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MIXED PROBLEM FOR BOUSSINESQ EQUATION IN THE BOUNDED DOMAIN AND BEHAVIOUR OF ITS SOLUTION AS $t \to +\infty$

Abstract

In the paper solvability of a mixed problem for Boussinesq linear equation in bounded domain of multidimensional space and behaviour of its solution as $t \to +\infty$ were studied.

Introduction. One of the main questions of the theory of nonstationary differential equations is studying the behaviour of solutions of mixed problems for large values of time. In [1] Muckenhoupt proved the almost periodicity of solution of a mixed problem on the segment for the wave equation. Later Bochner S. and Neumann J.V. proved almost periodicity of solution of a mixed problem for the general form of hyperbolic equation. In set of papers [4] S.L.Sobolev established almost periodicity of solutions of mixed boundary value problem for the wave equation with constant and variable coefficients in bounded domain with the Dirichlet or Neumann boundary condition. Asymptotic almost periodicity of solution of a mixed problem for the wave equation with the coefficients depending on spatial and time coordinates and with the Dirichlet boundary condition was established in the paper [5]. In the paper [6] Gabov S.A. and Orazov B.B. obtained the expansion for solution of the mixed problems for Boussinesq equation in the one-dimensional case. In the paper [7] solvability of mixed problem for the Boussinesq linear equation in infinite multi-dimensional cylindrical domain and asymptotics of its solution for large values of time were studied.

1. Definitions, notation, and uniqueness theorem for a mixed problem. Let R_n be n-dimensional Euclidean space with elements $x = (x_1, x_2, ..., x_n)$; $\Omega \subset R_n$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. Denote $Q = \Omega \times [0, \infty)$ and denote by $C^{(0,0)}(\Omega)$ a space of functions which are defined in Q and continuous with respect to x, t.

Definition 1. *We denote by $B^{(2,2)}(Q)$ a space of functions defined in Q such that $D_t^{\beta} D_x^{|\alpha|} u(x,t) \in C^{(0,0)}(\Omega)$ and for large t, they satisfy the estimate*

$$\left\| D_t^{\beta} D_x^{|\alpha|} u(x,t) \right\|_{C(\bar{\Omega})} \leq C t^{\beta}, \quad 0 \leq |\alpha|, \quad \beta \leq 2\ . \tag{$*$}$$

Consider in Q the following mixed problem

$$\left(\sigma^2 \Delta_n - 1\right) \frac{\partial^2}{\partial t^2} u(x,t) + \gamma^2 \Delta_n u(x,t) = f(x,t) \tag{1.1}$$

with initial conditions

$$u(x,0) = \psi_0(x)\,, \ \frac{\partial}{\partial t} u(x,0) = \psi_1(x)\,, \tag{1.2}$$

and with boundary condition

$$u\left(x,0\right)|_{\partial\Omega\times(0,\infty)}=0, \tag{1.3}$$

where $\psi_0\left(x\right),\ \psi_1\left(x\right)\in H^{2\left(\left[\frac{n}{2}\right]+2\right)}\left(\Omega\right),\ f\left(x,t\right)\in H^{2\left(\left[\frac{n}{2}\right]+1\right)}\left(\Omega\right)$ and

$$\left\|D_t^{\beta}f\left(x,t\right)\right\|_{H^{2\left(\left[\frac{n}{2}\right]+1\right)}(\Omega)}\leq C,$$

$\beta=0,1$ for $t>0$, σ,γ are real number, $H^v\left(\Omega\right)$ is Sobolev space.

Definition 2. *We call function $u\left(x,t\right)\in B^{(2,2)}\left(Q\right)$ a classical solution of problem (1.1)-(1.3) if it satisfies the equation, the initial conditions and the boundary condition in the ordinary sense.*

Denote by $H_D^v\left(\Omega\right)\quad\left(v\geq 1\right),\quad H_N^v\left(\Omega\right)\quad\left(v\geq 2\right)$, [8] (p.252) subspaces of Sobolev space $H^v\left(\Omega\right)$ for whose elements the conditions

$$\mathrm{F}\left(x\right)|_{\partial\Omega}=0,...,\Delta^{\left[\frac{v-1}{2}\right]}\mathrm{F}\left(x\right)\Big|_{\partial\Omega}=0,$$

and

$$\left.\frac{\partial\mathrm{F}\left(x\right)}{\partial\mu}\right|_{\partial\Omega}=0,...,\frac{\partial}{\partial\mu}\ \Delta^{\left[\frac{v}{2}\right]-1}\mathrm{F}\left(x\right)\Big|_{\partial\Omega}=0,$$

are satisfied respectively, where μ is a normal to $\partial\Omega$.

Everywhere below the sign $\tilde{}$ on the function denotes the Laplace transformation of this function with respect to t.

The following theorem is valid.

Theorem 1. *Classical solution of problem (1.1)-(1.3) is unique if it exists.*

Proof. Let us show that solution of the homogeneous problem corresponding to problem (1.1)-(1.3) is only trivial one. If we multiply equation (1.1) by $u_t\left(x,t\right)$ and integrate on $Q_t=\Omega\times[0,t)\quad(t\leq T)$, we obtain

$$\varepsilon\left(t\right)\equiv\int_0^t\int_\Omega\left[\left(\sigma^2\Delta_n-1\right)\frac{\partial^2}{\partial t^2}u\left(x,t\right)+\gamma^2\Delta_n u\left(x,t\right)\right]\frac{\partial}{\partial t}u\left(x,t\right)dxdt=0\ . \tag{1.4}$$

By Green's first formula

$$\varepsilon\left(t\right)\equiv\int_\Omega\left[\Delta_n\tfrac{\partial^2}{\partial t^2}u\left(x,t\right)\right]\frac{\partial}{\partial t}u\left(x,t\right)dx=$$

$$=-\int_\Omega\sum_{j=1}^n\frac{\partial}{\partial x_j}\left(\frac{\partial^2}{\partial t^2}u\left(x,t\right)\right)\frac{\partial}{\partial x_j}\left(\frac{\partial u\left(x,t\right)}{\partial t}\right)s\Omega dt+$$

$$+\int_\Omega\frac{\partial}{\partial n}\left(\frac{\partial^2}{\partial t^2}u\left(x,t\right)\right)\frac{\partial}{\partial t}u\left(x,t\right)d\Omega\ . \tag{1.5}$$

