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ON UNIQUENESS OF STRONG SOLUTION OF DIRICHLET PROBLEM FOR SECOND ORDER QUASILINEAR ELLIPTIC EQUATIONS WITH CORDES CONDITION

Abstract

The first boundary value provlem is considered for second order quasilinear elliptic equations of non-divergent form when the leading part satisfies the Cordes condition. The uniqueness of strong (almost everywhere) solution of mentioned problem is proved for any $n \geq 2$.

Let $\mathbb{E}_n$ be n-dimensional Euclidean space of points $x = (x_1, ..., x_n)$, $n \geq 2$, D be a bounded convex domain in $\mathbb{E}_n$ with the boundary $\partial D \in C^2$. Consider the following Dirichlet problem in D

$$\mathcal{L}u = \sum_{i,j=1}^{n} a_{ij}(x, u)\, u_{ij} = f(x), \quad x \in D; \tag{1}$$

$$u|_{\partial D} = 0, \tag{2}$$

where $u_i = \dfrac{\partial u}{\partial x_i}$, $u_{ij} = \dfrac{\partial^2 u}{\partial x_i \partial x_j}$; $i, j = 1, ..., n$; $\|a_{ij}(x, z)\|$ is a real symmetric matrix whose elements are measurable in D for any fixed $z \in \mathbb{E}_1$, moreover,

$$\mu|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x, z)\, \xi_i \xi_j \leq \mu^{-1}|\xi|^2; \; x \in D, \; z \in \mathbb{E}_1, \; \xi \in \mathbb{E}_n, \tag{3}$$

$$\sigma = \sup_{x \in D, \; z \in \mathbb{E}_1} \frac{\sum\limits_{i,j=1}^{n} a_{ij}(x, z)}{\left[\sum\limits_{i=1}^{n} a_{ii}(x, z)\right]^2} < \frac{1}{n-1} \, . \tag{4}$$

Here $\mu \in (0, 1]$ is a constant. Condition (4) is called the Cordes condition, and it is understood to within equivalence and nonsingular linear transformation in the following sense: domain D can be covered by a finite number of subdomains $D^1, ..., D^l$ so that in each D^i, $i = 1, ..., l$, equation (1) can be replaced by the equivalent equation $\mathcal{L}'u = f'(x)$, and nonsingular linear transformation of coordinates can be made, at which the coefficients of the image of the operator $\mathcal{L}'$ satisfy condition (4) in the domain D^i.

The aim of the present paper is to prove the uniqueness of strong (almost everywhere) solution of the first boundary value problem (1)-(2) for $f(x) \in L_s(D)$ at $s \in (n;\infty)$ for any $n \geq 2$. Note that the analogous problem was considered in [12] for $n = 2, 3, 4$ and the strong (almost everywhere) solvability was proved for $f(x) \in L_s(D)$; $s \in [2;\infty)$, for $n = 2, 3$; $s \in (1,\infty)$ at $n = 4$. In this connection we refer to the papers [1-2], where the analogous results for the second order linear elliptic equations with continuous coefficients were obtained, and we also refer to the papers [3-7], in which some classes of above mentioned equations with discontinuous coefficients were considered. We notice the papers [8-10], where the questions of strong solvability of boundary value problems for the second order parabolic equations were investigated. Existence of strong solution of the first boundary value problem (1)-(2) was established in [11], at that it was done for more general class of equations than (1).

We now agree upon some denotation. For $p \in [1,\infty)$ we denote by $W_p^1(D)$ and $W_p^2(D)$ Banach spaces of functions $u(x)$ given on D with the finite norms

$$\|u\|_{W_p^1(D)} = \left(\int_D \left(|u|^p + \sum_{i=1}^n |u_i|^p\right) dx\right)^{\frac{1}{p}}$$

and

$$\|u\|_{W_p^2(D)} = \left(\int_D \left(|u|^p + \sum_{i=1}^n |u_i|^p + \sum_{i,j=1}^n |u_{ij}|^p\right) dx\right)^{\frac{1}{p}},$$

respectively. Further, let $\overset{\circ}{W}{}_p^1(D)$ be a completion of $C_0^\infty(D)$ by the norm of space $W_p^1(D)$, and $\dot{W}_p^2(D) = W_2^2(D) \cap \overset{\circ}{W}{}_p^1(D)$. Function $u(x) \in \dot{W}_p^2(D)$ is called strong solution of the first boundary-value problem (1)-(2) (at $f(x) \in L_p(D)$) if it satisfies equation (1) almost everywhere in D.

