Abstract

The conditions, providing, completeness of the decreasing elementary solutions of one class of fourth order operator-differential equations are found. In the work it is proved existence of the regular solution of corresponding homogeneous operator-differential equation, when the boundary conditions contain operators, and it is proved a completeness of some arbitrary chains, constructed by these boundary conditions.

Let $\mathcal{H}$ be a separable Hilbert space, $A$ positively defined self-adjoint operator in $\mathcal{H}$. It is known that, the domain of the operator $A^\gamma (\gamma > 0)$ becomes a Hilbert Space $\mathcal{H}_\gamma$ with respect to the scalar products $(x,y)_\gamma = (A^\gamma x,A^\gamma y), x,y \in D(A^\gamma)$. We'll denote by $L_2(R_+;\mathcal{H}_\gamma)$ a set of all measurable Bohner vector-functions with the values from $\mathcal{H}_\gamma$, for which

$$\|f\| = \left(\int_0^\infty \|f(t)\|_{\gamma}^2 \, dt\right)^{1/2} < \infty.$$ Further, let $L(X,Y)$ define a set of linear restrictions of the operators acting from the Hilbert Space $X$ to another $Y$; $\sum( \cdot )$ be a spectrum of the operator $(\cdot )$; $\sum_{n=1}^\infty$ be an ideal of the completely continuous operators in $L(H,H)$; $\sum_{n=1}^\infty s_n(A) < \infty$ where $s_n(A) - s$ are numbers of the operator $A$; in future everywhere $u, u', u'', u'''$ and $u^{(4)}$ are derivatives in the since of distributions theory [1].

Now let’s introduce the following sets:

$$W_2^4(R_+;\mathcal{H}) = \{ u : u \in L_2(R_+;\mathcal{H}_4), u^{(4)} \in L_2(R_+;\mathcal{H}) \},$$

$$\dot{W}_2^4(R_+;\mathcal{H}) = \{ u : u \in W_2^4(R_+;\mathcal{H}), u(0) = u'(0) = u''(0) = u'''(0) = 0 \},$$

$$W_2^4,T,k(R_+;\mathcal{H}) = \{ u : u \in W_2^4(R_+;\mathcal{H}), u(0) = Tu''(0), u'(0) = Ku'''(0), T \in L(\mathcal{H}_{3/2},\mathcal{H}_{7/2}), K \in L(\mathcal{H}_{1/2},\mathcal{H}_{5/2}) \}.$$  

Each of these sets provided with norm

$$\|u\|_{W_2^4} = \left(\|u\|_{L_2(R_+;\mathcal{H}_4)}^2 + \|u^{(4)}\|_{L_2(R_+;\mathcal{H})}^2\right)^{1/2},$$

becomes a Hilbert space [1,p.29].

Now we’ll pass to the statement of the problems, which we are studying. Let $B_1, B_2, B_3 \in L(\mathcal{H};\mathcal{H})$, then a domain of the operator bundle

$$P(\lambda) = \lambda^4 E + \lambda^3 B_3 A + \lambda^2 B_2 A^2 + \lambda B_1 A^3 + A^4$$ (1)
coincides with the space $H_d$; here $E$ single operator in $H$. In the theorem on the completeness of decreasing elementary solutions of the equation

$$P(d/dt)u = u^{(4)} + B_3 A u''' + B_2 A^2 u'' + B_1 A^3 u' + A^4 u = 0$$

by fulfilling the boundary conditions:

$$u(0) -Tu''(0) = \varphi, \varphi \in H_7, u'(0) - Ku'''(0) = \psi, \psi \in H_5$$

in the corresponding space of solutions of problem (2), (3) in supposition $A^{-1} \in \mathbb{S}_p$.

To this end, at first we shall consider the operator-differential equation:

$$P(d/dt) = u^{(4)} + B_3 A u''' + B_2 A^2 u'' + B_1 A^3 u' + A^4 u = f, \ t \in R_+$$

by fulfilling the boundary conditions

$$u(0) = Tu''(0), \ u'(0) = Ku'''(0)$$

where almost everywhere $f(t) \in H, u(t) \in H$.

The questions on the completeness of the elementary solutions in the case when the operators are in the boundary conditions are investigated for example, in the work [5] for second order equations.

**Definition 1.** Problem (4), (5) is called regular solvable, if for each vector-function $f(t) \in L^2(R_+; H)$ there exists a unique vector-function $u(t) \in W^{4,T,K}_2(R_+; H)$, which satisfies equation (4) almost everywhere in $R_+$, boundary conditions (5) are fulfilled in the sense of convergence of the space $H_{7/2}, H_{5/2}$ and it holds the inequality

$$\|u\|_{W^4_2} \leq \text{const} \|f\|_{L^2}.$$  

Let’s find conditions, providing regular solvability of problem (4), (5).

