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ON SOLVABILITY OF ONE BOUNDARY-VALUE PROBLEM FOR THE SECOND ORDER OPERATOR-DIFFERENTIAL EQUATION

Abstract

In the paper the sufficient conditions are obtained in terms of coefficients of one class operator-differential equation of the second order of elliptic type. These conditions uniquely and correctly provide the solvability of some boundary-value problem.

Let H be a separable Hilbert space, A be invertible operator in H. Then A has polar expansion $A = U\,|A|$, where U is a unitary operator, $|A|$ is positively defined operator in H. Denote by H_α the scale of Hilbert space generated by the operator $|A|$, i.e., $H_\alpha = D\left(|A|^\alpha\right),\ (\varphi,\psi)_\alpha = \left(|A|^\alpha \varphi,\ |A|^\alpha \psi\right),\ \varphi,\psi \in D\left(|A|^\alpha\right).$

Let $t \in R^1_+ = (0,\infty),\ x \in R^1 = (-\infty,\infty)$ and consider in the half-space $R^2_+ = (0,,\infty)\times(-\infty,\infty) \equiv R^1_+ \times R^1$ the boundary-value problem

$$\begin{cases} -\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + A^2 u + A_{1,0}\frac{\partial u}{\partial t} + A_{0,1}\frac{\partial u}{\partial x} + A_{1,1}\frac{\partial^2 u}{\partial t\partial x} + A_{0,0}u = f, \quad (t,x)\in R^2_+ & (1) \\ u(0,x) = 0, & (2) \end{cases}$$

where relative to the operators $A, A_{1,0}, A_{0,1}A_{1,1}, A_{0,0}$ the fulfilment of the following conditions is assumed:

1) A is a normal invertible operator, whose spectrum is contained in the angular domain $S_\varepsilon \left\{\lambda : |\arg\lambda| \le \varepsilon\right\},\ 0 \le \varepsilon < \frac{\pi}{2}$;

2) The operators $A_{1,0}A^{-1}, A_{0,1}A^{-1}, A_{1,1}, A_{0,0}A^{-2}$ are bounded in H.

We assume, that $f(t,x) \in L_2\left(R^2_+;H\right)$ where $L_2\left(R^2_+;H\right)$ is Hilbert space of vector-functions defined on R^2_+ with the value from H measurable with the finite norm

$$\|f\|_{L_2\left(R^2_+;H\right)} = \left(\int_0^\infty\int_{-\infty}^{+\infty} \|f(t,x)\|^2\,dtdx\right)^{1/2} < \infty,$$

and $u(t,x) \in W^2_2\left(R^2_+;H\right)$, where $W^2_2\left(R^2_+;H\right)$ is a Hilbert space of vector-functions obtained by the completion of infinitely differentiable vector-functions with the value from H_2 having compact supports in R^2_+ with the norm

$$\|u\|_{W^2_2\left(R^2_+;H\right)} = \left(\left\|\frac{\partial^2 u}{\partial t^2}\right\|^2_{L_2\left(R^2_+;H\right)} + \left\|\frac{\partial^2 u}{\partial x^2}\right\|^2_{L_2\left(R^2_+;H\right)} + \left\|A^2 u\right\|^2_{L_2\left(R^2_+;H\right)}\right)^{1/2}.$$

Denote by

$$\mathring{W}_2^2\left(R_+^2;H\right)=\left\{u\left|u\in W_2^2\left(R_+^2;H\right),\quad u\left(0,x\right)=0\right.\right\}.$$

Besides this we have to consider the space $L_2\left(R_+^1;H\right)$ (see [1]) and

$$W_2^2\left(R_+^1;H\right)=\left\{\vartheta\left|\frac{d^2\vartheta}{dt^2}\in L_2\left(R_+^1;H\right),\ A^2\vartheta\in L_2\left(R_+^1;H\right)\right.\right\}$$

and

$$\mathring{W}_2^2\left(R_+^1;H\right)=\left\{\vartheta\left|\vartheta\left(t\right)\in W_2^2\left(R_+^1;H\right),\ \vartheta\left(0\right)=0\right.\right\}$$

with the norm

$$\left\|\vartheta\right\|_{W_2^2\left(R_+^1;H\right)}=\left(\left\|\frac{d^2\vartheta}{dt^2}\right\|_{L_2\left(R_+^1;H\right)}^2+\left\|A^2\vartheta\right\|_{L_2\left(R_+^1;H\right)}^2\right)^{1/2}.$$

The spaces $W_2^2\left(R^2;H\right)$ and $W_2^2\left(R^1;H\right)$ are determined analogously.

