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ESTIMATIONS OF THE BEST APPROXIMATION OF CONVOLUTION OF FUNCTIONS BY MEANS OF THEIR SMOOTHNESS MODULES IN $L_p(\mathbb{T})$

Abstract

In the paper the upper estimations of the best (in $L_r(\mathbb{T})$) approximation $E_{n-1}(h)_r$ of the convolution $h = f * g$ of two 2π periodic functions $f \in L_p(\mathbb{T})$ and $g \in L_q(\mathbb{T})$ are obtained by means of the product $\omega_l(f;\delta)_p\,\omega_k(g;\delta)_q$ of smoothess modules of these functions, where $p,q \in [1,\infty]$, $l,k \in \mathbb{N}$, $1/r = 1/p + 1/q - 1 \geq 0$. It is proved in the case $p,q \in (1,\infty)$ and the case $p = 1$, $q = r \in (1,\infty)$ that the obtained estimations are exact in the terms of order on the classes of convolutions with given majorants of smoothness modules of functions forming the convolution.

In what follows we use the following notation.

- $\mathbb{T}$ is the interval $(-\pi,\pi]$ in $\mathbb{R}$.
- $L_p(\mathbb{T})$, $1 \leq p < \infty$, is the space of all measurable 2π periodic functions $f : \mathbb{R} \to \mathbb{C}$ with finite L_p-norm $\|f\|_p = \left((2\pi)^{-1}\int_{\mathbb{T}} |f(x)|^p\,dx\right)^{1/p} < \infty$.
- $C(\mathbb{T}) \equiv L_\infty(\mathbb{T})$ is the space of all continuous 2π periodic functions with norm $\|f\|_\infty \equiv \max\{|f(x)| : x \in \mathbb{T}\}$.
- $E_n(f)_p$ is the best approximation of a function f in the metric of $L_p(\mathbb{T})$ by the trigonometric polynomials of order $\leq n \in \mathbb{Z}_+$.
- $T_{n,p}(f)$ is the polynomial of the best approximation of a function f in the metric $L_p(\mathbb{T}) : \|f - T_{n,p}(f)\|_p = E_n(f)_p$, $n \in \mathbb{Z}_+$.
- $S_n(f;\cdot)$ is the partial sum of order $n \in \mathbb{Z}_+$ of the Fourier-Lebesgue series of a function $f \in L_1(\mathbb{T}) : S_n(f;x) = \sum_{|\nu|=0}^{n} c_\nu(f)\,e^{i\nu x}$, $x \in \mathbb{T}$.
- $\omega_l(f;\delta)_p$ is the smoothness module of l-th order of a function $f \in L_p(\mathbb{T})$:
  $\omega_l(f;\delta)_p = \sup\left\{\left\|\Delta_t^l f\right\|_p : t \in \mathbb{R},\ |t| \leq \delta\right\}$, $l \in \mathbb{N}$, $\delta \geq 0$, where $\Delta_t^l f(x) = \sum_{\nu=0}^{l} (-1)^{l-\nu}\binom{l}{\nu} f\ (x + \nu t)$, $x \in \mathbb{R}$.
- $\Omega_l(0,\pi] \equiv \Omega_l$ is the class of all functions $\omega(\delta)$ defined on $(0,\pi]$ and satisfying the conditions: $0 < \omega(\delta) \downarrow 0\ (\delta \downarrow 0)$ and $\delta^{-l}\omega(\delta) \downarrow (\delta \uparrow)$.

Denote, for $1 \leq p \leq \infty$, $l \in \mathbb{N}$, $\omega \in \Omega_l$,

$$H_p^l[\omega] = \left\{f \in L_p(\mathbb{T}) : \omega_l(f;\delta)_p \leq \omega(\delta),\ \delta \in (0,\pi]\right\}.$$

The convolution $h = f * g$ of $f \in L_1(\mathbb{T})$ and $g \in L_1(\mathbb{T})$ is defined by the formula: $h(x) = (f * g)(x) = (1/2\pi)\int_{\mathbb{T}} f(x-y)\,g(y)\,dy$; it is known (see f.e. [1], v.1, § 2.1, pp.64-65, [2], v.1, § 3.1, pp.65-66) that the function h is defined almost everywhere, 2π periodic, measurable and $\|h\|_1 \leq \|f\|_1\,\|g\|_1$ (whence it follows in particular that $h = f * g \in L_1(\mathbb{T})$). The last statement is a particular case of the following result known as the W.Young's inequality (see, f.e. [1], v.1, Theorem (1.15), pp.67-68; [2], v.2, Theorem 13.6.1, pp.176-177; [2], v.1, Theorem 3.1.4, p.70, Theorem 3.1.6, p.72).

Given $p \in [1, \infty]$, let $p' = p/(p-1)$ be the exponent conjugate to p. As usual, we assume that $p' = 1$ for $p = \infty$ and $p' = \infty$ for $p = 1$.

Theorem A. *Let* $h = f * g$ be the convolution of $f \in L_p(\mathbb{T})$ and $g \in L_q(\mathbb{T})$ for $1 \le p,\ q \le \infty$. *Then, for* $1/r = 1/p + 1/q - 1$,

- *If* $1/r > 0$ *then* h *belongs to* $L_r(\mathbb{T})$ *and* $\|h\|_r \le \|f\|_p \|g\|_q$.
- *If* $1/r = 0$ *then* h *belongs to* $C(\mathbb{T}) \equiv L_\infty(\mathbb{T})$ *and* $\|h\|_\infty \le \|f\|_p \cdot \|g\|_{p'}$.

Recall that the Fourier coefficients $c_n(h)$ of $h = f * g$ of two arbitrary functions $f \in L_1(\mathbb{T})$ and $g \in L_1(\mathbb{T})$ are calculated by the formula (see [1], v.1, Theorem (1.5), p.64; [2], v.1, p.66, formula (3.1.5)) $c_n(h) = c_n(f * g) = c_n(f) \cdot c_n(g)$ for every $n \in \mathbb{Z}$.

