ASYMPTOTIC BEHAVIOR OF EIGEN-VALUES OF A BOUNDARY VALUE PROBLEM WITH SPECTRAL PARAMETER IN THE BOUNDARY CONDITIONS FOR THE SECOND ORDER ELLIPTIC DIFFERENTIAL-OPERATOR EQUATION

Abstract

In this present paper we obtain the asymptotic formula for eigen values of the following boundary value problems

\[-u''(x) + Au(x) = \lambda u(x), \quad x \in (0, b),\]

\[u'(0) - \lambda u(0) = 0, \quad u'(b) + \lambda u(b) = 0,\]

where \(A = A^* \geq \omega^2 I\) in \(H, A^{-1}\) is completely continuous in \(H, \lambda > 0\) is a spectral parameter, \(H\) is a separable Hilbert space.

Let \(H\) be a separable Hilbert space. By \(L^2((0, b) ; H)\) \((0 < b < \infty)\) we denote a set of all vector-functions \(x \rightarrow u(x) : (0, b) \rightarrow H,\) strongly measurable and such that

\[\int_0^b \|u(x)\|_H^2 \, dx < +\infty.\]

As is known \(L^2((0, b) ; H)\) is a Hilbert space with respect to the scalar product

\[(u, v)_{L^2((0, b) ; H)} = \int_0^b (u(x), v(x))_H \, dx.\]

Let \(A\) be a self-adjoint positive-definite operator in \(H\) \((A = A^* \geq \omega^2 I, \omega > 0, I\) be a unit operator in \(H)\) with domain of definition \(D(A).\) Since \(A^{-1}\) is bounded in \(H,\) then

\[H(A) = \left\{ u : u \in D(A), \|u\|_{H(A)} = \|Au\|_H \right\}\]

is a Hilbert space whose norm is equivalent to the norm of the graph of the operator \(A.\)

\[W^2_2((0, b) ; H(A), H) = \left\{ u : Au, u'' \in L^2((0, b) ; H) ; \right\}\]

\[\|u\|_{W^2_2((0, b) ; H(A), H)}^2 = \|Au\|_{L^2((0, b) ; H)}^2 + \|u''\|_{L^2((0, b) ; H)}^2\]

is a Hilbert space \([1, \text{p}.23].\)

In the space \(L^2((0, b) ; H)\) let’s consider the boundary value problem

\[-u''(x) + Au(x) = \lambda u(x), \quad x \in (0, b),\]

\[u'(0) - \lambda u(0) = 0,\]

\[u'(b) + \lambda u(b) = 0.\]
\begin{equation}
    u'(b) + \lambda u(b) = 0, \tag{2}
\end{equation}

where $A = A^* \geq \omega^2 I$ in $H$, $A^{-1}$ is completely continuous in $H$, $\lambda > 0$ is a spectral parameter.

The goal of the paper is to study the asymptotic behavior of eigen-values of problem (1)-(2) knowing the asymptotic distribution of eigen-values of the operator $A$.

Note that the asymptotics of eigen-values of boundary value problems for Sturm-Liouville differential operator equation on a finite segent with the same spectral parameter in the equation and in one of the boundary conditions was studied in papers of V.I. Gorbachuk and M.A.Rybak \cite{2}, M.A.Rybak \cite{3}. More precisely, in particular in the papers \cite{2}-\cite{3} it was studied the asymptotic distribution of eigen-values of the following boundary-value problem

\begin{equation}
    -u''(x) + Au(x) = \lambda u(x), \quad x \in (0, b), \tag{1}
\end{equation}

\begin{equation}
    u'(0) + \lambda u(0) = 0, \tag{3}
\end{equation}

\begin{equation}
    u(b) = 0
\end{equation}
in $L^2((0, b); H) \oplus H$. It is proved that if the spectrum of the operator $A$ is discrete, then the operator generated by the boundary value problem (1), (3) have also discrete spectrum. Eigen-values of the problem (1), (3) form two infinite sequences $\lambda_k \sim \sqrt{\mu_k}$: $\lambda_{n,k} \sim \mu_k + \frac{\pi^2}{b^2} n^2$, where $\mu_k = \mu_k(A)$ are eigen-values of the operator $A$.