By virtue of boundary condition (1.3) the integral on $\partial\Omega$ in (1.5) equals zero. Then from (1.5) we have

$$\varepsilon_1\left(t\right)=-\frac{1}{2}\int_0^t\int_\Omega\sum_{j=1}^n\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial t\partial x_j}u\left(x,t\right)\right)^2d\Omega dt\ . \tag{1.6}$$

Changing the order of integration in (1.6) and taking into account that for the homogeneous problem, initial functions are equal to zero, from (1.6) we obtain

$$\varepsilon_1(t) = -\frac{1}{2}\int_\Omega \sum_{j=1}^n \left(\frac{\partial^2 u(x,t)}{\partial t \partial x_j}\right)^2 d\Omega \ . \tag{1.7}$$

Now let us consider the second term in (1.4)

$$\varepsilon_2(t) = \int_0^t \int_\Omega \left[\frac{\partial^2}{\partial t^2} u(x,t)\right] \frac{\partial u(x,t)}{\partial t} d\Omega dt = \frac{1}{2}\int_0^t \int_\Omega \frac{\partial}{\partial t}\left(\frac{\partial u(x,t)}{\partial t}\right)^2 d\Omega dt \ .$$

Taking into account that initial functions are equal to zero, we have

$$\varepsilon_2(t) = \frac{1}{2}\int_\Omega \left(\frac{\partial u(x,t)}{\partial t}\right)^2 d\Omega \ . \tag{1.8}$$

Proceeding in the same way as when we derived the equality in (1.5) we obtain

$$\varepsilon_3(t) = \int_0^t \int_\Omega [\Delta_n u(x,t)] \frac{\partial u(x,t)}{\partial t} dx dt = -\frac{1}{2}\int_\Omega \sum_{j=1}^n \left(\frac{\partial u(x,t)}{\partial x_j}\right)^2 d\Omega \ . \tag{1.9}$$

From (1.4), (1.7), (1.8), (1.9) we obtain

$$\begin{aligned} \varepsilon(t) = -\tfrac{\sigma^2}{2}\int_\Omega \sum_{j=1}^n \left(\frac{\partial^2 u(x,t)}{\partial t \partial x_j}\right)^2 d\Omega - \\ -\tfrac{1}{2}\int_\Omega \left(\frac{\partial u(x,t)}{\partial t}\right)^2 d\Omega - \frac{\gamma^2}{2}\int_\Omega \sum_{j=1}^n \left(\frac{\partial u(x,t)}{\partial x_j}\right)^2 d\Omega \equiv 0 \ . \end{aligned} \tag{1.10}$$

$\varepsilon(t)$ is called the energy integral of the homogeneous mixed problem. If we introduce the notation

$$\int_\Omega \sum_{j=1}^n \left(\frac{\partial^2 u(x,t)}{\partial t \partial x_j}\right)^2 d\Omega = \|\nabla_x u_t\|^2_{L_2(\Omega)} \ , \quad \int_\Omega \sum_{j=1}^n \left(\frac{\partial u(x,t)}{\partial x_j}\right)^2 d\Omega = \|\nabla_x u\|^2_{L_2(\Omega)} \ ,$$

then for the energy integral we obtain

$$\varepsilon(t) = -\frac{\sigma^2}{2}\|\nabla_x u_t\|^2_{L_2(\Omega)} - \frac{1}{2}\|u_t\|^2_{L_2(\Omega)} - \frac{\gamma^2}{2}\|\nabla_x u\|^2_{L_2(\Omega)} = 0 \ .$$

From here and equality to zero of the initial conditions it follows that

$$u(x,t) \equiv 0 \ .$$

The theorem is proved.

2. Existence of solution of mixed problem (1.1)-(1.3) and its estimate. We represent solution of (1.1)-(1.3) $u(x,t)$ in the form

$$u(x,t) = u_0(x,t) + u_f(x,t) \ , \tag{**}$$

where $u_0(x,t)$ is a solution of problem $(1.1)_0$, (1.2) , (1.3) and $u_f(x,t)$ is a solution of problem (1.1), $(1.2)_0$, (1.3); zero at the number of data denotes that they are equal to zero.

Taking into account estimate (*) for $u_0(x,t)$ in $\bar{Q}$ we make the Laplace transformation with respect to t in problem (1.1) $_0$, (1.2), (1.3). Then we obtain the following boundary value problem

$$\left(\sigma^2 k^2 + \gamma^2\right) \Delta_n \tilde{u}_0(x,k) - k^2 \tilde{u}_0(x,k) = \Phi(x,k) \ , \tag{2.1}$$

$$\tilde{u}_0(x,k)|_{\partial\Omega} = 0 \ , \tag{2.2}$$

where

$$\Phi(x,k) = \left(\sigma^2 \Delta_n - 1\right) \psi_1(x) + k \left(\sigma^2 \Delta_n - 1\right) \psi_0(x) = f_1(x) + k f_0(x) \ .$$

Let us consider the differential expression $\tilde{A} = \Delta_n$ with the domain of definition

$$D(\tilde{A}) = \left\{W(x) : W(x) \in C^2(\Omega) \cap C\left(\bar{\Omega}\right), \right.$$

$$\left. \Delta_n W(x) \in L_2(\Omega), W(x)|_{\partial\Omega} = 0\right\}. \tag{2.3}$$

The differential expression $\tilde{A}$ with the domain of definition $D\left(\tilde{A}\right)$ generates a negative-defined self-adjoint operator A in space $L_2(\Omega)$. It is known ([9], p.117-178) that the spectrum of this operator is discrete and for its eigenvalues λ_l the following inequality holds

$$0 > \lambda_1 \geq \lambda_2 \geq ...\lambda_l \geq ..., \quad \lim \lambda_l = -\infty$$

Eigen functions $\varphi_l(y)$ of the operator A corresponding to the eigenvalues λ_l form the basis in space $L_2(\Omega)$. Using the above said, we prove the following theorem.