Between the best approximation and the smoothness modulus of a function $f \in L_p(\mathbb{T})$ there exists the known connection expressed by the following direct theorem of the approximation theory (see [3; p.226, Theorem 1], [4; p.338, Inequality (1)] and references therein).

Theorem B. *Let* $f \in L_p(\mathbb{T})$ with $1 \le p \le \infty$, and $l \in \mathbb{N}$. *Then*

$$E_{n-1}(f)_p \le C_1(l)\, \omega_l(f; \pi/n)_p \ \text{ for every } n \in \mathbb{N} \tag{1}$$

(*where* $C_1(l)$ *is a positive constant depending only on the parameter* l).

Estimation (1) is exact in the terms of order on $H_p^l[\omega]$, that is, there exists a function $f_0(x; p; \omega) \in H_p^l[\omega]$ such that $E_{n-1}(f_0)_p \ge C_2(l, p)\, \omega(\pi/n)$ for every $n \in \mathbb{N}$. The *individual* function $f_0(x; p; \omega)$ is extremal for $p = 1$ (see [5; p.575], [6; p.24]) and for $p = \infty$ (see [7; p.73], [8; p.292], [9; p.52], [10; p.503]; see the both of the cases in [11; pp.378-380] and [12; Lemma 1, pp.44-45]). For the case $1 < p < \infty$, exactness of estimation (1) is realized by means of some sequence $\{f_n(x; p; \omega)\}_{n=1}^{\infty} \subset H_p^l[\omega]$ (see [12; Lemma 2, pp.45-46], [13; Lemma 2.4, p.104], [14; Lemma 4, pp.69-70], [15; Lemma 3, pp.221-223]). Moreover, given $p \in (1, \infty)$, for the existence of an individual function $f_0 \in H_p^l[\omega]$ that realizes the estimation $E_{n-1}(f_0)_p \ge C_2(l, p)\, \omega(\pi/n)$, $n \in \mathbb{N}$, it is necessary and sufficient that the majorant $\omega \in \Omega_l$ satisfies the S_l - Stechkin condition $\omega \in S_l$: there exists a number $\gamma \in (0, l)$ such that $\delta^{-(l-\gamma)}\omega(\delta) \downarrow (\delta \uparrow)$ (see [12; Remark 1, p.50], [13; Remark 6, pp.94-95], [14; Theorem 2, pp.70-72], [15; Remark 6, pp.231-232]). Recall that there is a series of equivalent descriptions of the condition $\omega \in S_l$ in [10; § 2, p.493].

In the present paper the analogous questions are considered for the convolution $h = f * g$ of two arbitrary functions $f \in L_p(\mathbb{T})$ and $g \in L_q(\mathbb{T})$.

Theorem 1. *Let* $h = f * g$ be the convolution of $f \in L_p(\mathbb{T})$ and $g \in L_q(\mathbb{T})$ for $p, q \in [1, \infty]$. Then, for $1/r = 1/p + 1/q - 1$ and $l, k \in \mathbb{N}$,

(i) *If* $1/r > 0$ *then* $h \in L_r(\mathbb{T})$ *and*

$$E_{n-1}(h)_r \le C_3(l, k)\, \omega_l(f; \pi/n)_p\ \omega_k(g; \pi/n)_q \ \text{ for every } n \in \mathbb{N}.$$

(ii) *If* $1/r = 0$ *then* $h \in C(\mathbb{T}) \equiv L_\infty(\mathbb{T})$ *and*

$$E_{n-1}(h)_\infty \le C_4(l, k)\, \omega_l(f; \pi/n)_p\, \omega_k(g; \pi/n)_q \ \text{ for every } n \in \mathbb{N},$$

where $q = p'$ *and* $C_3(l, k) = C_4(l, k) = C_1(l)\, C_1(k)$.

Proof. Note that $r = pq/(p+q-pq) \in [1,\infty)$ for $1/r > 0$ and $r = \infty$ for $1/r = 0$. By Theorem A, $h \in L_r(\mathbb{T})$ for $1/r > 0$ and $h \in C(\mathbb{T})$ for $1/r = 0$. Denote by $\mathbb{P}_n(\mathbb{T})$ a set of all trigonometric polynomials of degree $\le n \in \mathbb{Z}_+$. Since $T_{n,p}(f),\ T_{n,q}(g) \in \mathbb{P}_n(\mathbb{T})$ then $T_{n,p}(f) * g,\ T_{n,q}(g) * f,\ T_{n,p}(f) * T_{n,q}(g) \in \mathbb{P}_n(\mathbb{T})$ and therefore $T_{n,p}(f) * g + T_{n,q}(g) * f - T_{n,p}(f) * T_{n,q}(g) \in \mathbb{P}_n(\mathbb{T})$. Further, by distributivity and commutativity of convolution operation, we have that

$$f * g - (T_{n,p}(f) * g + T_{n,q}(g) * f - T_{n,p}(f) * T_{n,q}(g)) = \\ = (f - T_{n,p}(f)) * (g - T_{n,q}(g)),$$

and, applying W.Young's inequality (see Theorem A), we obtain that

$$E_n(h)_r \le \|f * g - \{T_{n,p}(f) * g + T_{n,q}(g) * f - T_{n,p}(f) * T_{n,q}(g)\}\|_r = \\ = \|(f - T_{n,p}(f)) * (g - T_{n,q}(g))\|_r \le \\ \le \|f - T_{n,p}(f)\|_p \|g - T_{n,q}(g)\|_q = E_n(f)_p E_n(g)_q,$$

whence

$$E_n(h)_r \le E_n(f)_p E_n(g)_q, \quad n \in \mathbb{Z}_+. \tag{2}$$

Applying inequality (1) in (2), we obtain the required estimations in (i) and (ii). Theorem 2 is proved.