First of all, we shall consider the equation

$$P_0(d/dt)u = u^{(4)} + A^4 u = f$$

where $f(t) \in L^2(R_+; H)$. Let’s denote by $P_0$ the operator, acting from space $W^{4,T,K}_2(R_+; H)$ in $L^2(R_+; H)$ by the following way:

$$P_0 u = P_0(d/dt)u, u \in W^{4,T,K}_2(R_+; H).$$

It’s true.

**Theorem 1.** Let $C = A^{7/2} T A^{-3/2}, S = A^{5/2} K A^{-1/2},$ these operators are commutative, i.e. $CS = SC$ and point $-1 \notin \sum(CS-S+C)$. Then operator $P_0$ realizes an isomorphism from the space $W^{4,T,K}_2(R_+; H)$ on $L^2(R_+; H)$.

**Proof.** The condition $-1 \notin \sum(CS-S+C)$ implies that homogeneous $P_0(d/dt) = 0$ has just a zero solution from the space $W^{4,T,K}_2(R_+; H)$, but at any $f(t) \in L^2(R_+; H)$
the operator $P$ on $P$.

Thus, an operator space $W$.

Consequently, the second and the third number in equality (8) also belong to the isomorphism by these spaces.

Equation (7) has solution from the space $W^{A,T,K}(R_+; H)$, representable in the form

$$u(t) = \frac{1}{4\sqrt{2}} \int_0^\infty \left( (1 + i)e^{-\frac{i\pi}{\sqrt{2}} t |s|A} + (1 - i)e^{-\frac{\pi}{\sqrt{2}} t |s|A} \right) A^{-3} f(s) ds -$$

$$- \frac{i}{4\sqrt{2}} e^{-\frac{i\pi}{\sqrt{2}} tA} A^{-7/2}(CS - S + C + E)^{-1} \times$$

$$\times \left[ ((C + iE)(S - iE) + (E + iC)(S + iE)) \times$$

$$\times A^{1/2} \int_0^\infty e^{-\frac{i\pi}{\sqrt{2}} tA} f(s) ds + 2(E + iC)(S - iE) A^{1/2} \int_0^\infty e^{-\frac{i\pi}{\sqrt{2}} sA} f(s) ds \right] +$$

$$+ \frac{i}{4\sqrt{2}} e^{-\frac{i\pi}{\sqrt{2}} tA} A^{-7/2}(CS - S + C + E)^{-1} \times$$

$$\times \left[ 2(E - iC)(S + iE) A^{1/2} \int_0^\infty e^{-\frac{i\pi}{\sqrt{2}} sA} f(s) ds +$$

$$+ ((E - iC)(S - iE) + (E + iC)(E - iS)) A^{1/2} \int_0^\infty e^{-\frac{i\pi}{\sqrt{2}} sA} f(s) ds \right].$$

It is easy to check, that first number satisfies equation (7) and belongs to the space $W_2^2(R_+; H)$ (see [2,3]). Further, from the inequality [6, p.208]

$$\left\| A^{1/2} \int_0^\infty \exp(-tA) f(t) dt \right\|_H \leq \frac{1}{\sqrt{2}} \|f\|_{L^2},$$

(9)

$$\left\| A^{1/2} \exp(-tA) \psi \right\|_{L^2} \leq \frac{1}{\sqrt{2}} \|\psi\|, \psi \in H,$$

(10)

implies the inequality:

$$\left\| A^{1/2} \int_0^\infty \exp(-\frac{1 \pm i}{\sqrt{2}} tA) f(t) dt \right\|_H \leq \frac{1}{\sqrt{2}} \|f\|_{L^2},$$

(11)

$$\left\| A^4 \exp(-\frac{1 \pm i}{\sqrt{2}} tA) \psi \right\|_{L^2} \leq \frac{1}{\sqrt{2}} \|\psi\|_{L^2}, \psi \in H_{7/2}.$$  

(12)

Consequently, the second and the third number in equality (8) also belong to the space $W_2^2(R_+; H)$.

Fulfiment of boundary conditions (5) can be checked directly. Boundedness of the operator $P_0$ follows from the inequality

$$\|P_0 u\|_{L^2}^2 = \|u^{(4)} + A^4 u\|_{L^2}^2 \leq 2 \|u\|_{W_2^4}^2$$

(13)

Thus, an operator $P_0$ is bounded and one-to-one acts from the space $W_2^4(R_+; H)$ on $L^2(R_+; H)$ and by the Banach’s theorem on the inverse operator realizes isomorphism by these spaces.
Then problem (4), (5) is regular solvable.