Theorem 2. *Let* $\partial\Omega \in C^{2\left(\left[\frac{n}{2}\right]+2\right)}$, $\psi_0(x), \psi_1(x) \in H_D^{2\left(\left[\frac{n}{2}\right]+2\right)}(\Omega)$, $f(x,t) \in$ $\in H_D^{2\left(\left[\frac{n}{2}\right]+1\right)}(\Omega)$ *be continuously differentiable with respect to* t *and*

$$\left\|D_t^\beta f(x,t)\right\|_{H^{2\left(\left[\frac{n}{2}\right]+1\right)}(\Omega)} \leq C,$$

for $t > 0$ $\beta = 0,1$. *Then solution of mixed problem (1.1)-(1.3) exists and for it, representation (2.16) and estimate (2.26) are valid.*

Proof. Using theorem 3.6 from ([9], p.177) for the solution of problem (2.1), (2.2), we obtain

$$\tilde{u}_0(x,k) = \sum_{l=1}^{\infty} \frac{c_l(k)\varphi_l(x)}{(\sigma^2 k^2 + \gamma^2)\lambda_l - k^2} \ , \tag{2.4}$$

where

$$c_l(k) = \int_\Omega \Phi(x,k)\,\varphi_l(x)\,dx\ .$$

Solution of problem (1.1) $_0$, (1.2), (1.3) is defined as inverse Laplace transformation from $\tilde{u}_0(x,k)$

$$u_0(x,t) = \frac{1}{2\pi i}\sum_{l=1}^{\infty}\varphi_l(x)\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\frac{c_l(k)\,e^{kt}dk}{(\sigma^2k^2+\gamma^2)\,\lambda_l-k^2} \tag{2.5}$$

here the termwise integration is valid by virtue of uniform convergence of series (2.4) and uniform convergence of series (2.5) ([8], p.231). If we put the value of $c_l(k)$ into (2.5), we obtain

$$c_l(k) = \int_\Omega f_1(y)\,\varphi_l(y)\,dy + k\int_\Omega f_2(y)\,\varphi_l(y)\,dy \equiv c_l^{(1)} + kc_l^{(0)}\ .$$

Putting the value of $c_l(k)$ into (2.5), we have

$$u_0(x,t) = \tfrac{1}{2\pi i}\left\{\sum_{l=1}^{\infty}c_l^{(1)}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\tfrac{e^{kt}dk}{k^2(\sigma^2\lambda_l-1)+\gamma^2\lambda_l}\varphi_l(x)+\right.$$
$$\left.+\sum_{l=1}^{\infty}c_l^{(0)}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\tfrac{ke^{kt}dk}{k^2(\sigma^2\lambda_l-1)+\gamma^2\lambda_l}\varphi_l(x)\right\}\ . \tag{2.6}$$

The integrands in (2.6) have poles at the points

$$k_l^{\pm} = \pm i\gamma\left(\frac{|\lambda_l|}{1+\sigma^2|\lambda_l|}\right)^{1/2}\ .$$

Since the integrand in the first integral of (2.6) decreases as $k\to\infty,\ 0\le \operatorname{Re}k\le\varepsilon$ as k^{-2} but at the left half-space exponentially decreases, then applying the Cauchy theorem going out to the left half-space, we obtain

$$J_{1l}(t) \equiv \frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\frac{e^{kt}dk}{k^2(\sigma^2\lambda_l-1)+\gamma^2\lambda_l} = \frac{1}{2\pi i(\sigma^2\lambda_l-1)}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\frac{e^{kt}dk}{k^2+\frac{\gamma^2\lambda_l}{\sigma^2\lambda_l-1}} =$$
$$= -\frac{1}{(\sigma^2|\lambda_l|+1)}\left[\frac{e^{it\gamma\left(\frac{|\lambda_l|}{|\lambda_l|+1}\right)^{1/2}}}{2i\gamma\left(\frac{|\lambda_l|}{\sigma^2|\lambda_l|+1}\right)^{1/2}} - \frac{e^{-it\gamma\left(\frac{|\lambda_l|}{\sigma^2|\lambda_l|+1}\right)^{1/2}}}{2i\gamma\left(\frac{|\lambda_l|}{\sigma^2|\lambda_l|+1}\right)^{1/2}}\right] = \tag{2.7}$$
$$= -\frac{1}{\gamma\left(|\lambda_l|\left(\sigma^2|\lambda_l|+1\right)\right)^{1/2}}\sin t\gamma\left(\frac{|\lambda_l|}{\sigma^2|\lambda_l|+1}\right)^{1/2}\ .$$

Reasoning as above, we have

$$J_{2l}(t) \equiv \frac{1}{2\pi i}\int\limits_{\varepsilon-i\infty}^{\varepsilon+i\infty} \frac{ke^{kt}dk}{k^2(\sigma^2\lambda_l-1)+\gamma^2\lambda_l} = -\frac{1}{(\sigma^2|\lambda_l|+1)}\cos t\gamma\left(\frac{|\lambda_l|}{\sigma^2|\lambda_l|+1}\right)^{1/2}.$$

Putting expression for $J_{1l}(t)$ and $J_{2l}(t)$ into (2.6) for solution of problem (1.1)-(1.3) we obtain

$$u_0(x,t) =$$

$$= -\sum_{l=1}^{\infty}\left[\frac{c_l^{(1)}\sin t\gamma\left(\frac{|\lambda_l|}{\sigma^2|\lambda_l|+1}\right)}{\gamma\left(|\lambda_l|\left(\sigma^2|\lambda_l|+1\right)\right)^{1/2}} + c_l^{(0)}\frac{\cos t\gamma\left(\frac{|\lambda_l|}{\sigma^2|\lambda_l|+1}\right)^{1/2}}{\sigma^2|\lambda_l|+1}\right]\varphi_l(x) \; . \qquad (2.8)$$

Now we transform the expressions for the coefficients $c_l^{(1)}$ and $c_l^{(0)}$

$$c_l^{(1)} = \int\limits_{\Omega}\varphi_l(x)\left(\sigma^2\Delta-1\right)\psi_1(x)\,dx.$$