Remark 1. Estimation (2) for $p, q \in (1,\infty)$ can be obtained with the help of the known M.Riesz inequality (see, f.e. [4; § 5.11, p.339, Inequality (6)], [16; § 8.20, p.594], [1; v.1, § 7.6, p.423], [2; v.2, § 12.10, p.120])

$$\|\psi - S_n(\psi)\|_p \le C_5(p) E_n(\psi)_p \text{ for } 1 < p < \infty,\ \psi \in L_p(\mathbb{T}),\ n \in \mathbb{Z}_+, \tag{3}$$

if we take into account the obvious equality $f * g - S_n(f * g) = [f - S_n(f)] * [g - S_n(g)]$ (see, f.e. [17; p.138, Remark 2]) in the following chain of inequalites

$$E_n(h)_r \le \|h - S_n(h)\|_r = \|[f - S_n(f)] * [g - S_n(g)]\|_r \le \\ \le \|f - S_n(f)\|_p \|g - S_n(g)\|_q \le C_5(p) E_n(f)_p \cdot C_5(q) E_n(g)_q,$$

whence $E_n(h)_r \le C_5(p) C_5(q) E_n(f)_p E_n(g)_q$ for $n \in \mathbb{Z}_+$.

Denote, for $p, q \in [1,\infty],\ l, k \in \mathbb{N},\ \omega \in \Omega_l,\ \varphi \in \Omega_k$,

$$H_p^l[\omega] * H_q^k[\varphi] = \left\{h = f * g :\ f \in H_p^l[\omega],\ g \in H_q^k[\varphi]\right\}.$$

Estimations (i) and (ii) of Theorem 1 are exact in the terms of order on $H_p^l[\omega] * H_q^k[\varphi]$ for $p, q \in (1,\infty)$.

Theorem 2. *Let $p, q \in (1,\infty)$, $1/r = 1/p + 1/q - 1 \ge 0$, $l, k \in \mathbb{N}$, $\omega \in \Omega_l$ and $\varphi \in \Omega_k$. Then*

$$\sup\left\{E_{n-1}(h)_r : h \in H_p^l[\omega] * H_q^k[\varphi]\right\} \asymp \omega(\pi/n)\,\varphi(\pi/n) \ \ for\ n \in \mathbb{N}. \tag{4}$$

The upper estimates in (4) follow from inequalities (i) and (ii) of Theorem 1. The lower estimates in (4) are realized by some sequence $\{h_n(x;p;q;\omega;\varphi)\}_{n=1}^{\infty} \subset H_p'[\omega] * H_q^k[\varphi]$, $h_n(x;p;q;\omega;\varphi) = C_6^{-1}(l,p) f_n(x;p;\omega) * C_6^{-1}(k,q) g_n(x;q;\varphi)$ for every $n \in \mathbb{N}$ (see Lemma 1 below). If we put some restrictions on the behavior of majorants $\omega \in \Omega_l$ and $\varphi \in \Omega_k$ then the lower estimates in (4) are realized by means of an individual function (see Lemma 3 below) $h_0(x;p;q;\omega;\varphi) = C_{14}^{-1}(l,p) f_0(x;p;\omega) * C_{15}^{-1}(k,q) g_0(x;q;\varphi) \subset H_p^l[\omega] * H_q^k[\varphi]$.

Lemma 1. *Let $p, q \in (1,\infty)$, $1/r = 1/p + 1/q - 1 \ge 0$, $l,\ k \in \mathbb{N}$, $\omega \in \Omega_l$ and $\varphi \in \Omega_k$. There exist sequences $\{f_n(\cdot;p;\omega)\}_{n=1}^{\infty} \subset L_p(\mathbb{T})$ and $\{g_n(\cdot;q;\varphi)\}_{n=1}^{\infty} \subset L_q(\mathbb{T})$ such that*

(i) $\omega_l(f_n;\delta)_p \le C_6(l,p)\,\omega(\delta)\,,\delta\in(0,\pi] \Rightarrow \left\{C_6^{-1}(l,p)\,f_n(x;p;\omega)\right\}\subset H_p^l[\omega]\,,$

$\omega_k(g_n;\delta)_q \le C_6(k,q)\,\varphi(\delta)\,,\delta\in(0,\pi] \Rightarrow \left\{C_6^{-1}(k,q)\,g_n(x;q;\varphi)\right\}\subset H_q^k[\varphi].$

(ii) $E_{n-1}(h_n)_r \ge C_7(r)\,\omega(\pi/n)\,\varphi(\pi/n)$ *for* $h_n = f_n * g_n$ and every $n\in\mathbb{N}$.

Proof. Put, for every $n\in\mathbb{N}$, $f_n(x;p;\omega) = n^{1/p-1}\omega(\pi/n)\,d_{4n}(x)$ and $g_n(x;q;\varphi) = n^{1/q-1}\varphi(\pi/n)\,d_{4n}(x)$, where $d_{4n}(x) = \sum_{\nu=1}^{4n} e^{i\nu x}$ for $x\in\mathbb{T}$. Then $h_n(x;p;q;\omega;\varphi) = n^{1/p+1/q-2}\omega(\pi/n)\,\varphi(\pi/n)\,d_{4n}(x)$. In the paper [15; p.221, Formula (11)] the estimation $\|\mathrm{Re}\,d_{4n}\|_p \le [2p/(p-1)]^{1/p}(4n)^{1-1/p} = C_8(p)\,(4n)^{1-1/p}$ was proved. It follows from this estimation that

$$\|d_{4n}\|_p \le \|\mathrm{Re}\,d_{4n}\|_p + \|\mathrm{Im}\,d_{4n}\|_p \le (1+C_9(p))\,C_8(p)\,(4n)^{1-1/p} = C_{10}(p)\,n^{1-1/p},$$