Proof. Let’s write problem (4), (5) in the form of operator equation $(\mathcal{P}_0 + \mathcal{P}_1) u = f$, where $f(t) \in L_2(R_+; \mathcal{H}), u(t) \in W_2^{4,T,K}(R_+; \mathcal{H})$. $\mathcal{P}_1 u = \sum_{j=1}^{3} B_j A^{4-j} u(j)$ for $u \in W_2^{4,T,K}(R_+; \mathcal{H})$. Since, the operator $\mathcal{P}_0$ has a bounded inverse $\mathcal{P}_0^{-1}$ by theorem 1, acting from $L_2(R_+; \mathcal{H})$ on $W_2^{4,T,K}(R_+; \mathcal{H})$, then after substitution $u = \mathcal{P}_0^{-1} v$ we shall obtain the following equation in $L_2(R_+; \mathcal{H})$:

$$(E + \mathcal{P}_1 \mathcal{P}_0^{-1}) v = f.$$ 

On the other hand

$$\| \mathcal{P}_1 \mathcal{P}_0^{-1} v \|_{L_2} = \| \mathcal{P}_1 u \|_{L_2} \leq \sum_{j=1}^{3} \| B_j \| \| A^{4-j} u(j) \|_{L_2} \leq$$

$$\leq \sum_{j=1}^{3} \| B_j \| N_{T,K,j} \| \mathcal{P}_0 u \| = \sum_{j=1}^{3} \| B_j \| N_{T,K,j} \| v \|_{L_2}.$$ 

Therefore, by fulfilling the inequality $\sum_{j=1}^{3} \| B_j \| N_{T,K,j} < 1$ the operator $E + \mathcal{P}_1 \mathcal{P}_0^{-1}$ is reversible and we can find $u(t)$. The theorem is proved.

Let’s denote by $N_{0,j} = \sup_{0 \neq u \in W_2^2(R_+; \mathcal{H})} \left( \| A^{4-j} u(j) \|_{L_2} / \| P_0 u \|_{L_2} \right), j = 1, 2, 3.$

**Remark 1.** It is obvious, that $N_{T,K,j} \geq N_{0,j}$ and

$$N_{0,j} = \left( \frac{4}{4-j} \right)^{4-j} \left( \frac{4}{j} \right)^{j-1/4}, j = 1, 2, 3,$$

[7]. In suppositions $A^{-1} \in \sum_p$ the operator bundle $P(\lambda)$ has a discrete spectrum, and let $\lambda_n (n = 1, 2, 3,...)$ be characteristic numbers of bundle $P(\lambda)$ from the left-plane $\Pi_-$, and $x_{0,n}, x_{1,n}, ..., x_{m,n}$ be eigen and joined vectors, responding to the characteristic number $\lambda_n$:

$$P(\lambda_n) x_{0,n} = 0,$$

$$\sum_{j=0}^{p} \frac{1}{j!} P^{(j)}(\lambda_n) x_{p-j,n} = 0, p = 1, ..., m.$$
Then the vector-functions

\[ u_{p,n}(t) = e^{\lambda_{n}t} \left( \frac{t^{p}}{p!} x_{0,n} + \frac{t^{p-1}}{(p-1)!} x_{1,n} + \ldots + x_{p,n} \right), \quad p = 0, 1, \ldots, m, \]

belong to the space \( W_{2}^{4}(R+; \mathcal{H}) \) and satisfy equation (2). They will be called elementary solutions of equation (2) \([5]\). It is obvious, that elementary solutions satisfy the following boundary conditions:

\[ u_{0,n}(0) - T u_{0,n}''(0) = x_{0,n} - \lambda_{n}^{2} T x_{0,n} \equiv \varphi_{0,n}, \]

\[ u_{1,n}(0) - T u_{1,n}''(0) = x_{1,n} - \lambda_{n}^{2} T x_{1,n} - 2\lambda_{n} T x_{0,n} \equiv \varphi_{1,n}, \]

\[ u_{p,n}(0) - T u_{p,n}''(0) = x_{p,n} - \lambda_{n}^{2} T x_{p,n} - 2\lambda_{n} T x_{p-1,n} - T x_{p-2,n} \equiv \varphi_{p,n}, \quad p = 2, \ldots, m, \]