Since $\psi_1(x)$, $\varphi_l(x)$ are equal to zero on $\partial\Omega$, then by Green's second formula we obtain

$$\begin{aligned} c_l^{(1)} &= \int\limits_{\Omega}\psi_1(x)\left(\sigma^2\Delta-1\right)\varphi_l(x)\,dx = -\left(\sigma^2|\lambda_l|+1\right)\int\limits_{\Omega}\psi_1(x)\,\varphi_l(x)\,dx \equiv \\ &\equiv -\left(\sigma^2|\lambda_l|+1\right)\tilde{c}_l^{(1)} \; . \end{aligned} \qquad (2.9)$$

$\psi_0(x)$ also equals zero on $\partial\Omega$, therefore we have

$$\begin{aligned} c_l^{(0)} &= \int\limits_{\Omega}\varphi_l(x)\left(\sigma^2\Delta-1\right)\psi_0(x)\,dx = -\left(\sigma^2|\lambda_l|+1\right)\int\limits_{\Omega}\psi_0(x)\,\varphi_l(x)\,dx \equiv \\ &\equiv -\left(\sigma^2|\lambda_l|+1\right)\tilde{c}_l^{(0)} \; . \end{aligned} \qquad (2.10)$$

If we put the expressions for $c_l^{(1)}$ and $c_l^{(0)}$ from (2.9), (2.10) into (2.8) then for solution of mixed problem $(1.1)_0$, (1.2), (1.3) we obtain

$$\begin{aligned} u_0(x,t) = \sum_{l=1}^{\infty}\Bigg[&\frac{\tilde{c}_l^{(1)}}{\gamma}\left(\frac{\sigma^2|\lambda_l|+1}{|\lambda_l|}\right)^{1/2}\sin\gamma t\left(\frac{|\lambda_l|}{\sigma^2|\lambda_l|+1}\right)^{1/2} + \\ &+\tilde{c}_l^{(0)}\cos\gamma t\left(\frac{|\lambda_l|}{\sigma^2|\lambda_l|+1}\right)^{1/2}\Bigg]\varphi_l(x) \; . \end{aligned} \qquad (2.11)$$

Let us consider now problem (1.1), $(1.2)_0$, (1.3). For solution $u_f(x,t)$ of this problem as above, we have

$$\tilde{u}_f(x,t) = \sum_{l=1}^{\infty}\frac{c_l(k)\,\varphi_l(x)}{(k^2\sigma^2+\gamma^2)\,\lambda_l - k^2} \; ,$$

where

$$c_l(k) = \int_\Omega \tilde f(x,k)\,\varphi_l(x)\,dx\ . \tag{2.12}$$

If we make the Laplace transformation of (2.12), we obtain

$$\begin{gathered} u_f(x,t) = \frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\left\{\sum_{l=1}^{\infty}\varphi_l(x)\int_\Omega \frac{\tilde f(y,k)\,\varphi_l(y)}{(k^2\sigma^2+\gamma^2)\,\lambda_l-k^2}dy\right\}e^{kt}dk = \\ = \frac{1}{2\pi i}\sum_{l=1}^{\infty}\varphi_l(x)\int_\Omega\varphi_l(y)\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\frac{\tilde f(y,k)\,e^{kt}dk}{(k^2\sigma^2+\gamma^2)\,\lambda_l-k^2}dy\ , \end{gathered} \tag{2.13}$$

here the termwise integration is valid by virtue of uniform convergence of series in (2.13) which will be shown later. According to the Borel theorem ([10], p.475) from (2.13) we obtain

$$u_f(x,t) = \sum_{l=1}^{\infty}\varphi_l(x)\int_\Omega\varphi_l(y)\left[\int_0^t f(y,\tau)\,\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\frac{e^{(t-\tau)k}dk}{k^2\left(\sigma^2\lambda_l-1\right)+\gamma^2\lambda_l}\right]dy\ . \tag{2.14}$$

Taking into account the value of integral (2.7) in (2.14) we obtain

$$\begin{gathered} u_j(x,t) = -\frac{1}{\gamma}\sum_{l=1}^{\infty}\frac{\varphi_l(x)}{\left[|\lambda_l|\left(\sigma^2|\lambda_l|+1\right)\right]^{1/2}}\times \\ \times\int_0^t f_l(\tau)\sin\gamma(t-\tau)\left(\frac{|\lambda_l|}{\sigma^2|\lambda_l|+1}\right)^{1/2}d\tau\ , \end{gathered} \tag{2.15}$$

where

$$f_l(\tau) = \int_\Omega f(y,\tau)\,\varphi_l(x)\,d\tau\ .$$

Taking into consideration (2.11) and (2.15), from (**) we obtain

$$\begin{gathered} u(x,t) = \sum_{l=1}^{\infty}\left\{\frac{\tilde c_l^{(1)}}{\gamma}\left(\frac{\sigma^2|\lambda_l|+1}{|\lambda_l|}\right)^{1/2}\sin\gamma t\left(\frac{|\lambda_l|}{\sigma^2|\lambda_l|+1}\right)^{1/2}+\right. \\ +\tilde c_l^{(0)}\cos\gamma t\left(\frac{|\lambda_l|}{\sigma^2|\lambda_l|+1}\right)^{1/2}-\frac{1}{\gamma\left[|\lambda_l|\left(\sigma^2|\lambda_l|+1\right)\right]^{1/2}}\times \\ \left.\times\int_0^t f_l(\tau)\sin\gamma(t-\tau)\left(\frac{|\lambda_l|}{\sigma^2|\lambda_l|+1}\right)^{1/2}d\tau\right\}\varphi_l(x)\ . \end{gathered} \tag{2.16}$$

Let us prove now the uniform convergence of series in (2.16) and its derivatives if the data of the problem satisfy the conditions of theorem 2

$$\begin{gathered} |u(x,t)|_{C(\bar\Omega)} \le C(\gamma,\sigma)\times \\ \times\left\{\sum_{l=1}^{\infty}\left(\left|\tilde c_l^{(1)}\right|+\left|\tilde c_l^{(0)}\right|+\frac{1}{|\lambda_l|}\int_0^t|f_l(\tau)|\,d\tau\right)\|\varphi_l(x)\|_{C(\bar\Omega)}\right\}\ . \end{gathered} \tag{2.17}$$