Definition 1. *If at any* $f\left(t,x\right)\in L_2\left(R_+^2;H\right)$, *there exists the vector-function* $u\left(t,x\right)\in W_2^2\left(R_+^2;H\right)$ *which satisfies equation (1) almost everywhere in* R_+^2, *the boundary condition (2) in the sense*

$$\lim_{t\to+0}\left\|u\left(t,x\right)\right\|_{3/2}=0,$$

and the inequality

$$\left\|u\right\|_{W_2^2\left(R_+^2;H\right)}\le const\left\|f\right\|_{L_2\left(R_+^2;H\right)},$$

then the problem (1), (2) we'll call regularly solvable.

In the present paper we'll find the sufficient conditions which provide regularly the solvability of the problem (1), (2). Note, that when A is a self-adjoint operator this problem is considered in [2], equation (1) is considered in [3, 4], and problem (1), (2) when the norm of disturbed part is sufficiently small in finite domain in [5]. In one-dimensional case the similar problems are thoroughly studied for example in [6, 7].

Denote by

$$P_0u=-\frac{\partial^2u}{\partial t^2}-\frac{\partial^2u}{\partial x^2}+A^2u,\quad u\in\mathring{W}_2^2\left(R_+^2;H\right),$$

$$P_1u=A_{1,0}\frac{\partial u}{\partial t}+A_{0,1}\frac{\partial u}{\partial x}+A_{1,1}\frac{\partial^2u}{\partial t\partial x}+A_{0,0}u,\quad u\in\mathring{W}_2^2\left(R_+^2;H\right)$$

and at any $\xi\in R^1$

$$L_0\left(\xi\right)\vartheta=-\frac{\partial^2\vartheta}{dt^2}+\left(\xi^2E+A^2\right)\vartheta,\quad\vartheta\in\mathring{W}_2^2\left(R_+^1;H\right).$$

It holds the following lemma.

Lemma 1. *Let the condition 1) be fulfilled. Then the operator* $L_0(\xi)$ *at any* $\xi \in R^1$ *maps the space* $\overset{\circ}{W}{}_2^2\left(R_+^1;H\right)$ *on* $L_2\left(R_+^1;H\right)$ *isomorphically.*

Proof. It is evident, that at any $\xi \in R^1$ the operator $\left(\xi^2 E + A^2\right)^{1/2}$ is normal and its spectrum is contained in the angular sector $S_\varepsilon = \{\lambda : |\arg \lambda| \le \varepsilon\}$, $0 \le \varepsilon < \frac{\pi}{2}$. It is easy to see, that the equation $L_0(\xi)\vartheta = 0$ has the general solution from $W_2^2\left(R_+^1 : H\right)$ in the form $\vartheta_0(t) = e^{-\left(\xi^2 E + A^2\right)^{1/2} t}\varphi$, where $\varphi \in H_{3/2}$. From the condition $\vartheta(0) = 0$ it follows, that $\vartheta_0(t) = 0$. Show, that at any $g(t) \in L_2\left(R_+^1;H\right)$ the equation $L_0(\xi)\vartheta = g$ has the solution from the space $\overset{\circ}{W}{}_2^2\left(R_+^1;H\right)$.

Really,

$$\vartheta_1(t,\xi) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\eta^2 E + \xi^2 E + A^2\right)^{-1}\int_{-\infty}^{+\infty} g(s)\, e^{i\eta(t-s)} ds d\eta$$

satisfies the equation $L_0(\xi)\vartheta = g$. From the Plansharel theorem it follows, that $\vartheta_1(t,\xi) \in W_2^2\left(R^1;H\right)$.