where $C_9(p)$ is the constant in the known M.Riesz inequality (see f.e. [4; § 3.11.1, p.169], [16; § 8.14, p.566], [1; v.1, § 7.2, p.404], [2; v.2, § 12.9.1, p.113]) $\left\|\tilde{\psi}\right\|_p \le C_9(p)\,\|\psi\|_p$ for the function $\tilde{\psi}$ trigonometric conjugate to a function $\psi\in L_p(\mathbb{T})\,,\ 1<p<\infty$. By the estimation for $\|d_{4n}\|_p$, we obtain that

$$\|f_n(\cdot;p;\omega)\|_p = n^{1/p-1}\omega(\pi/n)\,\|d_{4n}\|_p \le C_{10}(p)\,\omega(\pi/n) \le C_{10}(p)\,\omega(\pi) < \infty,$$

whence $\{f_n(\cdot;p;\omega)\}_{n=1}^{\infty}\subset L_p(\mathbb{T})$. We have similarly that

$$\|g_n(\cdot;q;\varphi)\|_q = n^{1/q-1}\varphi(\pi/n)\,\|d_{4n}\|_q \le C_{10}(q)\,\varphi(\pi/n) \le C_{10}(q)\,\varphi(\pi) < \infty.$$

Therefore $\{g_n(\cdot;q;\varphi)\}_{n=1}^{\infty}\subset L_q(\mathbb{T})$.

We prove (i). For an arbitrary fixed $n\in\mathbb{N}$ and any $\delta\in(0,\pi]$, either $\delta\le\pi/n$ or $\delta>\pi/n$.

For the case $\delta\le\pi/n$, taking into account that $\delta^{-l}\omega(\delta)\downarrow(\delta\uparrow)$ and using S.N.Bernstein-M.Riesz-F.Riesz-A.Zygmund inequality for L_p-norms of derivatives of trigonometric polynomials (see [1; v.2, § 10.3, p.20, § 16.7, p.414], [4; § 4.8, p.223, p.228, p.230], [16; p.47, p.895], [18; § 2.11, p.115]) we obtain that

$$\omega_l(f_n;\delta)_p \le \delta^l\left\|f_n^{(l)}\right\|_p = \delta^l n^{1/p-1}\omega(\pi/n)\left\|d_{4n}^{(l)}\right\|_p \le$$
$$\le \delta^l n^{1/p-1}\omega(\pi/n)\,(4n)^l\,\|d_{4n}\|_p \le$$
$$\le \delta^l n^{1/p-1}\omega(\pi/n)\,(4n)^l\,C_{10}(p)\,n^{1-1/p} =$$
$$= C_{10}(p)\,4^l\delta^l n^l\omega(\pi/n) \le C_{10}(p)\,4^l\pi^l\omega(\delta)\,.$$

For $\delta>\pi/n$, taking into account that $\omega(\delta)\uparrow(\delta\uparrow)$, we obtain that

$$\omega_l(f_n;\delta)_p \le 2^l\,\|f_n\|_p = 2^l n^{1/p-1}\omega(\pi/n)\,\|d_{4n}\|_p \le$$
$$\le 2^l n^{1/p-1}\omega(\pi/n)\,C_{10}(p)\,n^{1-1/p} =$$
$$= 2^l C_{10}(p)\,\omega(\pi/n) \le 2^l C_{10}(p)\,\omega(\delta)\,.$$

By the estimations obtained, for every $\delta\in(0,\pi]$ we have that

$$\omega_l(f_n;\delta)_p \le C_{10}(p)\,2^l\left(2^l\pi^l+1\right)\omega(\delta) = C_6(l,p)\,\omega(\delta)\,,$$

whence it follows that $\left\{C_6^{-1}(l,p)\, f_n(x;p;\omega)\right\}_{n=1}^{\infty} \subset H_p^l[\omega]$.

The estimation $\omega_k(g_n;\delta)_q \le C_{10}(q)\, 2^k\left(2^k\pi^k+1\right)\varphi(\delta) = C_6(k,q)\,\varphi(\delta)$, $\delta \in (0,\pi]$ is similar. Thus $\left\{C_6^{-1}(k,q)\, g_n(x;q;\varphi)\right\}_{n=1}^{\infty} \subset H_q^k[\varphi]$.

Now we prove (ii). In the case $r \in (1,\infty)$, by (3) and the estimation ([15; p.221, Formula (11)]) $\|\operatorname{Im} d_{4n} - S_n(\operatorname{Im} d_{4n})\|_r \ge C_{11}(r)\, n^{1-1/r}$ for every $n \in \mathbb{N}$, we obtain that

$$C_5(r)\, E_{n-1}(h_n)_r \ge C_5(r)\, E_n(h_n)_r \ge \|h_n - S_n(h_n)\|_r =$$
$$= n^{1/p+1/q-2}\omega(\pi/n)\,\varphi(\pi/n)\,\|d_{4n} - S_n(d_{4n})\|_r \ge$$
$$\ge n^{1/p+1/q-2}\omega(\pi/n)\,\varphi(\pi/n)\,\|\operatorname{Im} d_{4n} - S_n(\operatorname{Im} d_{4n})\|_r \ge$$
$$\ge n^{1/p+1/q-2}\omega(\pi/n)\,\varphi(\pi/n)\, C_{11}(r)\, n^{1-1/r} =$$
$$= C_{11}(r)\, n^{-[1/r-(1/p+1/q-1)]}\omega(\pi/n)\,\varphi(\pi/n) =$$
$$= C_{11}(r)\,\omega(\pi/n)\,\varphi(\pi/n)$$

for every $n \in \mathbb{N}$.

In the case $r = \infty$ ($\Longrightarrow 1/r = 1/p + 1/q - 1 = 0 \Leftrightarrow 1/p + 1/q = 1$) we note first that, for a complex valued function $\psi \in C(\mathbb{T})$,

$$E_n(\operatorname{Re}\psi)_\infty = \|\operatorname{Re}\psi - T_{n,\infty}(\operatorname{Re}\psi)\|_\infty \le \|\operatorname{Re}\psi - \operatorname{Re}(T_{n,\infty}(\psi))\|_\infty =$$
$$= \|\operatorname{Re}[\psi - T_{n,\infty}(\psi)]\|_\infty \le \|\psi - T_{n,\infty}(\psi)\|_\infty = E_n(\psi)_\infty,$$

whence $E_n(\psi)_\infty \ge E_n(\operatorname{Re}\psi)_\infty$, $n \in \mathbb{Z}_+$.

