Makhmud A. AKHMEDOV

THE IMBEDDING THEOREM FOR A CLASS OF WEIGHT ANISOTROPIC PREUDONORMED SPACES

Abstract

In this work the sufficient conditions are obtained on weight functions at which the imbedding of some classes of weight anisotropic pseudonormed spaces in the weight Lebesgue spaces holds.

The work is dedicated to investigation of some properties and questions of imbedding for a class of pseudonormed spaces [1]. Study of the nonlinear spaces caused by questions connected with nonlinear boundary value problems. Sufficiently many works are dedicated to this theme. Detailed study of properties of the preudonormed spaces and imbedding theorem for them, and its application of obtained results to differential equations have been considered by K.N.Soltanov in works [1-4].

The weight spaces appear also as the "classical" Sobolev spaces at studying the differential equations, but now with degeneration and singularity. Similarly,the nonlinear weight spaces appear at researching the nonlinear differential equations with coefficients which have degeneration and/or singularities .

In the works, mentioned above, the properties of nonlinear spaces in weightless case have been researched. The paper [5] was dedicated to the question of imbedding for weight nonlinear pn-space $S^1_{\alpha,p,q}(\Omega,\upsilon,h)$, in which sufficient conditions have been obtained on weight functions, for which corresponding imbedding holds.

In the given paper, applying the methods of the works [5-7] the research of some properties and questions of imbedding for one class of weight anisotropic pn-spaces are considered.There are obtained sufficient conditions on weight, in which imbedding theorems holds.The results of the present work generalize the results of the work [5], on wider class of weight pn-spaces.

The main concepts and notation of the paper are in section 1. Some preparatory results and main imbedding theorem are given in section 2.

1. Preliminary information.

Let $\mathbf{R}^n$ $(n \geq 1)$ be n - dimensional Euclidean space, Ω be a bounded set in $\mathbf{R}^n$.

Arbitrary measurable on Ω function ν, that $0 < \nu(x) < +\infty$ almost everywhere on Ω we call weight function or just weight. Set of all weights on Ω we denote by $W(\Omega)$. For $\nu \in W(\Omega)$, ν - weight measure of the measurable set $E \subset \Omega$ we shall denote by $|E|_\nu = \nu(E) = \int\limits_E \nu(x)\,dx$. At $\nu(x) \equiv 1$ the Lebesgue measure of the set E we denote by $|E|$.

For $\nu \in W(\Omega)$ the space of measurable on Ω functions $u = u(x)$, for which the norm:

$$|u : L_p(\Omega,\nu)| = \begin{cases} \left(\int\limits_\Omega |u(x)|^p \nu(x)\,dx\right)^{1/p} & at \quad 1 \leq kp < +\infty \\ ess\sup\limits_\Omega |u(x)|\,\nu(x) & at \quad p = +\infty \end{cases}$$

is finite, we denote by $L_p(\Omega;\nu)$.

Remark. In the definition of space $L_p(\Omega;\nu)$ as usually it's assumed that the question is on the classes of equivalencies relative to the Lebesgue zero measure.

We denote by $C^1(\Omega)$ the set of all continuously differentiable functions on Ω.

Let $0 \le p_i < +\infty$, $q_i < +\infty$, $\nu_i \in W(\Omega)$, $i = \overline{0,n}$. We introduce a notation for the function $u \in C^1(\Omega)$

$$\left|u : S^1_{\bar{p},\bar{q}}(\Omega,\bar{\nu})\right| = |u : L_{p_0+q_0}(\Omega,\nu_0)| +$$
$$+\sum_{i=1}^{n}\left(\int_{\Omega} v_i(x)\,|u(x)|^{p_i}\left|\frac{\partial u}{\partial x_i}(x)\right|^{q_i}dx\right)^{1/(p_{oi}+q_i)} \qquad (1)$$

where $\bar{p} = (p_o,...,p_n)$, $\bar{q} = (q_o,...,q_n)$, $\bar{\nu} = (\nu_o,...,\nu_n)$.

Let's consider a set of all $u \in C^1(\Omega)$, for which the expression (1) is finite. We denote by $S^1_{\bar{p},\bar{q}}$ closure of this set related to the pseudonorm (1) ([1, 2]).