Really, it is easy to see, that

$$\left\|\frac{d^2\vartheta}{dt^2}\right\|^2_{L_2(R^1;H)} + \left\|A^2\vartheta\right\|^2_{L_2(R^1;H)} = \left\|\eta^2\left(\eta^2 E + \xi^2 E + A^2\right)^{-1}\hat{g}(\eta)\right\|^2_{L_2(R^1;H)} +$$

$$+\left\|A^2\left(\eta^2 E + \xi^2 E + A^2\right)^{-1}\hat{g}(\eta)\right\|^2_{L_2(R^1;H)} \le$$

$$\le \sup_{\eta\in R^1}\left\|\eta^2\left(\eta^2 E + \xi^2 E + A^2\right)^{-1}\right\|^2 \cdot \|\hat{g}\|^2_{L_2} +$$

$$+\sup_{\eta\in R^1}\left\|\eta^2\left(\eta^2 E + \xi^2 E + A^2\right)^{-1}\right\|^2 \|\hat{g}\|^2_{L_2},$$

where $\hat{g}(\eta) -$ is Fourier transformation of the vector-function $g(t)$. From the spectral expansion of the operator A it follows, that

$$\sup_{\eta\in R^1}\left\|\eta^2\left(\eta^2 E + \xi^2 E + A^2\right)^{-1}\right\| \leqslant c_0(\varepsilon),$$

$$\sup_{\eta\in R^1}\left\|A^2\left(\eta^2 E + \xi^2 E + A^2\right)^{-1}\right\| \leqslant c_0(\varepsilon),$$

where $c_0(\varepsilon) = 1$, at $0 \leqslant \varepsilon \leqslant \frac{\pi}{4}, c_0(\varepsilon) = (2\cos\varepsilon)^{-1}$, at $\frac{\pi}{4} \leqslant \varepsilon < \frac{\pi}{2}$ (see the proof of inequality (1.2)). Then using Plansharel's theorem we obtain, that $\vartheta_1(t,\xi) \in W_2^2\left(R^1;H\right)$. Further, denote by $\vartheta_2(t,\xi)$ the contraction of the vector-function

$\vartheta_1(t,\xi)$ on $R_+^1=[0,\infty)$. Then $\vartheta_2(t,\xi)\in W_2^2\left(R_+^1;H\right)$, and $\vartheta_2(0,\xi)\in H_{3/2}$. Then the general solution of the equation $L_0(\xi)\vartheta=g$ has the form

$$\vartheta(t,\xi)=\vartheta_2(t,\xi)+e^{-\left(\xi^2E+A^2\right)^{1/2}t}\varphi,$$

where $\varphi\in H_{3/2}$. From the condition $\vartheta(0,\xi)=0$, it follows, that $\varphi=-\vartheta_2(0,\xi)$. Thus, the domain of the values of operator $L_0(\xi)$ coincides with the space $L_2\left(R_+^1;H\right)$. Then, the assertion of the lemma follows from the Banach theorem on the inverse operator. The lemma is proved.

Lemma 2. *At any $\xi\in R^1$ and $\vartheta\in \mathring{W}_2^2\left(R_+^1;H\right)$ it holds the inequality*

$$\|L_0(\xi)\vartheta\|^2_{L_2\left(R_+^1;H\right)}\geqslant\left\|\left(\xi^2E+A^2\right)\vartheta\right\|^2_{L_2\left(R_+^1;H\right)}+\left\|\frac{d^2\vartheta}{dt^2}\right\|^2_{L_2\left(R_+^1;H\right)}+$$

$$+2\cos 2\varepsilon\left\|\left(\xi^2E+A^2\right)^{1/2}\frac{d\vartheta}{dt}\right\|^2_{L_2\left(R_+^1;H\right)}\tag{3}$$

Proof. It is evident, that at any $\vartheta\in \mathring{W}_2^2\left(R_+^1;H\right)$ it holds the equality

$$\|L_0(\xi)\vartheta\|^2_{L_2\left(R_+^1;H\right)}=\left\|\left(\xi^2E+A^2\right)\vartheta\right\|^2_{L_2\left(R_+^1;H\right)}+\left\|\frac{d^2\vartheta}{dt^2}\right\|^2_{L_2\left(R_+^1;H\right)}-$$

$$-2\operatorname{Re}\left(\left(\xi^2E+A^2\right)\vartheta,\frac{d^2\vartheta}{dt^2}\right)_{L_2\left(R_+^1;H\right)}.\tag{4}$$