\[ u_{0,n}''(0) - K u_{0,n}'''(0) = \lambda_{n} x_{0,n} - \lambda_{n}^{3} K x_{0,n} \equiv \psi_{0,n}, \]

\[ u_{1,n}''(0) - K u_{1,n}'''(0) = \lambda_{n} x_{1,n} - \lambda_{n}^{3} K x_{1,n} + x_{0,n} - 3\lambda_{n}^{2} K x_{0,n} \equiv \psi_{1,n}, \]

\[ u_{2,n}''(0) - K u_{2,n}'''(0) = \lambda_{n} x_{2,n} - \lambda_{n}^{3} K x_{2,n} + x_{1,n} - 3\lambda_{n}^{2} K x_{1,n} - 3\lambda_{n} K x_{0,n} \equiv \psi_{2,n}, \]

\[ u_{p,n}''(0) - K u_{p,n}'''(0) = \lambda_{n} x_{p,n} - \lambda_{n}^{3} K x_{p,n} + x_{p-1,n} - 3\lambda_{n}^{2} K x_{p-1,n} - 3\lambda_{n} K x_{p-2,n} - K x_{p-3,n} \equiv \psi_{p,n}, \quad p = 3, \ldots, m. \]

By fulfilling the condition of theorem 2, it is easy to see, that problem (2), (3) has a unique solution from the space \( W_{2}^{4}(R+; \mathcal{H}) \) at any \( \varphi \in \mathcal{H}_{7/2}, \psi \in \mathcal{H}_{5/2} \). A set of all such solutions we’ll denote by \( W_{4}(P) \).

From the theorem on the intermediate derivatives and about traces implies, that a set \( W_{4}(P) \) is closed subspace of the space \( W_{2}^{4}(R+; \mathcal{H}) \). There it is stated, a problem: when elementary solutions of problem (2) are complete in the space \( W_{4}(P) \)? It holds.

**Theorem 3.** Let \( C = A^{7/2} T A^{-3/2}, S = A^{5/2} K A^{-1/2}, \) these operators are commutative, i.e. \( CS = SC, -1 \sum (CS - S + C), \sum_{j=1}^{3} \| B_{j} \| N_{T,K,j} < 1 \) and one of the conditions is fulfilled:

a) \( A^{-1} \in \sum_{\rho}(0 < \rho \leq 2) \) or b) \( B_{j} \in \sum_{\infty}, j = 1, 2, 3, A^{-1} \in \sum_{\rho}(0 < \rho < \infty). \)

Then a system of elementary solutions of problem (2), (3) is complete in the space \( W_{4}(P) \).

**Proof.** First of all, we shall prove, that the system \( \{ \varphi_{p,n}, \psi_{p,n} \} \), defined from equality (15) is complete in the space \( \mathcal{H}_{7/2} \oplus \mathcal{H}_{5/2} \). If it is not so, then there exists
a non-zero vector \((\tilde{\varphi}, \tilde{\psi}) \in \mathcal{H}_{7/2} \oplus \mathcal{H}_{5/2}\) such that \((\tilde{\varphi}, \varphi_{p,n})_{7/2} + (\tilde{\varphi}, \psi_{p,n})_{5/2} = 0\). Then from the expansion of the main part of resolvent at the neighborhoods of characteristic numbers it follows, that \((A_{7/2}(E - \lambda^2 T)P^{-1}(\lambda))\overline{A_{7/2}\tilde{\varphi}} + (A_{5/2}(\lambda E - \lambda^3 K)P^{-1}(\lambda))\overline{A_{5/2}\tilde{\psi}}\) will be holomorphic vector-function in the left half-plane \(\Pi_+\).

If \(u(t)\) is a solution of problem (2), (3), then it can be represented in the form
\[
\begin{align*}
  u(t) &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \hat{u}(\lambda) \exp(\lambda t) d\lambda, \\
  &\quad \text{(16)}
\end{align*}
\]

where
\[
\begin{align*}
  \hat{u}(\lambda) &= P^{-1}(\lambda) \left( (\lambda^3 E + \lambda^2 B_3 A + \lambda B_2 A^2 + B_1 A^3) u(0) + \\
  &\quad + (\lambda^2 E + \lambda B_3 A + B_2 A^2) u'(0) + (\lambda E + B_3 A) u''(0) + u'''(0) \right).
\end{align*}
\]

Taking into account Remark 1 from theorem 5 of the work [4] we'll obtain that by fulfilling the condition of the theorem the following statement holds:

1) \(P^{-1}(\lambda)\) is represented in the form of ratio of two \(\rho\) order entire and minimal type functions at order \(\rho\);