As it was mentioned above the definitions of pseudonormed (pn) spaces and their properties are sufficiently stated in details in works [1-4]. Here we'll need just the following definition, which gives us the concept of imbedding in these spaces.

Let X, Y are local convergens topological spaces, B, $B \cap Y \ne \varnothing$ is Banach space and $g : X \to Y$ is a mapping from X into Y. We denote by $S_{gB}(X) = \{x : x \in X \text{ and } g(x) \in B \cap Y\}$.

Definition ([1,2]). *Let $S_1 = S_{g_1 b_1(x)}$ and $S_2 = S_{g_2 b_2}(x)$ be pn-spaces. We'll say that S_2 is imbedded to the S_1, $S_2 \subset S_1$, if the following relations hold:*

(i) $S_2 \subset S_1$ in the sense of sets theory;

(ii) There is a non-negative increasing function $\varphi(\tau) \ge 0$, that for any $x \in S_2$ the inequality

$$[x]_{S_1} \le \varphi\left([x]_{S_2}\right)$$

is valid, where $\varphi \uparrow$, at that $\varphi(0) = 0$, φ depends on mappings g_1 and g_2 and the pair of Banach spaces B_1 and B_2 and for $g_1 \equiv g_2 \implies \varphi(\tau) = C\tau$, $C > 0$ is a constant, and $[\cdot]_S$ is a pseudonorm in space S.

2. Imbedding theorem.

Before to formulate and to prove the main result of the work, we'll introduce some auxiliary statements.

Lemma 1. *Let $\Omega \subset \mathbf{R}^n$ be a bounded open set and $\beta > 1$ be some number. Then from family of closed balls*

$$\left\{\bar{B}(x, r(x)) : \beta r(x) = dist(x, \partial\Omega),\quad x \in \Omega\right\}$$

it is possible to choose at most countable subfamily $\left\{\bar{B}^k_j\right\}$ such, that

(i) $\Omega = \mathop{U}\limits_{k=1}^{\xi_n}\mathop{U}\limits_{j\ge1}\bar{B}^k_j$ where number ξ_n depends only on dimension of n.

(ii) $\bar{B}^k_i \cap \bar{B}^k_j = \varnothing$, $i \ne j$, $k = \overline{1,\xi_n}$.

Lemma 1 is a simple consequence of the Besichovitch's covering theorem (see, for example [8])

Lemma 2. *Let $1 \leq q < +\infty$, $B(y,r)$ be an arbitrary ball in $\mathbf{R}^n$ and $\omega \in W(B(y,r))$. Then for any $u \in C^1(B(y,r))$ such that $\int\limits_{B(y,r)} u(z)\,dz = 0$ an inequality*

$$|u : L_q(\Omega;\omega)| \leq C \sum_{k=1}^{n} \left| \int\limits_{B(y,r)} \frac{|D_k u(z)|}{|x-z|^{n-1}} : dx L_q(\Omega;\Omega) \right| \qquad (2)$$

holds, where C is a constant, depending only on n.

To prove lemma 2 it is enough to use the integral representation from the book [9] (lemma 1, page 436) and perform simple estimations.

Let's consider the integral operators

$$Ku(x) = \int\limits_{\Omega} k(x,y)\,u(y)\,dy \qquad x \in \Omega$$

$$K^* u(x) = \int\limits_{\Omega} k^*(x,y)\,u(y)\,dy \qquad x \in \Omega$$

where $k : \Omega \to \mathbf{R}$ is non-negative measurable function, $k^*(x,y) = k(y,x)$.

At $\omega, \nu \in W(\Omega)$. For operator K we suppose

$$[K]_{p\nu^{1-p'},q\omega} =$$

$$= \sup \left\{ \left| K\nu^{1-p'} \chi_Q : L_q(\Omega;\omega) \right| \; \left| \chi_Q : L_p\left(\Omega;\nu^{1-p'}\right) \right|^{-1} : dyadic\ Q \subset \Omega \right\}, \qquad (3)$$

where upper boundary is taken by all diadic cubes $Q \subset \Omega$.

