, x_n'\right).$$

We have

$$\int_{R^n} \frac{\mathcal{V}_N^2(x)}{\rho(x)} dx = \int_{R^n} \frac{\mathcal{V}_\delta^2(N^{-\frac{1}{p_1}}x_1, \dots, N^{-\frac{1}{p_n}}x_n)}{\rho(x_1, \dots, x_n)} dx =$$

$$= \int_{R^n} \frac{\mathcal{V}_\delta^2(x')}{\rho(x')} N^{2+|\lambda|} dx',$$

$$\sum_{i=1}^{n} \int_{R^n} \left|\frac{\partial^{p_i}\mathcal{V}_N}{\partial x_i^{p_i}}\right|^2 dx = \int_{R^n} \sum_{i=1}^{n} \left|\frac{\partial^{p_i}\mathcal{V}_\delta(N^{-\frac{1}{p_1}}x_1, \dots, N^{-\frac{1}{p_n}}x_n)}{\partial x_i^{p_i}}\right|^2 dx =$$

$$= \int_{R^n} \sum_{i=1}^{n} \left| \frac{\partial^{p_i} v_\delta(x')}{\partial x_i'^{p_i}} \right|^2 N^{-2+|\lambda|} dx' .$$

Hence

$$\int_{R^n} \sum_{i=1}^{n} \left| \frac{\partial^{p_i} v_N}{\partial x_i^{p_i}} \right|^2 dx < \left(\gamma_{\varpi,n}^{-1} + \varepsilon \right) \int_{R^m} \frac{v_N^2}{\rho(x)} dx .$$

By the condition of the theorem

$$\varliminf_{|x| \to \infty} \rho(x) Q_-(x) > \gamma_{m,n}^{-1} .$$

It means that outside of sufficiently large parallelepiped

$$Q_-(x) > \frac{\gamma_{\varpi,n} + \mu}{\rho(x)} \quad \text{i.e.}$$

$$Q_+(x) \equiv 0 .$$

Consider

$$(Lv_N, v_N)$$

$$(Lv_N, v_N) = (L_0 v_N, v_N) + (Q(x) v_N, v_N) =$$

$$= \sum_{i=1}^{n} \int_{R^n} \left| \frac{\partial^{p_i} v_N}{\partial x_i^{p_i}} \right| dx - \int_{R^n} Q_-(x) v_N^2 dx < \left(\gamma_{\varpi,n}^{-1} + \varepsilon \right) \int_{R^n} \frac{v_N^2}{\rho(x)} dx -$$

$$- \left(\gamma_{\varpi,n}^{-1} + \mu \right) \int_{R^n} \frac{v_N^2}{\rho(x)} dx < 0,$$

Since $\varepsilon > 0$ may be arbitrary small. Thus on the infinite dimensional manifold $\{v_N\} \subset \overset{\circ}{C}{}^{\infty}(R^n)$

$$(Lv_N, v_N) < 0 .$$

The theorem is proved.

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