Since at any $\xi\in R^1$ and $\vartheta\in \mathring{W}_2^2\left(R_+^1;H\right)$, then after integrating by parts we have

$$-2\operatorname{Re}\left(\left(\xi^2E+A^2\right)\vartheta,\frac{d^2\vartheta}{dt^2}\right)_{L_2\left(R_+^1;H\right)}=$$

$$=2\operatorname{Re}\left(\left(\xi^2E+A^2\right)^{1/2}\frac{d\vartheta}{dt},\left(\xi^2E+A^{*2}\right)^{1/2}\frac{d\vartheta}{dt}\right)_{L_2\left(R_+^1;H\right)}\geqslant$$

$$\geqslant 2\cos 2\varepsilon\left\|\left(\xi^2E+A^2\right)^{1/2}\frac{d\vartheta}{dt}\right\|^2_{L_2\left(R_+^1;H\right)}$$

Allowing for this inequality in (4) we obtain the assertion of the lemma.

Lemma 3. *At any $\xi\in R^1$ and $\vartheta\in \mathring{W}_2^2\left(R_+^1;H\right)$ it hold the following estimations*

$$\left\|A^2\vartheta\right\|_{L_2\left(R_+^1;H\right)}\leqslant c_0^2(\varepsilon)\|L_0(\xi)\vartheta\|_{L_2\left(R_+^1;H\right)},\tag{5}$$

$$\left\|A\frac{d\vartheta}{dt}\right\|_{L_2\left(R_+^1;H\right)}\leqslant c_0^{1/4}(\varepsilon)\,c_1(\varepsilon)\|L_0(\xi)\vartheta\|_{L_2\left(R_+^1;H\right)},\tag{6}$$

$$\|i\xi A\vartheta\|_{L_2\left(R_+^1;H\right)}\leqslant c_0(\varepsilon)\,c_1(\varepsilon)\|L_0(\xi)\vartheta\|_{L_2\left(R_+^1;H\right)},\tag{7}$$

$$\left\| i\xi \frac{d\vartheta}{dt} \right\|_{L_2\left(R_+^1;H\right)} \leqslant c_0(\varepsilon)\, c_1(\varepsilon) \left\| L_0(\xi)\vartheta \right\|_{L_2\left(R_+^1;H\right)}, \tag{8}$$

where

$$c_0(\varepsilon) = \begin{cases} 1, & 0 \le \varepsilon \le \frac{\pi}{4} \\ \frac{1}{\sqrt{2}\cos\varepsilon}, & \pi/4 \le \varepsilon < \frac{\pi}{2}, \end{cases} \tag{9}$$

$$c_1(\varepsilon) = \frac{1}{2\cos\varepsilon}, \qquad 0 \le \varepsilon < \frac{\pi}{2}. \tag{10}$$

Proof. Since the normal operator $\left(\xi^2 E + A^2\right)^{1/2}$ has polar expansion $U(\xi)\left|\left(\xi^2 E + A^2\right)^{1/2}\right|$, where $U(\xi)$ is unitary at any $\xi \in R^1$ and $\left|\left(\xi^2 E + A^2\right)^{1/2}\right|$ is a positive part of the operator $\left(\xi^2 E + A^2\right)^{1/2}$. It is evident, that at any $\xi \in R^1$ and $\overset{\circ}{W}{}_2^2\left(R_+^1;H\right)$

$$\left\| \left|\left(\xi^2 E + A^2\right)^{1/2}\right| \frac{d\vartheta}{dt} \right\|_{L_2}^2 = \int_0^\infty \left(\left|\xi^2 E + A^2\right|^{1/2} \frac{d\vartheta}{dt}, \left|\xi^2 E + A^2\right|^{1/2} \frac{d\vartheta}{dt} \right) dt =$$

$$-\int_0^\infty \left(\left|\xi^2 E + A^2\right| \vartheta, \frac{d^2\vartheta}{dt^2} \right) dt \le \left\| \left|\xi^2 E + A^2\right| \vartheta \right\|_{L_2\left(R_+^1;H\right)} \cdot \left\| \frac{d^2\vartheta}{dt^2} \right\|_{L_2\left(R_+^1;H\right)} \le .$$

$$\le \frac{1}{2} \left(\left\| \left(\xi^2 E + A^2\right)\vartheta \right\|_{L_2}^2 + \left\| \frac{d^2\vartheta}{dt^2} \right\|_{L_2}^2 \right).$$