Involving inequality (132) in [18; p.117]: $3E_n(\psi)_\infty \ge \|\psi - \sigma_{n,n}(\psi)\|_\infty$, where $\sigma_{n,n}(\psi;\cdot)$ is the Vallèe-Poussin sum [18; p.51, Formula (49)] of a real valued function $\psi \in C(\mathbb{T})$, and noting that $\cos x = 1$ at $x = 0$, we obtain (see also [15; Remark 2, p.222]) that

$$3E_n(\operatorname{Re} d_{4n})_\infty \ge \|\operatorname{Re} d_{4n} - \sigma_{n,n}(\operatorname{Re} d_{4n})\|_\infty \ge$$
$$\ge \left\|\sum_{\nu=1}^{4n}\cos\nu x - \left\{\sum_{\nu=1}^{n}\cos\nu x + \sum_{\nu=n+1}^{2n}\left(1-\frac{\nu-n}{n}\right)\cos\nu x\right\}\right\|_\infty \ge$$
$$\ge \left|\sum_{\nu=1}^{4n}1 - \left\{\sum_{\nu=1}^{n}1 + \sum_{\nu=n+1}^{2n}\left(1-\frac{\nu-n}{n}\right)\right\}\right| = \frac{5n+1}{2} > \frac{5}{2}n$$

for every $n \in \mathbb{N}$. Taking into account the last estimation, we have

$$E_{n-1}(h_n)_\infty \ge E_n(h_n)_\infty \ge E_n(\operatorname{Re} h_n)_\infty =$$
$$= n^{1/p+1/q-2}\omega(\pi/n)\,\varphi(\pi/n)\, E_n(\operatorname{Re} d_{4n})_\infty \ge$$
$$\ge n^{1/p+1/q-2}\omega(\pi/n)\,\varphi(\pi/n)\,(5/6)\, n =$$
$$= (5/6)\, n^{1/p+1/q-1}\omega(\pi/n)\,\varphi(\pi/n) = (5/6)\,\omega(\pi/n)\,\varphi(\pi/n),$$

for every $n \in \mathbb{N}$. Lemma 1 is proved.

Let M_0 be the class of all sequences $\lambda = \{\lambda_n\}_{n=1}^{\infty}$ of reals such that $0 < \lambda_n \downarrow 0$ as $n \uparrow \infty$. Given numbers $\theta \in [1,\infty)$ and $l \in \mathbb{N}$, we put

$$D^{(\theta)} = \left\{\lambda \in M_0 : \sum\nolimits_{n=1}^{\infty} n^{-1}\lambda_n^{\theta} < \infty\right\},$$

$$B^{(\theta)}=\left\{\lambda\in M_0:\left(\sum\nolimits_{\nu=n}^{\infty}\nu^{-1}\lambda_\nu^\theta\right)^{1/\theta}=O\left(\lambda_n\right),\ n\in\mathbb{N}\right\},$$

$$B_l^{(\theta)}=\left\{\lambda\in M_0:n^{-l}\left(\sum\nolimits_{\nu=1}^{n}\nu^{\theta l-1}\lambda_\nu^\theta\right)^{1/\theta}=O\left(\lambda_n\right),\ \ n\in\mathbb{N}\right\}.$$

Note for example that the sequence of $\lambda_n=n^{-\alpha},\ n\in\mathbb{N}$, belongs to $D^{(\theta)}$ and $B^{(\theta)}$ for every $\alpha>0$ (it is clear that $B^{(\theta)}\subset D^{(\theta)}$) and belongs to the class $B_l^{(\theta)}$ for $0<\alpha<l$, where $\theta\in[1,\infty)$.

Lemma 2. *Let* $p\in(1,\infty)$, $p'=p/(p-1)$, $l\in\mathbb{N}$ *and* $\lambda=\{\lambda_n\}\in M_0$. *Then the function* $f_0(x;p;\lambda)=\sum_{n=1}^{\infty}\lambda_n n^{-1/p'}e^{inx}$ for $x\in\mathbb{T}$, *satisfies the following conditions*

(i) $f_0\in L_p(\mathbb{T})$ *for* $\lambda\in D^{(p)}$.

(ii) $E_{n-1}(f_0)_p=O(\lambda_n)\,,\ n\in\mathbb{N}$, *for* $\lambda\in B^{(p)}$.

(iii) $\omega_l(f_0;\pi/n)_p=O(\lambda_n)\,,\ n\in\mathbb{N}$, *for* $\lambda\in B_l^{(\theta)}\cap B^{(p)}$, *where* $\theta=\min\{2,p\}$.

Proof. (i) Since $\lambda\in D^{(p)},\ c_n(f_0)=n^{-1/p'}\lambda_n\downarrow 0\,(n\uparrow\infty)$ and

$$\sum\nolimits_{n=1}^{\infty}n^{p-2}c_n^p(f_0)=\sum\nolimits_{n=1}^{\infty}n^{p-2}n^{-p/p'}\lambda_n^p=\sum\nolimits_{n=1}^{\infty}n^{-1}\lambda_n^p<\infty,$$

then, by the Hardy-Littlewood theorem (see f.e. [16; § 10.3, p.657-658]; [1; v.2, § 12.6, Lemma (6.6) on p.193]; [2; v.1, § 7.3.5, pp.148-149]), $f_0\in L_p(\mathbb{T})$ and $\|f_0\|_p\asymp\left(\sum_{n=1}^{\infty}n^{-1}\lambda_n^p\right)^{1/p}$.