For the kernel k of operator K we introduce the quantity

$$[k]_{p\nu^{1-p'},p\omega} = \sup_{x\in\Omega} \sup_{r>0} \left\{ |\Omega \cap B(x,r)|_{\omega}^{1/q} \; \left| \chi_{\Omega \backslash B(x,r)} k(x\cdot) : L_p\left(\Omega;\nu^{1-p'}\right) \right| \right\}. \qquad (4)$$

We shall denote the kernel k of operator K by $k_{h,a}$ if it has the following form:

$$k_{h,a}(x,y) = \chi_{[h,a]}(|x-y|)\,k(x,y) \qquad 0 \leq h \leq a < +\infty .$$

We denote the operator K with kernel $k_{h,a}$ $\left(k^*_{h,a}\right)$ by $K_{h,a}$ $\left(K^*_{h,a}\right)$.

The given notation (3) and (4) are taken from the work [10].

Also we need the following statements, which are obtained by applying theorem 1 from [10] and lemma 2.

Lemma 3. *Let $\alpha \geq 0$, $1 \leq p_0 \leq p < +\infty$, $1 < p_i \leq p < +\infty$, $B(y,r) \subset \mathbf{R}^n$ be an arbitrary ball, $\omega, \nu_0, \nu_i \in W(B(y,r))$ and the following conditions be fulfilled:*

1) $C_0(B(y,r)) = \dfrac{1}{|B(y,r)|} \left(\int\limits_{B(y,r)} \omega(z)\,dz \right)^{1/p} \left(\int\limits_{B(y,r)} \nu_0^{1-p'_0}(z)\,dz \right)^{1/p'_0} < +\infty$

2) $\left. \begin{array}{l} C_i(B(y,r)) = \left[K_{0,6\sqrt{n}r}\right]_{p_i\nu_i^{1-p'_i},p\omega} < +\infty \\ C_i^*(B(y,r)) = \left[K^*_{0,6\sqrt{n}r}\right]_{p'\omega,p'_i\nu_i^{1-p'_i}} < +\infty \end{array} \right\}$ *if* $1 < p_i \leq p < +\infty$

or

$$\left.\begin{array}{l} C_i\left(B\left(y,r\right)\right)=\left[k_{0,6\sqrt{n}r}\right]_{p_i\nu_i^{1-p_i'},p\omega}<+\infty \\ \\ C_i^*\left(B\left(y,r\right)\right)=\left[k^*_{0,6\sqrt{n}r}\right]_{p'\omega,p_i'\nu_i^{1-p_i'}}<+\infty \end{array}\right\} if\ 1<p_i<p<+\infty$$

$i=\overline{1,n}$, where $k\left(x,y\right)=\frac{1}{|x-y|^{n-1}}$ is the kernel of operator K.

Then for any $u\in C^1\left(B\left(y,r\right)\right)$ it follows an inequality

$$\left|u:L_{(\alpha+1)p}\left(B\left(y,r\right)\omega\right)\right|\leq$$

$$\leq C\max_{i=\overline{1,n}}\left\{C_0\left(B\left(y,r\right)\right),C_i\left(B\left(y,r\right)\right),C_i^1\left(B\left(y,r\right)\right)\right\}\times \tag{5}$$

$$\times\left\{\left|u:L_{(x+1)p_0}\left(B\left(y,r\right);v_0\right)\right|+\sum_{i=1}^{n}\left|\left|u\left(x\right)\right|^{\alpha}D_iu\left(x\right):L_{p_i}\left(B\left(y,r\right);\nu_i\right)\right|\right\},$$

where C is a constant, which is independent on $u,\omega,v_0,v_i,i=\overline{1,n}$ and $B\left(y,r\right)$.

Proof. Let $u\in C^1\left(B\left(y,r\right)\right)$ be an arbitrary function. Then applying Minkowski inequality, we obtain

$$\left(\int\limits_{B(y,r)}\left|u\left(x\right)\right|^{(\alpha+1)p}\omega\left(x\right)dx\right)^{1/p}=$$

$$\left(\int\limits_{B(y,r)}\left|\left|u\left(x\right)\right|^{\alpha}u\left(x\right)-\frac{1}{|B(y,r)|}\int\limits_{B(y,r)}\left|u\left(z\right)\right|^{\alpha}u\left(z\right)dz\right|^{p}\omega\left(x\right)dx\right)^{0/p}+ \tag{6}$$

$$+\frac{1}{|B(y,r)|}\int\limits_{B(y,r)}\left|u\left(x\right)\right|^{\alpha+1}dx\left(\int\limits_{B(y,r)}\omega\left(x\right)dx\right)^{1/p}=I_1+I_2\ .$$