Allowing for inequality (3) in the last inequality we obtain

$$\left\| \left(\xi^2 E + A^2\right)^{1/2} \frac{d\vartheta}{dt} \right\|_{L_2\left(R_+^1;H\right)}^2 \le$$

$$\le \frac{1}{2} \left(\left\| L_0(\xi)\vartheta \right\|_{L_2\left(R_+^1;H\right)}^2 - 2\cos 2\varepsilon \left\| \left(\xi^2 E + A^2\right)^{1/2} \frac{d\vartheta}{dt} \right\|_{L_2\left(R_+^1;H\right)}^2 \right).$$

Hence we obtain that

$$(1+\cos 2\varepsilon) \left\| \left(\xi^2 E + A^2\right)^{1/2} \frac{d\vartheta}{dt} \right\|_{L_2\left(R_+^1;H\right)}^2 \le \frac{1}{2} \left\| L_0(\xi)\vartheta \right\|_{L_2\left(R_+^1;H\right)}^2$$

or

$$\left\| \left(\xi^2 E + A^2\right)^{1/2} \frac{d\vartheta}{dt} \right\|_{L_2\left(R_+^1;H\right)}^2 \le \frac{1}{4\cos^2\varepsilon} \left\| L_0(\xi)\vartheta \right\|_{L_2\left(R_+^1;H\right)}^2. \tag{11}$$

It is evident, that

$$\left\| A \frac{d\vartheta}{dt} \right\|_{L_2\left(R_+^1;H\right)}^2 \le \sup_{\xi \in R^1} \left\| A\left(\xi^2 E + A^2\right)^{-1/2} \right\| \cdot \left\| \left(\xi^2 E + A^2\right)^{1/2} \frac{d\vartheta}{dt} \right\|_{L_2\left(R_+^1;H\right)}^2.$$

Using spectral expansion of the operator A we obtain that at any $\xi \in R^1$

$$\left\| A\left(\xi^2 E + A^2\right)^{-1/2} \right\| = \sup_{\mu>0} \left| \mu^2 \left(\mu^2 + \xi^2\right)^{-1/2} \right| \leq$$

$$\leq \sup_{\mu>0} \left| \mu \left(\mu^4 + \xi^4 + 2\xi^2\mu^2 \cos 2\varepsilon\right)^{-1/4} \right| .$$

Since at $0 \leq \varepsilon \leq \frac{\pi}{4}$, $\left| \mu \left(\mu^4 + \xi^4 + 2\xi^2\mu^2 \cos 2\varepsilon\right)^{-1/4} \right| \leq 1$ and at $\frac{\pi}{4} \leq \varepsilon < \frac{\pi}{2}$ $(\cos 2\varepsilon \leq 0)$ it holds the inequality

$$\left| \mu \left(\mu^4 + \xi^4 + 2\xi^2\mu^2 \cos 2\varepsilon\right)^{-1/4} \right| \leq \left| \mu \left(\xi^4 + \mu^4\right)^{-1/4} (1 + \cos 2\varepsilon)^{-1/4} \right| \leq$$

$$\leq \left(2\cos^2 \varepsilon\right)^{-1/4} = 2^{-1/4} \cos^{-1/2} \varepsilon,$$

then using inequality (11) we obtain

$$\left\| A\frac{d\vartheta}{dt} \right\|_{L_2}^2 \leq c_0^{1/2}(\varepsilon) \left\| \left(\xi^2 E + A^2\right)^{1/2} \frac{d\vartheta}{dt} \right\|_{L_2\left(R_+^1;H\right)}^2 \leq$$

$$\leq c_0^{1/2}(\varepsilon)\, c_1^2(\varepsilon) \left\| L_0(\xi)\, \vartheta \right\|_{L_2\left(R_+^1;H\right)}^2$$

or

$$\left\| A\frac{d\vartheta}{dt} \right\|_{L_2\left(R_+^1;H\right)} \leq c_0^{1/4}(\varepsilon)\, c_1(\varepsilon) \left\| L_0(\xi)\, \vartheta \right\|_{L_2\left(R_+^1;H\right)}^2 . \tag{12}$$

Thus, inequality (6) is proved.