In [11] it was shown that

$$\|\varphi_l(x)\|_{H^{[\frac{n}{2}]+1}(\Omega)} \le C|\lambda_l|^{\frac{1}{2}([\frac{n}{2}]+1)}, \tag{2.18}$$

where C is a constant independent of l. Hence by means of Sobolev imbedding theorem we obtain

$$\|\varphi_l(y)\|_{C(\bar{\Omega})} \le C\|\varphi_l(y)\|_{H^{[\frac{n}{2}]+1}(\Omega)} \;. \tag{2.19}$$

It is known that [9] (p.200)

$$c_0 l^{\frac{2}{n}} \le |\lambda_l| \le c_1 l^{\frac{2}{n}} \tag{2.20}$$

c_0, c_1 are constants independent of l. (2.18)-(2.19) imply that

$$\|\varphi_l(y)\|_{C(\bar{\Omega})} \le C|\lambda_l|^{\frac{1}{2}([\frac{n}{2}]+1)}. \tag{2.21}$$

Since of $\Delta^v\varphi_l(y) \quad (v \ge 1)$ is also an eigenfunction of the operator A with the eigenvalue λ_l^v, then as above on can show that

$$\|\varphi_l(y)\|_{C^v(\bar{\Omega})} \le C|\lambda_l|^{\frac{1}{2}([\frac{n}{2}]+v+1)}. \tag{2.22}$$

(2.19)-(2.21) implies that

$$\|\varphi_l(y)\|_{C^v(\bar{\Omega})} \le C|l|^{\frac{1}{n}([\frac{n}{2}]+v+1)}, \quad v = 0,1,2,... \tag{2.23}$$

From (2.16) by virtue of estimate (2.22) we have

$$\begin{gathered}\|u(x,t)\|_{C(\bar{\Omega})} \le C(\gamma,\sigma)\sum_{l=1}^{\infty}\Big\{|\lambda_l|^{([\frac{n}{2}]+1)}+ \\ +|\lambda_l|^{2([\frac{n}{2}]+1)}\left(|\tilde{c}_l^{(1)}|+|\tilde{c}_l^{(0)}|^2\right)+t\int_0^t|\lambda_l|^{2[\frac{n}{2}]}\,|f_l(\tau)|^2\,d\tau\Big\}\;,\end{gathered} \tag{2.24}$$

From (2.24) by virtue of theorem 8 from [8] (p.253) we obtain

$$\begin{gathered}\|u(x,t)\|_{C(\bar{\Omega})} \le C(\gamma,\sigma)\Big[J_0+\|\psi_l(x)\|^2_{H^{2([\frac{n}{2}]+1)}(\Omega)}+ \\ +\|\psi_0(x)\|^2_{H^{2([\frac{n}{2}]+1)}(\Omega)}+t\int_0^t\|f(x,\tau)\|^2_{H^{2([\frac{n}{2}]+1)}(\Omega)}\,dt\Big]\;,\end{gathered} \tag{2.25}$$

where $J_0 = \sum\limits_{l=1}^{\infty}|\lambda_l|^{-n}$. This series converges because for any natural n,

$$\frac{2}{n}\left(\left[\frac{n}{2}\right]+1\right) \ge 1+\frac{1}{n}$$

and by virtue of estimate (2.20)

$$\sum_{l=1}^{\infty}|\lambda_l|^{-([\frac{n}{2}]+1)} \le c_1\sum_{l=1}^{\infty}l^{-(1+\frac{1}{n})}\;.$$

Let us now estimate the derivatives of $u(x,t)$ contained in the equation. Calculating derivatives of $u(x,t)$ from (2.16) and estimating them like in (2.25), we obtain

$$\left\| D_t^{\beta} D_x^{|\alpha|} u(x,t) \right\|_{C(\bar{\Omega})} \leq$$

$$\leq C(\gamma,\sigma) \left[J_0 + \|\psi_1(x)\|^2_{H^{2([\frac{n}{2}]+1)+|\alpha|}(\Omega)} + \|\psi_0(x)\|^2_{H^{2([\frac{n}{2}]+1)+|\alpha|}(\Omega)} + \right. \tag{2.26}$$

$$\left. \left\| D_t^{\beta-1} f(x,t) \right\|^2_{H^{2[\frac{n}{2}]+|\alpha|}(\Omega)} + t \int_0^t \|f(x,\tau)\|^2_{H^{2[\frac{n}{2}]+|\alpha|}(\Omega)} dt \right] ,$$

where $0 \leq \beta,\ |\alpha| \leq 2$. For $\beta = 0,1$ the fourth term in the right-hand side of (2.26) is absent.

The theorem is proved.

3. The mixed problem in bounded domain for the Boussinesq equation with time derivative in the boundary condition.

Now let us consider problem (1.1), (1.2) with boundary condition

$$D_\mu \left[D_t^2 + E \right] u(x,t) \left|_{\partial\Omega \times (0,\infty)} \right. = 0 , \tag{3.1}$$

where μ is a normal to the boundary $\partial\Omega$. Here we will investigate a classical solution of problem (1.1), (1.2), (3.1), whose definition is given similar to the above mentioned one. If we apply Lapalce transformation to this problem, we obtain the following boundary value problem

$$\left(\sigma^2 k^2 + \gamma \right) \Delta_n \tilde{u}(x,k) - k^2 \tilde{u}(x,k) = \Phi(x,k) \tag{3.2}$$

$$\left(k^2 + 1 \right) D_\mu \tilde{u}(x,k) \left|_{\partial\Omega} \right. = 0 , \tag{3.3}$$

where $\operatorname{Re} k > 0$. If we cancel out $1 + k^2$ in the boundary condition, we obtain the Neumann boundary condition

$$D_\mu \tilde{u}(x,k) \left|_{\partial\Omega} \right. = 0 . \tag{3.4}$$

Problem (3.2), (3.4) is solved in the same way as problem (2.1), (2.2) only with the distinction that in formula (2.4) $\varphi_l(x)$ and λ_l will be eigenfunctions and eigenvalues, respectively, of the Neumann problem for the Laplace operator

$$\begin{gathered} \Delta \varphi_l(x) = \lambda_l \varphi_l(x) \ , \\ \left. \frac{\partial \varphi_i(x)}{\partial \mu} \right|_{\partial\Omega} = 0 . \end{gathered} \tag{3.5}$$

At boundary condition (3.1) to transform the expressions $c_l^{(0)},\ c_l^{(1)}$ we take into account that normal derivatives of fucntions $\psi_0(x),\ \psi_l(x)$ and $\psi_l(x)$ equal zero on $\partial\Omega$.