First of all we'll estimate I_2. Applying the Hölder inequality, we obtain

$$I_2=\frac{1}{\left|B\left(y,r\right)\right|}\int\limits_{B(y,r)}\left|u\left(x\right)\right|^{\alpha+1}\nu_0^{\frac{1}{p_0}}\left(x\right)\nu_0^{-\frac{1}{p_0}}\left(x\right)dx\left(\int\limits_{B(y,r)}\omega\left(x\right)dx\right)^{1/q}\leq$$

$$\leq\left(\int\limits_{B(y,r)}\left|u\left(x\right)\right|^{(\alpha+1)p_0}\nu_0\left(x\right)dx\right)^{1/p_0}\frac{1}{\left|B\left(y,r\right)\right|}\times$$

$$\times\left(\int\limits_{B(y,r)}\nu_0^{1-p_0'}\left(x\right)dx\right)^{1/p_0'}\left(\int\limits_{B(y,r)}\omega\left(x\right)dx\right)^{1/q}=$$

$$=C_0\left(B\left(y,r\right)\right)\ \left|u:L_{(\alpha+1)p_0}\left(B\left(y,r\right),\nu_0\right)\right|\ . \tag{7}$$

Now we'll estimate I_2. Taking into account, that

$$\frac{1}{\left|B\left(y,r\right)\right|}\int\limits_{B(y,r)}\left(\left|u\left(x\right)\right|^{\alpha}u\left(x\right)-\frac{1}{\left|B\left(y,r\right)\right|}\int\limits_{B(y,r)}\left|u\left(z\right)\right|^{\alpha}u\left(z\right)dz\right)dx=0$$

and applying lemma 2, we shall obtain

$$I_1 \leq C\sum_{i=1}^{n} \left| \int\limits_{B(y,r)} \frac{|u(z)|^{\alpha} |D_i u(z)|}{|x-z|^{n-1}} dz : L_p \left(B\left(y,r\right);\omega\right) \right| =$$

$$= C\sum_{i=1}^{n} \left| \int\limits_{B(y,r)} \frac{\chi_{[0,2r]}(|z-x|)}{|x-z|^{n-1}} \left|u\left(z\right)\right|^{\alpha} \left|D_i u\left(z\right)\right| dz : L_p\left(B\left(y,r\right);\omega\right) \right| .$$

Taking into account condition (2) and theorem 1 from [10], we get:

$$I_1 \leq C\sum_{i=1}^{n} \left[C_i\left(B(y,r)\right) + C_i^*\left(B(y,r)\right)\right] \times$$

$$\times \left| |u(x)|^{\alpha} D_i u\left(x\right) : L_{p_i}\left(B(y,r);\nu_i\right) \right| \leq \tag{8}$$

$$\leq C \max_{i=\overline{1,n}} \left\{C_i\left(B\left(y,r\right)\right), C_i^*\left(B\left(y,r\right)\right)\right\} \sum_{i=1}^{n} \left| |u\left(x\right)|^{\alpha} D_i u\left(x\right) : L_{p_i}\left(B\left(y,r\right);\nu_i\right) \right|$$

Taking into account (7) and (8) in (6), we get (5).

The lemma is proved.

Now we shall give the main result of the work:

Theorem. *Let $\Omega \subset \mathbf{R}^n$ be an arbitrary bounded open set with non-empty interior, $\alpha \geq 0$, $1 \leq p_0 \leq p < +\infty$, $1 < p_i \leq p < +\infty$, $\omega, \nu_0, \nu_i \in W(\Omega)$ and the following conditions be fulfilled:*

1) $\tilde{C}_0\left(\Omega\right) = \sup\limits_{\substack{B(y,r) \\ y\in\Omega,\ r=\frac{\rho(y)}{12\sqrt{n}}}} C_0\left(B\left(y,r\right)\right) < +\infty$

2) $\tilde{C}_i\left(\Omega\right) = \sup\limits_{\substack{B(y,r) \\ y\in\Omega,\ r=\frac{\rho(y)}{12\sqrt{n}}}} C_0\left(B\left(y,r\right)\right) < +\infty$

3) $\tilde{C}_i^*\left(\Omega\right) = \sup\limits_{\substack{B(y,r) \\ y\in\Omega,\ r=\frac{\rho(y)}{12\sqrt{n}}}} C_0^*\left(B\left(y,r\right)\right) < +\infty$

where C_0, C_i, C_i^*, $i = \overline{1,n}$ were defined in lemma 3.