From inequality (3) it follows that at $0 \leq \varepsilon \leq \frac{\pi}{4}$ it holds the inequality

$$\left\| \left(A^2 + \xi^2 E\right) \vartheta \right\|_{L_2\left(R_+^1;H\right)} \leq \left\| L_0(\xi)\, \vartheta \right\|_{L_2\left(R_+^1;H\right)}^2 . \tag{13}$$

And at $\pi/4 \leq \varepsilon < \pi/2$ from inequality (3) subject to inequality (12) it follows that

$$\left\| \left(A^2 + \xi^2 E\right) \vartheta \right\|_{L_2\left(R_+^1;H\right)} \leq \left\| L_0(\xi)\, \vartheta \right\|_{L_2\left(R_+^1;H\right)}^2 -$$

$$-2\cos 2\varepsilon \left\| \left(\xi^2 E + A^2\right)^{1/2} \frac{d\vartheta}{dt} \right\|_{L_2\left(R_+^1;H\right)}^2 \leq \left\| L_0(\xi)\, \vartheta \right\|_{L_2\left(R_+^1;H\right)}^2 -$$

$$-2\cos 2\varepsilon \cdot \frac{1}{4\cos^2 \varepsilon} \left\| L_0(\xi)\, \vartheta \right\|_{L_2\left(R_+^1;H\right)}^2 \leq \frac{1}{2\cos^2 \varepsilon} \left\| L_0(\xi)\, \vartheta \right\|_{L_2\left(R_+^1;H\right)}^2 .$$

Thus

$$\left\| \left(A^2 + \xi^2 E\right) \vartheta \right\|_{L_2\left(R_+^1;H\right)} \leq c_0(\varepsilon) \left\| L_0(\xi)\, \vartheta \right\|_{L_2\left(R_+^1;H\right)} . \tag{14}$$

Hence we have

$$\left\| A^2 \vartheta \right\|_{L_2} \leq \sup_{\xi \in R^1} \left\| A^2 \left(A^2 + \xi^2\right)^{-1} \right\| \cdot \left\| \left(A^2 + \xi^2 E\right) \vartheta \right\|_{L_2} \leq$$

$$\leq \sup_{\xi\in R^1}\sup_{\mu\geq 0}\left|\mu^2\left(\mu^2+\xi^2\right)^{-1}\right| c_0(\varepsilon)\left\|L_0(\xi)\vartheta\right\|_{L_2\left(R_+^1;H\right)} \leq$$

$$\leq c_0^2(\varepsilon)\left\|L_0(\xi)\vartheta\right\|_{L_2\left(R_+^1;H\right)}.$$

Inequality (5) is proved.

It is evident, that

$$\left\|i\xi A\vartheta\right\|^2_{L_2\left(R_+^1;H\right)} \leq \left\|i\xi A\left(\xi^2E+A^2\right)^{-1}\left(\xi^2E+A^2\right)\vartheta\right\|^2_{L_2\left(R_+^1;H\right)} \leq$$

$$\leq \sup_{\xi\in R^1}\left\|\xi A\left(\xi^2E+A^2\right)^{-1}\right\|^2\left\|\xi^2E+A^2\vartheta\right\|^2_{L_2\left(R_+^1;H\right)}. \tag{15}$$

Using the spectral expansion of the operator A we obtain that

$$\left\|\xi A\left(\xi^2E+A^2\right)^{-1}\right\| \leq \sup_{\mu>0}\left|\xi\mu\left(\xi^2+\mu^2\right)^{-1}\right| =$$

$$= \sup_{\mu>0}\left|\xi\mu\left(\xi^4+\mu^4+2\xi^2\mu^2\cos 2\varepsilon\right)^{-1/2}\right| \leq$$

$$\leq \sup_{\mu>0}\left|\xi\mu\left(2\xi^2\mu^2\left(1+\cos 2\varepsilon\right)\right)^{-1/2}\right| \leq \left(2\cos 2\varepsilon\right)^{-1}.$$