(ii) Taking into account that $\lambda\in B^{(p)}$ and applying the Hardy-Littlewood Theorem, we obtain that

$$E_{n-1}(f_0)_p\le\|f_0-S_{n-1}(f_0)\|_p=\left\|\sum\nolimits_{\nu=n}^{\infty}\nu^{-1/p'}\lambda_\nu e^{i\nu x}\right\|_p\asymp$$

$$\asymp\left(\sum\nolimits_{\nu=n}^{\infty}\nu^{p-2}\nu^{-p/p'}\lambda_\nu^p\right)^{1/p}=\left(\sum\nolimits_{\nu=n}^{\infty}\nu^{-1}\lambda_\nu^p\right)^{1/p}=O(\lambda_n)$$

for every $n\in\mathbb{N}$.

(iii) By inequality $\omega_l(\psi;\pi/n)_p\le C_{13}(l,p)\,n^{-l}\left(\sum_{\nu=1}^{n}\nu^{\theta l-1}E_{\nu-1}^\theta(\psi)_p\right)^{1/\theta}$, (see [19; Lemma 1, p.502] for $p=2,\ l=1$; [20; Theorem 1, p.126] for $p\in(1,\infty)\,,\ l\in\mathbb{N}$) where $\psi\in L_p(\mathbb{T})$, $p\in(1,\infty)$, $\theta=\min\{2,p\}$, and taking into account that $\lambda\in B^{(p)}\cap B_l^{(\theta)}$, we have

$$\omega_l(f_0;\pi/n)_p\le C_{13}(l,p)\,n^{-l}\left(\sum\nolimits_{\nu=1}^{n}\nu^{\theta l-1}E_{\nu-1}^\theta(f_0)_p\right)^{1/\theta}=$$

$$=O\left(n^{-l}\left(\sum\nolimits_{\nu=1}^{n}\nu^{\theta l-1}\lambda_\nu^\theta\right)^{1/\theta}\right)=O(\lambda_n)$$

for every $n\in\mathbb{N}$. Lemma 2 is proved.

Lemma 3. *Let* $p,q\in(1,\infty)$, $1/r=1/p+1/q-1\ge0$, $l,k\in\mathbb{N}$, $\theta=\min\{2,p\}$, $\gamma=\min\{2,q\}$, and let $\omega\in\Omega_l$, $\varphi\in\Omega_k$, $\{\omega(\pi/n)\}_{n=1}^{\infty}\in B^{(p)}\cap B_l^{(\theta)}$ *and* $\{\varphi(\pi/n)\}_{n=1}^{\infty}\in B^{(q)}\cap B_k^{(\gamma)}$. *Then there exist functions* $f_0(x;p;\omega)\in L_p(\mathbb{T})$ *and* $g_0(x;q;\varphi)\in L_q(\mathbb{T})$ *such that*

(i) $\omega_l(f_0;\delta)_p \le C_{14}(l,p)\ \omega(\delta), \delta\in(0,\pi] \Rightarrow C_{14}^{-1}(l,p)\, f_0(\cdot;p;\omega)\in H_p^l[\omega]$,

$\omega_k(g_0;\delta)_q \le C_{15}(k,q)\,\varphi(\delta), \delta\in(0,\pi] \Rightarrow C_{15}^{-1}(k,q)\, g_0(\cdot;q;\varphi)\in H_q^k[\varphi]$.

(ii) $E_{n-1}(h_0)_r \ge C_{16}(r,l,k)\,\omega(\pi/n)\,\varphi(\pi/n)$, $n\in\mathbb{N}$, *for* $h_0 = f_0 * g_0$.

Proof. Put $\omega_n = \omega(\pi/n)$ and $\varphi_n = \varphi(\pi/n)$ for every $n\in\mathbb{N}$. Let $f_0(x;p;\omega) = \sum_{n=1}^{\infty} n^{-1/p'}\omega_n e^{inx}$ and $g_0(x;q;\varphi) = \sum_{n=1}^{\infty} n^{-1/q'}\varphi_n e^{inx}$ for every $x\in\mathbb{T}$, where $p' = p/(p-1)$ and $q' = q/(q-1)$. Then, by (i) of Lemma 2, taking into account that $\{\omega_n\}\in B^{(p)}\subset D^{(p)}$and $\{\varphi_n\}\in B^{(q)}\subset D^{(q)}$, and by (iii) of Lemma 2, taking into account that $\{\omega_n\}\in B_l^{(\theta)}\cap B^{(p)}$ and $\{\varphi_n\}\in B_k^{(\gamma)}\cap B^{(q)}$, we obtain that $f_0\in L_p(\mathbb{T})$, $g_0\in L_q(\mathbb{T})$, $\omega_l(f_0;\pi/n)_p = O(\omega_n)$ and $\omega_k(g_0;\pi/n)_q = O(\varphi_n)$ for every $n\in\mathbb{N}$. Hence $\omega_l(f_0;\delta)_p \le 2^l C_{17}(l,p)\,\omega(\delta)$ and $\omega_k(g_0;\delta)_q \le 2^k C_{18}(k,q)\,\varphi(\delta)$ for every $\delta\in(0,\pi]$.

Further, for the convolution, we have that

$$h_0(x;p;q;\omega;\varphi) = (f_0(\cdot;p;\omega) * g_0(\cdot;q;\varphi))(x) = \sum_{n=1}^{\infty} n^{-(1/p'+1/q')}\omega_n\varphi_n e^{inx}.$$