Since $\lambda_1 = 0$ is the eigenvalue corresponding to eigenfunction $\varphi_1 \equiv 1$, then in (2.16) for the first term we obtain

$$\lim_{\lambda_1\to 0}\left(\frac{\sigma^2|\lambda_1|+1}{|\lambda_1|}\right)^{1/2}\sin\gamma t\left(\frac{|\lambda_1|}{\sigma^2|\lambda_1|+1}\right)^{1/2}=\gamma t\ ,$$

$$\lim_{\lambda_1\to 0}\frac{1}{\gamma\left[|\lambda_1|\left(\sigma^2\left|\lambda_1\right|+1\right)\right]^{1/2}}\int_0^t f_1(\tau)\sin\gamma(t-\tau)\left(\frac{|\lambda_1|}{\sigma^2|\lambda_1|+1}\right)^{1/2}d\tau=$$

$$=\int_0^t (t-\tau)\,f_1(\tau)\,d\tau=\int_0^t\int_0^\tau f_1(\xi)\,d\xi d\tau\ .$$

Then

$$u(x,t)=\tilde{c}_1^{(1)}t+\tilde{c}_1^{(0)}-\int_0^t\int_0^\tau f_1(\xi)\,d\xi d\tau+$$

$$+\sum_{l=2}^{\infty}\left\{\frac{\tilde{c}_l^{(1)}}{\gamma}\left(\frac{\sigma^2|\lambda_l|+1}{|\lambda_l|}\right)^{1/2}\sin\gamma t\left(\frac{|\lambda_l|}{\sigma^2|\lambda_l|+1}\right)^{1/2}+\right.$$

$$+\tilde{c}_l^{(0)}\cos\gamma t\left(\frac{|\lambda_l|}{\sigma^2|\lambda_l|+1}\right)^{1/2}-\frac{1}{\gamma\left[|\lambda_l|\left(\sigma^2\left|\lambda_l\right|+1\right)\right]^{1/2}}\times \tag{3.6}$$

$$\left.\times\int_0^t f_l(\tau)\sin\gamma(t-\tau)\left(\frac{|\lambda_l|}{\sigma^2|\lambda_l|+1}\right)^{1/2}d\tau\right\}\varphi_l(x)\ ,$$

where

$$\tilde{c}_1^{(1)}=\int_\Omega\psi_1(x)\,dx\ ,\quad \tilde{c}_1^{(0)}=\int_\Omega\psi_0(x)\,dx\ ,\quad f_1(\xi)=\int_\Omega f(x,\xi)\,dx\ .$$

Thus, the solution of problem (1.1), (1.2), (3.1) is constructed.

4. On almost periodicity of solution of mixed problem (1.1)-(1.3) as $t\to+\infty$.

Theorem 3. *Let the conditions of theorem 2 be fulfilled and*

$$\int_0^\infty(1+\tau)^2\ \ \|f(x,\tau)\|^2_{H^{2\left[\frac{n}{2}\right]}(\Omega)}\,d\tau<+\infty\ .$$

Then for solution of problem (1.1)-(1.3) the following representation holds

$$u(x,t)=W_1(x,t)+W_2(x,t)\ ,$$

where $W_1(x,t)$ is a uniform almost periodic function with respect to t, and

$$\lim_{t\to\infty}W_2(x,t)=0$$

uniformly with respect to $x \in \Omega$.

Proof. We represent solution $u(x,t)$ of problem (1.1)-(1.3) from (2.16) in the following form

$$u(x,t)=\sum_{l=1}^{\infty}\left\{\frac{\tilde{c}_l^{(1)}}{\gamma}\left(\frac{\sigma^2|\lambda_l|+1}{|\lambda_l|}\right)^{1/2}\sin\gamma t\left(\frac{|\lambda_l|}{\sigma^2|\lambda_l|+1}\right)^{1/2}+\right.$$

$$+\tilde{c}_l^{(0)}\cos\gamma t\left(\frac{|\lambda_l|}{\sigma^2|\lambda_l|+1}\right)^{1/2}-\frac{1}{\gamma\left[|\lambda_l|\left(\sigma^2|\lambda_l|+1\right)\right]^{1/2}}\times$$

$$\times\left[\sin\gamma t\left(\frac{|\lambda_l|}{\sigma^2|\lambda_l|+1}\right)^{1/2}\int_0^{\infty}f_l(\tau)\cos\gamma\tau\left(\frac{|\lambda_l|}{\sigma^2|\lambda_l|+1}\right)^{1/2}d\tau-\right.$$

$$-\cos\gamma t\left(\frac{|\lambda_l|}{\sigma^2|\lambda_l|+1}\right)^{1/2}\int_0^{\infty}f_l(\tau)\sin\gamma\tau\left(\frac{|\lambda_l|}{\sigma^2|\lambda_l|+1}\right)^{1/2}d\tau-$$

$$-\sin\gamma t\left(\frac{|\lambda_l|}{\sigma^2|\lambda_l|+1}\right)^{1/2}\int_t^{\infty}f_l(\tau)\cos\gamma\tau\left(\frac{|\lambda_l|}{\sigma^2|\lambda_l|+1}\right)^{1/2}d\tau+$$

$$\left.\left.+\cos\gamma t\left(\frac{|\lambda_l|}{\sigma^2|\lambda_l|+1}\right)^{1/2}\int_t^{\infty}f_l(\tau)\sin\gamma\tau\left(\frac{|\lambda_l|}{\sigma^2|\lambda_l|+1}\right)^{1/2}d\tau\right]\right\}\varphi_l(x)\equiv$$