Everywhere below notation $C(...)$ means that the positive constant C depends only on what in parentheses.

We now give some known facts which we will need further on.

Theorem 1 ([1]). *Let $1 < p < n$, $1 \leq q \leq \frac{np}{n-p}$. Then for any function $u(x) \in \overset{\circ}{W}{}_p^1(D)$, the following estimate holds*

$$\|u\|_{L_q(D)} \leq C_1(p,q,n) \, \||u_x|\|_{L_p(D)} .$$

If $p > n$, then

$$\|u\|_{L_\infty(D)} \leq C_2(p,n) \, \||u_x|\|_{L_p(D)} .$$

Here $u_x = (u_1, ..., u_n)$.

Consider the following equation in D

$$\mathcal{M}u=\sum_{i,j=1}^{n}a_{ij}\left(x,u,u_x\right)u_{ij}=f\left(x\right),\quad x\in D \tag{1'}$$

and suppose that the elements of the real symmetric matrix $\|a_{ij}\left(x,z,\upsilon\right)\|$ are measurable in D for any fixed $z\in\mathbb{E}_1,\ \upsilon\in\mathbb{E}_n$,

$$\mu|\xi|^2\leq\sum_{i,j=1}^{n}a_{ij}\left(x,z,\upsilon\right)\xi_i\xi_j\leq\mu^{-1}|\xi|^2;\ x\in D,\ z\in\mathbb{E}_1,\ \upsilon\in\mathbb{E}_n,\ \xi\in\mathbb{E}_n, \tag{3'}$$

$$\sigma=\sup_{x\in D,\ z\in\mathbb{E}_1,\upsilon\in\mathbb{E}_n}\frac{\sum\limits_{i,j=1}^{n}a_{ij}^2\left(x,z,\upsilon\right)}{\left[\sum\limits_{i=1}^{n}a_{ii}\left(x,z,\upsilon\right)\right]^2}<\frac{1}{n-1}\ ; \tag{4'}$$

and besides

$$\left|a_{ij}\left(x,z^1,\upsilon^1\right)-a_{ij}\left(x,z^2,\upsilon^2\right)\right|\leq H\left(\left|z^1-z^2\right|^{\alpha}+\left|\upsilon^1-\upsilon^2\right|^{\alpha}\right);$$

$$x\in D;\ z^1,\ z^2\in\mathbb{E}_1;\ \upsilon^1,\upsilon^2\in\mathbb{E}_n;\ i,j=1,...,n \tag{5}$$

with constants $H\geq 0$ and $\alpha\in(0,1]$.

Theorem 2 ([11]). *Let the coefficients of the operator $\mathcal{M}$ satisfy conditions (3′), (4′), (5′). Then there exists $p_1\left(\mu,\sigma,n\right)\in\left(\frac{3}{2},2\right)$ such that at any $p\in[p_1,2]$ for any function $u\left(x\right)\in\dot{W}_p^2\left(D\right)$ the following estimate holds*

$$\|u\|_{W_p^2(D)}\leq C_3\left(\mu,\sigma,n,\partial D\right)\|\mathcal{M}u\|_{L_p(D)}\ . \tag{6}$$

At that for any $A>0$ there exists $d_A=d_A\left(\mu,\sigma,n,\partial D,H,\alpha,A\right)$ such that if $mesD\leq d_A$, then the first boundary value problem (1′)-(2′) has a strong solution from the space $\dot{W}_2^2\left(D\right)$ for any function $f\left(x\right)\in L_2\left(D\right)$, whenever $\|f\|_{L_2(D)}\leq A$.