For $r\in(1,\infty)$, by inequality (3) and Hardy-Littlewood theorem, we have that

$$C_5(r)\,E_{n-1}(h_0)_r \ge \|h_0 - S_{n-1}(h_0)\|_r = \left\|\sum_{\nu=n}^{\infty}\nu^{-(1/p'+1/q')}\omega_\nu\varphi_\nu e^{i\nu x}\right\|_r \ge$$

$$\ge C_{19}(r)\left(\sum_{\nu=n}^{\infty}\nu^{r-2-(1/p'+1/q')r}\omega_\nu^r\varphi_\nu^r\right)^{1/r} = C_{19}(r)\left(\sum_{\nu=n}^{\infty}\nu^{-1}\omega_\nu^r\varphi_\nu^r\right)^{1/r} \ge$$

$$\ge C_{19}(r)\left(\sum_{\nu=n+1}^{2n}\nu^{-1}\omega_\nu^r\varphi_\nu^r\right)^{1/r} \ge C_{19}(r)\,\omega_{2n}\varphi_{2n}\left(\sum_{\nu=n+1}^{2n}\nu^{-1}\right)^{1/r} \ge$$

$$\ge C_{19}(r)\,\omega\left(\frac{\pi}{2n}\right)\varphi\left(\frac{\pi}{2n}\right)(2n)^{-1/r}n^{1/r} \ge C_{19}(r)\,2^{-(l+k+1/r)}\omega\left(\frac{\pi}{n}\right)\varphi\left(\frac{\pi}{n}\right),$$

whence $E_{n-1}(h_0)_r \ge C_{16}(r,l,k)\,\omega(\pi/n)\,\varphi(\pi/n)$ for every $n\in\mathbb{N}$.

For $r=\infty$, by the N.K.Bary inequality [8; p.293], we obtain that

$$4E_{n-1}(h_0)_\infty \ge 4E_n(h_0)_\infty \ge 4E_n(\operatorname{Re} h_0)_\infty \ge \sum_{\nu=2n}^{\infty}\nu^{-(1/p'+1/q')}\omega_\nu\varphi_\nu =$$

$$= \sum_{\nu=2n}^{\infty}\nu^{-1}\omega_\nu\varphi_\nu \ge \sum_{\nu=2n+1}^{3n}\nu^{-1}\omega_\nu\varphi_\nu \ge \omega_{3n}\varphi_{3n}\sum_{\nu=2n+1}^{3n}\nu^{-1} \ge$$

$$\ge \omega(\pi/3n)\,\varphi(\pi/3n)\,(3n)^{-1}n \ge 3^{-(l+k+1)}\omega(\pi/n)\,\varphi(\pi/n),$$

whence $E_{n-1}(h_0)_\infty \ge 4^{-1}3^{-(l+k+1)}\omega(\pi/n)\,\varphi(\pi/n)$ for every $n\in\mathbb{N}$. Lemma 3 is proved.

Remark 2. Theorem 2 holds also in the case $p = 1 < q < \infty$ ($\Rightarrow r = q\in(1,\infty)$) or $q = 1 < p < \infty$ ($\Rightarrow r = p\in(1,\infty)$). Moreover, the last case does not require a separate consideration by virtue of commutativity of convolution. The upper estimate follows from (i) of Theorem 1, and the lower estimate is realized by the family $\{h_n(x;1;q;\omega;\varphi)\} \subset H_1^l[\omega] * H_q^k[\varphi]$ (see Lemma 4 below).

Lemma 4. *Let $l,k \in \mathbb{N}$, $\omega \in \Omega_l$, $\varphi \in \Omega_k$, $1 < q < \infty$. There exist sequences $\{f_n(x;1;\omega)\}_{n=1}^{\infty} \subset L_1(\mathbb{T})$ and $\{g_n(x;q;\varphi)\}_{n=1}^{\infty} \subset L_q(\mathbb{T})$ such that*

(i) $\omega_l(f_n;\delta)_1 \le C_{20}(l)\,\omega(\delta)\,,\ \delta \in (0,\pi] \Rightarrow \left\{C_{20}^{-1}(l)\,f_n(x;1;\omega)\right\} \subset H_1^l[\omega]\,,$

$\omega_k(g_n;\delta)_q \le C_{21}(k,q)\,\varphi(\delta)\,,\delta \in (0,\pi] \Rightarrow \left\{C_{21}^{-1}(k,q)\,g_n(x;q;\varphi)\right\} \subset H_q^k[\varphi]\,.$

(ii) $E_{n-1}(h_n)_r \ge C_{22}(q)\,\omega(\pi/n)\,\varphi(\pi/n)\,,\ n \in \mathbb{N}$, *for* $h_n = f_n * g_n$.

Proof. Put $f_n(x;1;\omega) = \omega(\pi/n)\,F_{2n}(x)$ for every $n \in \mathbb{N}$, where $F_{2n}(x)$ is a Fejer kernel of order $2n$: $F_{2n}(x) = 1/2 + \sum_{\nu=1}^{2n}(1-\nu/(2n+1))\cos\nu x$. Put $g_n(x;q;\varphi) = n^{1/q-1}\varphi(\pi/n)\,\mathrm{Re}\,d_{2n}(x)$, where $d_{2n}(x) = \sum_{\nu=1}^{2n} e^{i\nu x}$. Since $\|F_{2n}\|_1 = 1$ for every $n \in \mathbb{N}$, then $\|f_n(\cdot;1;\omega)\|_1 = \omega(\pi/n)\,\|F_{2n}\|_1 = \omega(\pi/n) \le \omega(\pi) < \infty$, whence $\{f_n(\cdot;1;\omega)\} \subset L_1(\mathbb{T})$. Further, by estimation $\|\mathrm{Re}\,d_{2n}\|_q \le C_8(q)\,(2n)^{1-1/q}$ (see the proof of Lemma 1), we have that

$$\|g_n(\cdot;q;\varphi)\|_q = n^{1/q-1}\varphi(\pi/n)\,\|\mathrm{Re}\,d_{2n}\|_q \le$$

$$\le n^{1/q-1}\varphi(\pi/n)\,C_8(q)\,(2n)^{1-1/q} \le 2^{1-1/q}C_8(q)\,\varphi(\pi) < \infty,$$

and therefore $\{g_n(x;q;\varphi)\}_{n=1}^{\infty} \subset L_q(\mathbb{T})$.