$$\equiv W_1(x,t)+W_2(x,t)\equiv W_1^{(1)}(x,t)+W_1^{(2)}(x,t)+$$

$$+W_1^{(3)}(x,t)+W_1^{(4)}(x,t)+W_2^{(1)}(x,t)+W_2^{(2)}(x,t)\ , \tag{4.1}$$

where we denote by $W_1(x,t)$ the sum of the first four series in (4.1) which are denoted by $W_1^j(x,t)$ respectively $(j=1,2,3,4)$ and we denote by $W_2(x,t)$ the sum of the last two series in (4.1) which are denoted by $W_2^{(v)}(x,t)$, $v=1,2$. Let

$$W_1^{(1)}(x,t)=\frac{1}{\gamma}\sum_{l=1}^{\infty}\tilde{c}_l^{(1)}\left(\frac{\sigma^2|\lambda_l|+1}{|\lambda_l|}\right)^{1/2}\sin\gamma t\left(\frac{|\lambda_l|}{\sigma^2|\lambda_l|+1}\right)^{1/2}\varphi_l(x)\ .$$

Then

$$\left\|W_1^{(1)}(x,t)\right\|_{C(\bar{\Omega})}\leq C(\sigma,\gamma)\sum_{l=1}^{\infty}\left|\tilde{c}_l^{(1)}\right|\ \|\varphi_l(x)\|_{C(\bar{\Omega})}\,. \tag{4.2}$$

From estimate (2.21) we obtain

$$\left\|W_1^{(1)}(x,t)\right\|_{C(\bar{\Omega})}\leq C(\sigma,\gamma)\sum_{l=1}^{\infty}\left|\tilde{c}_l^{(1)}\right|\ |\lambda_l|^{\frac{1}{2}\left(\left[\frac{n}{2}\right]+1\right)}\leq$$

$$\leq C(\sigma,\gamma)\left[\sum_{l=1}^{\infty}\left|\tilde{c}_l^{(1)}\right|^2|\lambda_l|^{2\left(\left[\frac{n}{2}\right]+1\right)}+\sum_{l=1}^{\infty}|\lambda_l|^{-\left(\left[\frac{n}{2}\right]+1\right)}\right]\ , \tag{4.3}$$

here and later on $C(\sigma,\gamma)$ is a constant depending on parameters σ,γ. By virtue of theorem 8 from [8] (p.253) we have

$$\sum_{l=1}^{\infty}\left|\tilde{c}_l^{(1)}\right|^2|\lambda_l|^{2\left(\left[\frac{n}{2}\right]+1\right)}\leq\|\psi_1(x)\|_{H^{2\left(\left[\frac{n}{2}\right]+1\right)}(\Omega)}\ . \tag{4.4}$$

Reasoning as in the previous theorem and using estimate (2.20), for eigenvalues we obtain

$$\sum_{l=1}^{\infty}|\lambda_l|^{-\left(\left[\frac{n}{2}\right]+1\right)}\leq C\sum_{l=1}^{\infty}l^{-\left(1+\frac{1}{n}\right)}\ , \tag{4.5}$$

the last series converges. It follows from (4.2)-(4.5) that the series in expression $W_1^{(1)}(x,t)$ uniformly converges with respect to x,t. One can analogously prove the uniform convergence of series

$$W_1^{(2)}(x,t)=\sum_{l=1}^{\infty}\tilde{c}_l^{(0)}\cos\gamma t\left(\frac{|\lambda_l|}{\sigma^2|\lambda_l|+1}\right)^{1/2}\varphi_l(x)$$

with respect to x,t.

Now consider

$$W_1^{(3)}(x,t)=-\frac{1}{\gamma}\sum_{l=1}^{\infty}\frac{\varphi_l(x)}{\left[|\lambda_l|\left(\sigma^2|\lambda_l|+1\right)\right]^{1/2}}\sin\gamma t\left(\frac{|\lambda_l|}{\sigma^2|\lambda_l|+1}\right)^{1/2}\times$$

$$\times\int_0^{\infty}f_l(\tau)\cos\gamma\tau\left(\frac{|\lambda_l|}{\sigma^2|\lambda_l|+1}\right)^{1/2}d\tau\ .$$

Estimating the norm and using estimate (2.21) we obtain

$$\left\|W_1^{(3)}(x,t)\right\|_{C(\bar{\Omega})}\leq C(\sigma,\gamma)\sum_{l=1}^{\infty}|\lambda_l|^{-1}\|\varphi_l(x)\|_{C(\bar{\Omega})}\int_0^{\infty}|f_l(\tau)|\,d\tau\leq$$

$$\leq C(\sigma,\gamma)\sum_{l=1}^{\infty}|\lambda_l|^{\frac{1}{2}\left(\left[\frac{n}{2}-1\right]\right)}\int_0^{\infty}|f_l(\tau)|\,d\tau\leq$$

$$\leq C(\sigma,\gamma)\left\{\sum_{l=1}^{\infty}|\lambda_l|^{2\left[\frac{n}{2}\right]}\left[\int_0^{\infty}|f_l(\tau)|\,d\tau\right]^2+\sum_{l=1}^{\infty}|\lambda_l|^{-\left(\left[\frac{n}{2}\right]+1\right)}\right\}\ .$$

By virtue of Cauchy-Bunyakovskii inequality we have

$$\left\|W_1^{(3)}(x,t)\right\|_{C(\bar{\Omega})}\leq$$

$$\leq C(\sigma,\gamma)\left\{\sum_{l=1}^{\infty}|\lambda_l|^{2\left[\frac{n}{2}\right]}\int_0^{\infty}(1+\tau)^{-2}\,d\tau\int_0^{\infty}(1+\tau)^{-2}|f_l(\tau)|^2\,d\tau+\sum_{l=1}^{\infty}|\lambda_l|^{-\left(\left[\frac{n}{2}\right]+1\right)}\right\}$$

further, by virtue of estimate (2.19) and theorem 8 from [8] (p.253) we obtain

$$\left\| W_1^{(3)}(x,t) \right\|_{C(\bar{\Omega})} \leq$$

$$\leq C(\sigma,\gamma)\left\{ \int_0^\infty (1+\tau)^2 \left[\sum_{l=1}^\infty |\lambda_l|^{2\left[\frac{n}{2}\right]} |f_l(\tau)|^2 \right] d\tau + \sum_{l=1}^\infty |\lambda_l|^{-\left(\left[\frac{n}{2}\right]+1\right)} \right\} \leq \tag{4.6}$$

$$\leq C(\sigma,\gamma)\left\{ \int_0^\infty (1+\tau)^2 \left\| f(x,\tau) \right\|^2_{H^{2\left[\frac{n}{2}\right]}(\Omega)} d\tau + \sum_{l=1}^\infty l^{-\left(1+\frac{1}{n}\right)} \right\} .$$

because in the space $L_2(0,\infty)$ we can change the order of integration and summation.