Allowing for this inequality and inequality (14) in inequality (15) we have

$$\left\|i\xi A\vartheta\right\|_{L_2\left(R_+^1;H\right)} \leq c_1(\varepsilon)\, c_0(\varepsilon)\left\|L_0(\xi)\vartheta\right\|_{L_2}.$$

Inequality (7) is also proved.

Let's prove inequality (8). Using inequality (11) we obtain

$$\left\|i\xi\frac{d\vartheta}{dt}\right\|^2_{L_2\left(R_+^1;H\right)} \leq \sup_{\xi\in R^1}\left\|\xi\left(\xi^2+A^2\right)^{-1/2}\right\|\left\|\left(\xi^2+A^2\right)^{1/2}\frac{d\vartheta}{dt}\right\|_{L_2\left(R_+^1;H\right)} \leq$$

$$\leq \sup_{\xi\in R^1}\sup_{\mu>0}\left|\xi\left(\xi^2+\mu^2\right)^{-1}\right| c_1(\varepsilon)\left\|L_0(\xi)\vartheta\right\|_{L_2\left(R_+^1;H\right)} =$$

$$= c_0(\varepsilon)\, c_1(\varepsilon)\left\|L_0(\xi)\vartheta\right\|_{L_2\left(R_+^1;H\right)}.$$

The lemma is proved.

Theorem 1. *The operator $P_0 : \overset{\circ}{W}{}_2^2\left(R_+^2;H\right) \to L_2\left(R_+^2;H\right)$ is an isomorphism.*

Proof. After the Fourier transformation by the variable x the equation $P_0u = 0$ has the form $L_0(\xi)\,\hat{u}(t,\xi) = 0$. By lemma 1 $\hat{u}(t,\xi) = 0$, i.e., $u(t,x) = 0$. Thus $KerP_0 = \{0\}$. Show that the domain of value P_0 coincides with the $L_2\left(R_+^2;H\right)$. Consider the equation $P_0u = f$, $u \in \overset{\circ}{W}{}_2^2\left(R_+^2;H\right)$, $f \in L_2\left(R_+^2;H\right)$. After the Fourier transformation by the variable x we obtain the equation $L_0(\xi)\,\hat{u}(t,\xi) = \hat{f}(t,\xi)$. By lemma 1 $\hat{u}(t,\xi)$ at every $\xi \in R^1$ belongs to the space $\overset{\circ}{W}{}_2^2\left(R_+^1;H\right)$. From the form $\hat{u}(t,\xi)$ it is easily obtained that $\xi^2\hat{u}(t,\xi)$, $A^2\hat{u}(t,\xi)$ belongs to the space $L_2\left(R_+^1;H\right)$. Hence, it follows that $u(t,x)\in \mathring{W}_2^2\left(R_+^2;H\right)$. The theorem is proved.

Theorem 2. *Let conditions 1), 2) be fulfilled and it holds the inequality*

$$\alpha(\varepsilon)=c_0^{1/4}(\varepsilon)\,c_1(\varepsilon)\left\|A_{1,0}A^{-1}\right\|+$$

$$+c_0(\varepsilon)\,c_1(\varepsilon)\left(\left\|A_{0,1}A^{-1}\right\|+\left\|A_{1,1}\right\|\right)+c_0^2(\varepsilon)\left\|A_{0,0}A^{-2}\right\|<1,$$

where $c_0(\varepsilon),c_1(\varepsilon)$ *are determined from equality (9) and (10) respectively. Then problem (1), (2) is regularly solvable.*

Proof. Write problem (1), (2) in the form:

$$P_0u+P_1u=f,\ u\in \mathring{W}_2^2\left(R_+^2;H\right),\quad f\in L_2\left(R_+^2;H\right).$$

After the substitution $P_0u=0$ we obtain the equation $\left(E+P_1P_0^{-1}\right)\omega=f$ in the space $L_2\left(R_+^2;H\right)\left(\omega,f\in L_2\left(R_+^2;H\right)\right)$.