Theorem 3 ([12]). *Let $3\leq n\leq 4$, and the coefficients of the operator $\mathcal{L}$ satisfy conditions (3)-(4) and*

$$|a_{ij}\left(x,z^1\right)-a_{ij}\left(x,z^2\right)|\leq H_1|z^1-z^2|;\ x\in D;\ z^1,z^2\in\mathbb{E}_1;\ i,j=1,...,n \tag{7}$$

with some constant $H_1\geq 0$. Then for any $s\in(2,\infty)$ at $n=4$; for any $s\in[2;\infty)$ at $n=3$ and $A>0$ *there exists $\rho_A=\rho_A\left(\mu,\sigma,\ n,\ \partial D,H_1,s,A\right)$ such that if $mesD\leq\rho_A$, $f\left(x\right)\in L_s\left(D\right)$ and $\|f\|_{L_S(D)}\leq A$ then the first boundary value problem (1)-(2) has a unique strong solution $u\left(x\right)\in\dot{W}_2^2\left(D\right)$.*

Theorem 4 ([12]). *Let $n=2$, and let the coefficients of the operator $\mathcal{M}$ satisfy conditions* (3′) *and (5) (for $\alpha=1$ Then for any $s\in(2,\infty)$ at $A>0$*

there exists $\rho_A = \rho_A(\mu, \partial D, H, s, A)$ *such that if* $mesD \le \rho_A,\ f(x) \in L_s(D)$ *and* $\|f\|_{L_s(D)} \le A$ *then the first boundary value problem (1)-(2) has a unique strong solution* $u(x) \in \dot{W}_2^2(D)$.

Theorem 5. *Let* $n \ge 5$, *and the coefficients of the operator* $\mathcal{L}$ *satisfy conditions (3)-(4) and (7). Then for any* $s \in (n, \infty)$ *at* $A > 0$ *there exists* $\rho_A = \rho_A(\mu, \sigma, n, \partial D, H_1, s, A)$ *such that if* $mesD \le \bar{\rho}_A,\ f(x) \in L_s(D)$ *and* $\|f\|_{L_s(D)} \le A$ *then the first boundary value problem (1)-(2) has a unique strong solution* $u(x) \in \dot{W}_2^2(D)$.

Proof. The constant $\bar{\rho}_A$ will be chosen such that $\bar{\rho}_A \le d_A$. Therefore, according to theorem 2 we must prove only the uniqueness of the solution. Let $u^1(x)$ and $u^2(x)$ be two strong solutions of the first boundary problem (1)-(2) from the space $\dot{W}_2^2(D)$, and

$$\mathcal{L}_{(1)} = \sum_{i.j=1}^{n} a_{ij}(x)\, u^{(1)}(x) \frac{\partial^2}{\partial x_i \partial x_j}\ .$$

We have

$$\mathcal{L}_{(1)}\left(u^1 - u^2\right) = \sum_{i,j=1}^{n} a_{ij}\left(x, u^1\right) u_{ij}^1 - \sum_{i,j=1}^{n} \left[a_{ij}\left(x, u^1\right) - a_{ij}\left(x, u^2\right)\right] u_{ij}^2 -$$

$$- f(x) = -\sum_{i,j=1}^{n} \left[a_{ij}\left(x, u^1\right) - a_{ij}\left(x, u^2\right)\right] u_{ij}^2 = F(x)\ . \tag{8}$$

On the other hand, according to (7)

$$|F(x)| \le H_1 \left|u^1 - u^2\right| \sum_{i,j=1}^{n} |u_{ij}^2|\ . \tag{9}$$

Let $q_1 = \frac{np_1}{n-p_1}$. From theorem 1 we obtain

$$\left\|u^1 - u^2\right\|_{L_{q_2}(D)} \le C_1(p_1) \left\|u^1 - u^2\right\|_{W_{q_1}^1(D)} \tag{10}$$

Now applying theorem 2, from (8)-(10) we conclude

$$\left\|u^1 - u^2\right\|_{L_{q_1}(D)} \le C_1 C_3 \|F\|_{L_{p_1}(D)} \le$$

$$\le C_1 C_3 H \left[\int_D \left|u^1 - u^2\right|^{p_1} \left(\sum_{i,j=1}^{n} |u_{ij}^2|\right)^{p_1} dx\right]^{\frac{1}{p_1}} \le$$

$$\le C_1 C_3 H \left\|u^1 - u^2\right\|_{L_{q_1}(D)} \left[\int_D \left(\sum_{i,j=1}^{n} |u_{ij}^2|\right)^{n} dx\right]^{\frac{1}{n}} \le$$