Then the imbedding

$$S^1_{\alpha\overline{p},\overline{p}}\left(\Omega;\bar{\nu}\right) \subset L_{(\alpha+1)p}\left(\Omega;\omega\right) \tag{9}$$

holds.

Proof. We should show, that for any function $u \in S^1_{\alpha\overline{p},\overline{p}}\left(\Omega;\bar{\nu}\right)$ it holds the inequality

$$\left|u : L_{(\alpha+1)p}\left(\Omega;\omega\right)\right| \leq$$

$$\leq \tilde{C}\left(\Omega\right) \left\{ \left|u : L_{(\alpha+1)p_0}\left(\Omega;\nu_0\right)\right| + \sum_{i=1}^{n} \left| |u\left(x\right)|^{\alpha} D_i u\left(x\right) : L_{p_i}\left(\Omega;\nu_i\right) \right| \right\}, \tag{10}$$

where constant $\tilde{C}\left(\Omega\right) = C \max\left\{\tilde{C}_0\left(\Omega\right), \tilde{C}_i\left(\Omega\right), \tilde{C}_i^*\left(\Omega\right),\ i = \overline{1,n}\right\}$ doesn't depend on the function u.

Taking into account definition of the considered pn-space, it is enough to prove inequality (10) for arbitrary function $u \in C^1\left(\Omega\right) \cap S^1_{\alpha\overline{p},\overline{p}}\left(\Omega;\bar{\nu}\right)$

Let's consider a family

$$\left\{ \bar{B}\left(y,r\right):\ y\in\Omega,\ r=\frac{\rho\left(y\right)}{12\sqrt{n}}\right\} \tag{11}$$

of closed balls, contained in Ω and its coverings.

By lemma 1, from family (11) it is possible to select at most countable subfamily $\left\{\overline{B_j^k}\right\}_{j\geq 1}$, $k=\overline{1,\xi_n}$, satisfying the conditions (i) and (ii) of lemma 1.

Taking into account the noted, we obtain

$$\left|u:L_{(\alpha+1)p}\left(\Omega;\omega\right)\right|\leq\left(\sum_{k=1}^{\xi_n}\sum_{j\geq 1}\int\limits_{\overline{B_j^k}}\left|u\left(x\right)\right|^{(\alpha+1)p}\omega\left(x\right)dx\right)^{1/p}\leq$$

$$\leq\sup_{k=\overline{1,\xi_n}}\left(\sum_{j\geq 1}\int\limits_{\overline{B_j^k}}\left|u\left(x\right)\right|^{(\alpha+1)p}\omega\left(x\right)dx\right)^{1/p}\xi_n^{1/p}.$$

As $\left|\partial\overline{B_j^k}\right|=0$, then in the last expression we can replace the domain of integration by open ball B_j^k. Taking this into account, we shall obtain

$$\left|u:L_{(\alpha+1)p}\left(\Omega;\omega\right)\right|\leq\xi_n^{1/p}\sup_{k=\overline{1,\xi_n}}\left(\sum_{j\geq 1}\int\limits_{B_j^k}\left|u\left(x\right)\right|^{(\alpha+1)p}\omega\left(x\right)dx\right)^{1/p}$$

Applying inequality (5) in the right hand side of this ratio and taking into account conditions (1) and (2), we get

$$\left|u:L_{(\alpha+1)p}\left(\Omega;\omega\right)\right|\leq\tilde{C}\left(\Omega\right)\left\{\sup_{k=\overline{1,\xi_n}}\left(\sum_{j\geq 1}\int\limits_{B_j^k}\left|u:L_{(\alpha+1)p_0}\left(\overline{B_j^k};\nu_0\right)\right|^p\right)^{1/p}+\right.$$

$$\left.+\sum_{i=1}^{n}\sup_{k=\overline{1,\xi_n}}\left(\sum_{j\geq 1}\left|\left|u\left(x\right)\right|^{\alpha}D_iu\left(x\right):L_{p_i}\left(\overline{B_j^k};\nu_i\right)\right|^p\right)^{1/p}\right\}.$$