2) there exists a number \(\mathcal{E} > 0\) such, that the resolvent \(P^{-1}(\lambda)\) is holomorphic at the angles \(S_{\pm \mathcal{E}} = \{\lambda : \lambda = r \exp(\pm i\theta), \pi/2 < \theta < \pi/2 + \mathcal{E}, r > 0\}\) and at the same angles admits the estimations \(\|A_{7/2}P^{-1}(\lambda)\| \leq c|\lambda|^{-1/2}, \|A_{5/2}^{-1}(\lambda)\| \leq c|\lambda|^{-3/2}\);

3) at the left half-plane there exists a system of rays \(\{\Omega\}\), including rays \(\Gamma_{\pm \mathcal{E}} = \{\lambda : \lambda = r \exp(\pm i(\pi/2 + \mathcal{E})), r > 0\}\), such that the angle between neighbour rays is less than \(\pi/\rho\) and on these rays of the functions \(\|A_{7/2}P^{-1}(\lambda)\|\) and \(\|A_{5/2}^{-1}(\lambda)\|\) grow no faster than \(|\lambda|^{-1/2}\) and \(|\lambda|^{-3/2}\) correspondingly.

Taking into account all of this in equality (16), a contour of integration we can substitute by \(\Gamma_{\pm \mathcal{E}}\). Then at \(t > 0\)
\[
(u(t) - Tu''(t), \tilde{\varphi})_{7/2} + (u'(t) - Ku'''(t), \tilde{\psi})_{5/2} =
\]
\[
= \frac{1}{2\pi i} \int_{\Gamma_{\pm \mathcal{E}}} \left( (\lambda^3 E + \lambda^2 B_3 A + \lambda B_2 A^2 + B_1 A^3) u(0) + \\
  &\quad + (\lambda^2 E + \lambda B_3 A + B_2 A^2) u'(0) + \\
  &\quad + (\lambda E + B_3 A) u''(0) + u'''(0), (A_{7/2}(E - \lambda^2 T)P^{-1}(\lambda))\overline{A_{7/2}\tilde{\varphi}} + \\
  &\quad + (A_{5/2}(\lambda E - \lambda^3 K)P^{-1}(\lambda))\overline{A_{5/2}\tilde{\psi}}\right) \exp(\lambda t) d\lambda.
\]

From the Frangmen-Lindelof’s theorem we obtain, that integrand function in front of \(\exp \lambda t\) is a polynomial, and therefore the integral equals zero at \(t > 0\), consequently, \((u(t) - Tu''(t), \tilde{\varphi})_{7/2} + \left( u'(t) - Ku'''(t), \tilde{\psi} \right)_{5/2} = 0, t > 0\).

Passing to the limit at \(t \to 0\), by the theorem about traces we shall obtain \((\varphi, \psi)_{7/2} + (\psi, \psi)_{5/2} = 0, \forall (\varphi, \psi) \in \mathcal{H}_{7/2} \oplus \mathcal{H}_{5/2}\). Therefore, \(\tilde{\varphi} = \tilde{\psi} = 0\). Further, from the uniqueness of the solution of problem (2), (3) and from the theorem about traces it holds the inequality
\[
(72) (\varphi, 0)^2_{7/2} + (\psi, 0)^2_{5/2}\leq \|u\|^2_{W_2^1} \leq c_1 \left( (\varphi, 0)^2_{7/2} + (\psi, 0)^2_{5/2} \right)^{1/2}.
\]
Since, the system \( \{ (\varphi_{p,n}, \psi_{p,n}) \} \) is complete in \( H_{7/2} \oplus H_{5/2} \), then for the given \( \mathcal{E} > 0 \) there exists a number \( N \) and numbers \( c_{p}^{N}(\mathcal{E}) \) such that

\[
\left( \left\| \varphi - \sum_{n=1}^{N} \sum_{p} c_{p}^{N}(\mathcal{E})\varphi_{p,n} \right\|_{7/2}^{2} + \left\| \psi - \sum_{n=1}^{N} \sum_{p} c_{p}^{N}(\mathcal{E})\psi_{p,n} \right\|_{5/2}^{2} \right)^{1/2} < \mathcal{E}. \tag{18}
\]

Taking into account equalities (3) and (5) in (17), from inequality (18) we’ll obtain

\[
\left\| u(t) - \sum_{n=1}^{N} \sum_{p} c_{p}^{N}(\mathcal{E})u_{p,n}(t) \right\|_{W_{2}^{4}} < c_{1}\mathcal{E}.
\]

The theorem is proved.

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References


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