$$\leq 2nC_1C_3H\left\|u^1-u^2\right\|_{L_{q_1}(D)}\left\|u^2\right\|_{W_2^2(D)}(mesD)^{\frac{2-n}{2n}}\leq$$

$$\leq 2nC_1C_3^2H\left\|u^1-u^2\right\|_{L_{q_1}(D)}\|f\|_{L_2(D)}(mesD)^{\frac{2-n}{2n}}\leq$$

$$\leq 2nC_1C_3^2H\left\|u^1-u^2\right\|_{L_{q_1}(D)}\|f\|_{L_s(D)}(mesD)^{\frac{2-n}{2n}+\frac{s-2}{2s}}\leq$$

$$\leq 2nC_1C_3^2HA\,(mesD)^{\frac{s-n}{ns}}\left\|u^1-u^2\right\|_{L_{q_1}(D)}\leq$$
$$\leq 2nC_1C_3^2HA\rho_A^{\frac{s-n}{ns}}\left\|u^1-u^2\right\|_{L_{q_1}(D)}. \qquad (11)$$

Let ρ' be such that

$$2C_1C_3^2HA\left(\rho'\right)^{\frac{s-2}{2s}}=\frac{1}{2}\,.$$

We choose $\rho_A=\min\{d_A,\rho'\}$. Then from (11) it follows that

$$\left\|u^1-u^2\right\|_{L_{q1}(D)}\leq\frac{1}{2}\left\|u^1-u^2\right\|_{L_{q_1}(D)},$$

i.e. $u^1(x)=u^2(x)$ a.e. in D.

Theorem 6. *Let $n\geq 3$, and the coefficients of the operator $\mathcal{M}$ satisfy conditions (3′)-(4′) and (5) (at $\alpha=1$). Then for any $s\in(n,\infty)$ at $A>0$ there exists $\rho_A=\rho_A(\mu,\sigma,n,\partial D,H,s,A)$ such that if $mesD\leq\rho_A$, $f(x)\in L_s(D)$ and $\|f\|_{L_s(D)}\leq A$ then the first boundary value problem (1′)-(2) has a unique strong solution $\dot W_2^2(D)$.*

Proof. Let $n\geq 3$, and $u^1(x)$ and $u^2(x)$ be two strong solutions of the first boundary value problem (1′)-(2) from the space $\dot W_2^2(D)$ and denote

$$\mathcal{M}_{(1)}=\sum_{i,j=1}^{n}a_{ij}\left(x,u^1(x),u_x^1(x)\right)\frac{\partial^2}{\partial x_i\partial x_j}\,.$$

We have

$$\mathcal{M}_{(1)}\left(u^1-u^2\right)=\sum_{i,j=1}^{n}a_{ij}\left(x,u^1,u_x^1\right)u_{ij}^1-$$
$$-\sum_{i,j=1}^{n}\left[a_{ij}\left(x,u^1,u_x^1\right)-a_{ij}\left(x,u^2,u_x^2\right)\right]u_{ij}^2-f(x)= \qquad (12)$$
$$=-\sum_{i,j=1}^{n}\left[a_{ij}\left(x,u^1,u_x^1\right)-a_{ij}\left(x,u^2,u_x^2\right)\right]u_{ij}^2=F_1(x)\,.$$

On the other hand, according to (5)

$$|F_1(x)|\leq H\left(|u^1-u^2|+|u_x^1-u_x^2|\right)\sum_{i,j=1}^{n}|u_{ij}^2|\,. \qquad (13)$$

Let $q_2 = \frac{np_1}{n-p_1}$. From theorem 1 we obtain

$$\left\|u^1 - u^2\right\|_{W^1_{q_4(D)}} \leq C_1\left(p_1\right)\left\|u^1 - u^2\right\|_{W^2_{p_1}(D)}. \tag{14}$$