Similarly we estimate

$$W_1^{(4)}(x,t) = -\frac{1}{\gamma}\sum_{l=1}^\infty \frac{\varphi_l(x)}{\left[|\lambda_l|\left(\sigma^2|\lambda_l|+1\right)\right]^{1/2}} \cos\gamma t\left(\frac{|\lambda_l|}{\sigma^2|\lambda_l|+1}\right)^{1/2} \times$$

$$\times \int_0^\infty f_l(\tau)\sin\gamma\tau\left(\frac{|\lambda_l|}{\sigma^2|\lambda_l|+1}\right)^{1/2} d\tau .$$

Proceeding in the same way as for estimation of $W_1^{(3)}(x,t)$ we obtain

$$\left\| W_1^{(4)}(x,t) \right\|_{C(\bar{\Omega})} \leq C(\sigma,\gamma)\times$$
$$\times\left\{ \int_0^\infty (1+\tau)^2 \left\| f(x,\tau) \right\|^2_{H^{2\left[\frac{n}{2}\right]}(\Omega)} d\tau + \sum_{l=1}^\infty l^{-\left(1+\frac{1}{n}\right)} \right\} . \tag{4.7}$$

Let us estimate now $W_2^{(1)}(x,t)$ and $W_2^{(2)}(x,t)$.

Since

$$W_2^{(1)}(x,t) = -\frac{1}{\gamma}\sum_{l=1}^\infty \frac{\varphi_l(y)}{\left[|\lambda_l|\left(\sigma^2|\lambda_l|+1\right)\right]^{1/2}} \sin\gamma t\left(\frac{|\lambda_l|}{\sigma^2|\lambda_l|+1}\right)^{1/2} \times$$

$$\times \int_0^\infty f_l(\tau)\cos\gamma\tau\left(\frac{|\lambda_l|}{\sigma^2|\lambda_l|+1}\right)^{1/2} d\tau$$

then we have

$$\left\| W_2^{(1)}(x,t) \right\|_{C(\bar{\Omega})} \leq C(\sigma,\gamma)\sum_{l=1}^\infty |\lambda_l|^{-1}\int_t^\infty |f_l(\tau)|\, d\tau .$$

By virtue of Cauchy-Bunyakovskii inequality we have

$$\left\| W_2^{(1)}(x,t) \right\|_{C(\bar\Omega)} \le C(\sigma,\gamma) \sum_{l=1}^{\infty} |\lambda_l|^{-1} \left(\int_t^{\infty} (1+\tau)^{-2} d\tau \right)^{1/2} \times$$

$$\times \left(\int_t^{\infty} (1+\tau)^2 |f_l(\tau)|^2 d\tau \right)^{1/2} \le C(\sigma,\gamma) \times$$

$$\times \left\{ \int_t^{\infty} (1+\tau)^{-2} d\tau \sum_{l=1}^{\infty} |\lambda_l|^{-\left(\left[\frac{n}{2}\right]+1\right)} + \sum_{l=1}^{\infty} |\lambda_l|^{\left[\frac{n}{2}\right]} \int_t^{\infty} (1+\tau)^2 |f_l(\tau)|^2 d\tau \right\} .$$

Reasoning as above, we obtain

$$\left\| W_2^{(1)}(x,t) \right\|_{C(\bar\Omega)} \le C(\sigma,\gamma) \times$$

$$\times \left\{ \int_t^{\infty} (1+\tau)^{-2} d\tau \sum_{l=1}^{\infty} l^{-\left(1+\frac{1}{n}\right)} + \int_t^{\infty} (1+\tau)^2 \|f(x,\tau)\|^2_{H^{\left[\frac{n}{2}\right]}(\Omega)} d\tau \right\} .$$

This estimate implies that

$$\left\| W_2^{(1)}(x,t) \right\|_{C(\bar\Omega)} \to 0 \tag{4.8}$$

as $t \to +\infty$.

In the same way we prove that

$$\left\| W_2^{(2)}(x,t) \right\|_{C(\bar\Omega)} \to 0 \tag{4.9}$$

as $t \to +\infty$.

By virtue of theorem 1.1.5 from the paper [11] (p.26) the uniform convergence of series in $W_1^{(j)}(x,t)$ $j=1,2,3,4$ implies that their sum $\sum_{j=1}^{4} W_1^{(j)}(x,t)$ is a uniform almost periodic function with respect to t. The proof of theorem 3 follows from (4.8), (4.9).

The following theorem follows from formula (3.6) which represents solution of mixed problem (1.1), (1.2), (3.1).

Theorem 4. *Let the conditions of theorem 3 be fulfilled and, moreover,*

$$\int_{\Omega} \psi_1(x)\,dx = 0\ , \quad \int_{\Omega} f(x,t)\,dx = 0\ . \tag{4.10}$$

Then for solution of mixed problem (1.1), (1.2), (3.1) the assertion of theorem 3 is also valid.

Under the mentioned conditions (4.10) theorem 4 is proved in the same way as theorem 3. If conditions (4.10) are not satisfied, then from formula (3.6) it follows, that solution of mixed problem (1.1), (1.2), (3.1) as $t \to +\infty$ at the conditions of theorem 3 will be the sum of linear and uniform almost-periodic function with respect to t function.

Estimate of derivatives of solution of mixed problem (1.1), (1.2), (3.1) is realized in the same way as for solution of problem (1.1), (1.2), (1.3).

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