(i) If $\delta \le \pi/n$ then, for arbitrary fixed $n \in \mathbb{N}$ and $\delta \in (0,\pi]$, by the condition $\delta^{-l}\omega(\delta) \downarrow (\delta \uparrow)$, we have that

$$\omega_l(f_n;\delta)_1 \le \delta^l \left\| f_n^{(l)} \right\|_1 = \delta^l\omega(\pi/n)\left\|F_{2n}^{(l)}\right\|_1 \le$$

$$\le \delta^l\omega(\pi/n)\cdot(2n)^l\,\|F_{2n}\|_1 = 2^l\delta^l n^l\omega(\pi/n) \le 2^l\pi^l\omega(\delta)\,.$$

If $\delta > \pi/n$ then, by the condition $\omega(\delta) \uparrow (\delta \uparrow)$, we obtain that

$$\omega_l(f_n;\delta)_1 \le 2^l\,\|f_n\|_1 = 2^l\omega(\pi/n)\,\|F_{2n}\|_1 = 2^l\omega(\pi/n) \le 2^l\omega(\delta)\,.$$

By the estimations obtained, we have that $\omega_l(f_n;\delta)_1 \le 2^l\left(\pi^l+1\right)\omega(\delta) = C_{20}(l)\,\omega(\delta)\,,\ \delta \in (0,\pi]$. Hence it follows that $\left\{C_{20}^{-1}(l)\,f_n(x;1;\omega)\right\}_{n=1}^{\infty} \subset H_1^l[\omega]$. The proof of the second estimation in (i) for $\omega_k(g_n;\delta)_q$ repeats the corresponding arguments of Lemma 1, and we obtain that

$$\omega_k(g_n;\delta)_q \le 2^{k+1-1/q}C_8(q)\left(\pi^k+1\right)\varphi(\delta)\ \text{ for every } \delta \in (0;\pi]\,.$$

(ii) According to Formula (1.9) of [1; v.1, p.65], we have that

$$h_n(x;1;q;\omega;\varphi) = (f_n(\cdot;1;\omega) * g_n(\cdot;q;\varphi))(x) = \omega(\pi/n)\,n^{1/q-1}\varphi(\pi/n)\,F_{2n}(x)\,.$$

Hence, by (3), we obtain that $(r = q \in (1,\infty))$

$$C_5(r)\,E_{n-1}(h_n)_r = C_5(q)\,E_{n-1}(h_n)_q \ge C_5(q)\,E_n(h_n)_q \ge \|h_n - S_n(h_n)\|_q =$$

$$= \omega\left(\frac{\pi}{n}\right)n^{1/q-1}\varphi\left(\frac{\pi}{n}\right)\|F_{2n} - S_n(F_{2n})\|_q \ge$$

$$\geq \omega\left(\frac{\pi}{n}\right) n^{1/q-1} \varphi\left(\frac{\pi}{n}\right) C_{23}(q)\, n^{1-1/q},$$

whence $E_{n-1}(h_n)_r \geq C_{22}(q)\,\omega(\pi/n)\,\varphi(\pi/n)$, $n \in \mathbb{N}$, where $C_{22}(q) = \frac{C_{23}(q)}{C_5(q)}$.

To complete the proof, we establish the estimation $\|F_{2n} - S_n(F_{2n})\|_q \geq$ $\geq C_{23}(q)\, n^{1-1/q}$, $n \in \mathbb{N}$, that was used above. Since

$$F_m(x) = \frac{1}{2} + \sum\nolimits_{\nu=1}^{m} \left(1 - \frac{\nu}{(m+1)}\right) \cos \nu x = \frac{\sin^2((m+1)\,x/2)}{2\,(m+1)\sin^2(x/2)}$$

for every $m \in \mathbb{N}$, we have that

$$F_{2n}(x) - S_n(F_{2n}; x) = \frac{(2n+1)\, F_{2n}(x) - (n+1)\, F_n(x)}{(2n+1)} =$$

$$= \frac{\sin^2((2n+1)\,x/2) - \sin^2((n+1)\,x/2)}{2\,(2n+1)\sin^2(x/2)} =$$

$$= \frac{(1/2)\,(\cos(n+1)\,x - \cos(2n+1)\,x)}{2\,(2n+1)\sin^2(x/2)} = \frac{\sin(nx/2)\cdot\sin((3n+2)\,x/2)}{2\,(2n+1)\sin^2(x/2)},$$

whence taking into account inequalities $\sin z \geq (2/\pi)\,z$ for every $z \in [0, \pi/2]$ and $|\sin z| \leq |z|$, $z \in R$, we obtain that

$$\|F_{2n} - S_n(F_{2n})\|_q^q = (2\pi)^{-1} \int_T |F_{2n}(x) - S_n(F_{2n}; x)|^q\, dx =$$

$$= \int_{-\pi}^{\pi} \frac{|\sin(n/2)\,x|^q\, |\sin((3n+2)/2)\,x|^q}{2\pi(2\,(2n+1)\sin^2(x/2))^q} dx \geq$$

$$\geq \int_0^{\pi/(3n+2)} \frac{(\sin(n/2)\,x)^q (\sin((3n+2)/2)\,x)^q}{2\pi(2\,(2n+1)\sin^2(x/2))^q} dx \geq$$

$$\geq \int_0^{\pi/(3n+2)} \frac{\left(\pi^{-1} nx\right)^q \left(\pi^{-1}(3n+2)\,x\right)^q 2^{2q}}{2\pi(2\,(2n+1))^q x^{2q}} dx =$$

$$= \pi^{-2q} 2^{q-1} (2n+1)^{-q}\, n^q\, (3n+2)^{q-1} \geq$$

$$\geq \pi^{-2q} 2^{q-1} (3n)^{-q}\, n^q\, (3n)^{q-1} = \pi^{-2q} 2^{q-1} 3^{-1} n^{q-1},$$

and therefore $\|F_{2n} - S_n(F_{2n})\|_q \geq \pi^{-2} 2^{1-1/q} 3^{-1/q} n^{1-1/q} = C_{23}(q)\, n^{1-1/q}$. Lemma 4 is proved.

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