Since

$$\left\|P_1P_0^{-1}\omega\right\|_{L_2\left(R_+^2;H\right)}=\left\|P_1u\right\|_{L_2\left(R_+^2;H\right)}=$$

$$=\left\|A_{1,0}\frac{\partial u}{\partial t}+A_{0,1}\frac{\partial u}{\partial x}+A_{1,1}\frac{\partial^2 u}{\partial t\partial x}+A_{0,0}u\right\|_{L_2\left(R_+^2;H\right)}\le$$

$$\le\left\|A_{1,0}A^{-1}\right\|\left\|A\frac{\partial u}{\partial t}\right\|_{L_2\left(R_+^2;H\right)}+\left\|A_{0,1}A^{-1}\right\|\left\|A\frac{\partial u}{\partial x}\right\|_{L_2\left(R_+^2;H\right)}+$$

$$+\left\|A_{1,1}\right\|\left\|\frac{\partial^2 u}{\partial t\partial x}\right\|_{L_2\left(R_+^2;H\right)}+\left\|A_{0,0}A^{-2}\right\|\left\|A^2u\right\|_{L_2\left(R_+^2;H\right)}. \tag{16}$$

From inequality (6), applying the Plansharel theorem we obtain

$$\left\|A\frac{\partial u}{\partial t}\right\|_{L_2\left(R_+^2;H\right)}=\left\|A\frac{\partial \hat{u}(t,\xi)}{\partial t}\right\|_{L_2\left(R_+^2;H\right)}\le$$

$$\le c_0^{1/4}(\varepsilon)\,c_1(\varepsilon)\left\|L_0(\xi)\,\hat{u}(t,\xi)\right\|_{L_2\left(R_+^2;H\right)}=$$

$$=c_0^{1/4}(\varepsilon)\,c_1(\varepsilon)\left\|P_0u\right\|_{L_2\left(R_+^2;H\right)}=c_0^{1/4}(\varepsilon)\,c_1(\varepsilon)\left\|\omega\right\|_{L_2\left(R_+^2;H\right)}. \tag{17}$$

Analogously, we obtain

$$\left\|A\frac{\partial u}{\partial x}\right\|_{L_2\left(R_+^2;H\right)}=\left\|i\xi A\hat{u}(t,\xi)\right\|_{L_2\left(R_+^2;H\right)}\le c_0(\varepsilon)\,c_1(\varepsilon)\left\|\omega\right\|_{L_2\left(R_+^2;H\right)},$$

$$\left\|\frac{\partial^2 u}{\partial x\partial t}\right\|_{L_2\left(R_+^2;H\right)}=\left\|i\xi\frac{\partial \hat{u}(t,\xi)}{\partial t}\right\|_{L_2\left(R_+^2;H\right)}\le c_0(\varepsilon)\,c_1(\varepsilon)\left\|\omega\right\|_{L_2\left(R_+^2;H\right)},$$

$$\left\|A^2u\right\|_{L_2\left(R_+^2;H\right)}=\left\|A^2\hat{u}(t,\xi)\right\|_{L_2\left(R_+^2;H\right)}\le c_0^2(\varepsilon)\left\|\omega\right\|_{L_2\left(R_+^2;H\right)}.$$

Allowing for all these inequalities in inequality (16) we obtain

$$\left\|P_1 P_0^{-1}\omega\right\|_{L_2\left(R_+^2;H\right)} \leq \alpha\left(\varepsilon\right)\left\|\omega\right\|_{L_2\left(R_+^2;H\right)}.$$

Since by the condition of the theorem $\alpha(\varepsilon) < 1$, the operator $E + P_1 P_0^{-1}$ is invertible and

$$u = P_0^{-1}\left(E + P_1 P_0^{-1}\right)^{-1} f.$$

It is evident that

$$\left\|u\right\|_{W_2^2\left(R_+^2;H\right)} \leq const\left\|f\right\|_{L_2\left(R_+^2;H\right)}.$$

The theorem is proved.

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