Applying theorem 2, from (12)-(14) we conclude

$$\left\|u^1 - u^2\right\|_{W^1_{q_2}(D)} \leq C_1C_3\left\|F_1\right\|_{L_{p_1}(D)} \leq$$

$$\leq C_1C_3H\left[\int\limits_D \left(|u^1-u^2| + |u^1_x - u^2_x|\right)^{p_1}\left(\sum_{i,j=1}^n |u^2_{ij}|\right)^{p_1} dx\right]^{\frac{1}{p_1}} \leq$$

$$\leq C_1C_3H\|u^1 - \underset{D}{u^2}\|_{W^1_{q_2}(D)}\left(\int\limits_D \sum_{i,j=1}^n |u^2_{ij}|^n dx\right)^{\frac{1}{n}} \leq \tag{15}$$

$$\leq 2nC_1C_3H\left\|u^1-u^2\right\|_{W^1_{q_2}(D)}\left\|u^2\right\|_{W^2_2(D)}(mesD)^{\frac{2-n}{2n}} \leq$$

$$\leq 2nC_1C_3^2H\left\|u^1-u^2\right\|_{W^1_{q_2}(D)}\left\|u^2\right\|_{L_s(D)}(mesD)^{\frac{s-n}{2s}} \leq$$

$$\leq 2nC_1C_3^2HA\rho_A^{\frac{s-n}{sn}}\left\|u^1-u^2\right\|_{W^1_{q_2}(D)}.$$

Let ρ'' be so that

$$2nC_1C_3^2HA\left(\rho''\right)^{\frac{s-n}{sn}} = \frac{1}{2}.$$

We choose $\rho_A = \min\{d_A, \rho''\}$. Then from (15) it follows that

$$\left\|u^1-u^2\right\|_{W^1_{q_2}(D)} \leq \frac{1}{2}\left\|u^1-u^2\right\|_{W^1_{q_2}(D)},$$

i.e. $u^1(x) = u^2(x)$ a.e. in D. The theorem is proved.

References

[1]. Ladyzhenskaya O.A., Ural'tseva N.N. *Linear and quasilinear elliptic equations.* // "Academic Press", New York, 1969.

[2]. Chicco M. *Solvability of the Dirichlet problem in $H^{2,p}(\Omega)$ for a class of linear second order elliptic partial differential equations.* // Boll. Un. Mat. Ital., 1971, v.4, No 4, pp.374-387.

[3]. Talenti G. *Sopra una classe di equazioni ellitiche a coefficienti misurableili.* // Ann. Mat. Pura Appl., 1965, v. 69, pp.285-304.

[4]. Mamedov I.T., Agayeva R.A., *The first boundary value problem for non-divergent linear second order elliptic equations of Cordes type.* // Trans. Nat. Acad. Sci. Azerb., 2002, v.XXII, No1, pp.150-167.

[5]. Mamedov I.T., Muradov T.R. *On the Dirichlet problem for Gilbarg-Serrin equation.* // Proc. Inst. Math. Mech. Nat. Acad. Sci. Azerb., 2002, v.XVI (XXIV), pp.81-93.

[6]. Vitanza C. $W^{2,p}$ *- regularity for a class of elliptic second order equations with discontinuous coefficients.* // Le Matematiche, 1992, v.47, pp.177-186.

[7]. Vitanza C. *A new contribution to the* $W^{2,p}$*-regularity for a class of elliptic second order equations with discontinuous coefficients.* // Le Matematiche, 1993, v.48, pp.287-296.

[8]. Alkhutov Yu.A., Mamedov I.T. *Some properties of the solutions of the first boundary value problem for parabolic equations with discontinuous coefficients.* // Sov. Math. Dokl., 1985, v.32, No 2, pp.343-347.

[9]. Alkhutov Yu.A., Mamedov I.T. *The first boundary value problem for nondivergence second order parabolic equations with discontinuous coefficients.* // Math. USSR Sbornik, 1988, v.59, No 2, pp.471-495.

[10]. Wen G.C. *Initial-mixed boundary value problems for parabolic equations of second order with measurable coefficients in a higher dimensional domain.* // Proc. of the Second ISAAK Congress, 2000, v.1, pp.185-192.

[11]. Mamedov I.T., Hassanpour S.A. *Strong solvability of the first boundary value problem for second order quasi-linear elliptic equations.* // Trans. Nat. Acad. Azerb., 2002, v.XXII, No4, pp.153-166.

[12]. Hassanpour S.A. *On uniqueness of strong solution of Dirichlet problem for second order quasilinear elliptic equations.* // Proc. of IMM of NAS of Azerb., 2003, v.XIX, pp.101-106.

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