Taking into account that $\frac{p}{p_i}\geq 1$, $i=\overline{0,n}$ and applying the known inequality:

$$\left(\sum_i\left|a_i\right|^{\alpha}\right)^{1/\alpha}\leq\left(\sum_i\left|a_i\right|^{\beta}\right)^{1/\beta}\qquad\alpha\geq\beta\geq 1$$

at $\alpha = \frac{p}{p_i}$ and $\beta = 1$, we get

$$\left|u : L_{(\alpha+1)p}(\Omega;\omega)\right| \le \tilde{C}(\Omega)\left\{\sup_{k=\overline{1,\xi_n}}\left(\sum_{j\ge 1}\int_{B_j^k}|u(x)|^{(\alpha+1)p_0}\nu_0(x)\,dx\right)^{1/p_0}+\right.$$

$$\left.+\sum_{i=1}^{n}\sup_{k=\overline{1,\xi_n}}\left(\sum_{j\ge 1}\int_{B_j^k}\left(|u(x)|^{\alpha}|D_i u(x)|\right)^{p_i}\nu_i(x)\,dx\right)^{1/p_i}\right\}\le$$

$$\le \tilde{C}(\Omega)\left|u : S^1_{\alpha p,\bar{p}}(\Omega;\bar{\nu})\right| .$$

The last inequality proves the estimation (10) therefore and imbedding (9).

The theorem is proved.

From the proved theorem in the case of $\alpha = 0$ the validity of corresponding theorem follows for weight anisotropic Sobolev spaces $W^1_{p_0,\dots,p_n}(\Omega;\nu_0,\dots,\nu_n)$.

The obtained result gives more common conditions on weight functions, at which the corresponding imbedding theorems are valid in the works [5-7].

References

[1]. Soltanov K.N. *Imbedding theorems for nonlinear spaces and solvability of some nonlinear coercitive equations.* Dep. VINITI, 1991, No3697-B-91, 72 p.(Russian)

[2]. Soltanov K.N. *Imbedding theorems for nonlinear spaces and solvability of some noncoercitive equations.* Proc. of IMM of AS of Azerb., 1996, v.V(XIII), pp.72-103.(Russian)

[3]. Soltanov K.N. *Some imbedding theorems and their applications to nonlinear equations.* Diff. urav., 1984, v.20, No12, pp.2181-2184.(Russian)

[4]. Soltanov K.N. *Some applications of nonlinear analysis to the differential equations.* Baku, Science, 2002, 296 p.(Russian)

[5]. Akhmedov M.A. *The imbedding theorems for one class of weight spaces.* Trans. AS of Azerb., ser. phys. math. and techn. sci., 1996, v.XVII, No1-3, pp.37-48.(Russian)

[6]. Akhmedov M.A. *Some weight spaces, weight boundary values and their applications.* Thesis for a candidate degree, Baku, BSU, 2000, 118 p.(Russian)

[7]. Akhmedov M.A. *On the imbedding and compactness theorems for anisotropic weight Sobolev's spaces.* Proc. of IMM of NASA, 2001, v.XV(XXIII), pp.3-8.

[8]. Gusman M. *Differentiation of the integrals in R^n.* M.: "Mir", 1973, 200 p.(Russian)

[9]. Konterovich M.B., Akilov G.P. *Fucntional analysis.* M., Science, 1984, 751p.(Russian)

[10]. Besov O.V. *Imbedding of spaces of differentiable functions of variable smoothness.* Proc. of M.I.Steklov math. Inst., 1997, v.214, pp.25-58.(Russian)

Makhmud A. Akhmedov
Institute of Mathematics and Mechanics of NAS of Azerbaijan.
9, F.Agayev str., AZ1141, Baku, Azerbaijan.
Tel.: (99412) 750 923 (apt.)
E-mail: akhmedovm@hotmail.com

Received June 02, 2003; Revised October 01, 2003.
Translated by Mamedova Sh.N.