In the paper substantially using the ideas and method of the papers \cite{2}, \cite{3} it is proved that the problem (1)-(2) has only one set of eigen-values: $\lambda_{n,k} \sim \mu_k + \frac{\pi^2}{b^2} n^2$, where $\mu_k$ are the eigen-values of the operator.

**Theorem 1.** Let $A = A^* \geq \omega^2 I$ in $H$ and $A^{-1}$ be completely continuous in $H$. Then for the eigen-values of the problem (1)-(2) it is valid the following asymptotic formula

\begin{equation}
    \lambda_{n,k} = \mu_k + \xi_n \quad (k = 1, 2, \ldots; \quad n = N, N + 1, \ldots),
\end{equation}

where $\mu_k = \mu_k(A)$ are the eigen-values of the operator $A$, $\xi_n \sim \frac{\pi^2}{b^2} n^2$, $N$ is a natural number.

**Proof.** The eigen-elements of the operator $A$ that correspond to eigen-values $\mu_k(A)$ we denote by $\varphi_k$. It is known that $\{\varphi_k\}$ forms an orthonormed basis in $H$. Then allowing for spectral expansion, for coefficients $u_k = (u, \varphi_k)$ we get the following problem

\begin{equation}
    -u''_k(x) + (\mu_k - \lambda) u_k(x) = 0, \tag{4}
\end{equation}

\begin{equation}
    u'_k(0) - \lambda u_k(0) = 0,
\end{equation}
The general solution of ordinary differential equations (4) is of the form:

\[ u_k(x) = c_1 e^{-x \sqrt{\mu_k - \lambda}} + c_2 e^{-(b-x) \sqrt{\mu_k - \lambda}}, \quad (6) \]

where \( c_i \) (\( i = 1, 2 \)) are arbitrary constants.

Having put (6) into (5) we get a system with respect to \( c_i \) (\( i = 1, 2 \)) whose determinant is of the form

\[ K(\lambda) = - \left( \sqrt{\mu_k - \lambda + \lambda} \right)^2 + \left( \sqrt{\mu_k - \lambda - \lambda} \right)^2 e^{-2b \sqrt{\mu_k - \lambda}}. \]

So, the eigen-values of the problem (1)-(2) these zero are for the equation

\[ e^{2b \sqrt{\mu_k - \lambda}} \left( \sqrt{\mu_k - \lambda + \lambda} \right)^2 - \left( \sqrt{\mu_k - \lambda - \lambda} \right)^2 = 0. \quad (7) \]

Write equation (7) in the form of a system of equations

\[ e^{b \sqrt{\mu_k - \lambda}} \left( \sqrt{\mu_k - \lambda + \lambda} \right) - \left( \sqrt{\mu_k - \lambda - \lambda} \right) = 0, \quad (8) \]

\[ e^{b \sqrt{\mu_k - \lambda}} \left( \sqrt{\mu_k - \lambda + \lambda} \right) + \left( \sqrt{\mu_k - \lambda - \lambda} \right) = 0. \quad (9) \]

Thus, the eigen-values of the problem (1)-(2) consist of that real \( \lambda \neq \mu_k \) that even at \( k \) satisfy the equations (8)-(9).

Look for the eigen-values of the problem (1)-(2) lesser than \( \mu_k \). Put \( \sqrt{\mu_k - \lambda} = y \). Equations (8) and (9) in this case are equivalent to the equations

\[ e^{by} \left( y^2 - y - \mu_k \right) + \left( y^2 + y - \mu_k \right) = 0, \quad 0 < y < \sqrt{\mu_k}, \quad (10) \]

\[ e^{by} \left( y^2 - y - \mu_k \right) - \left( y^2 + y - \mu_k \right) = 0, \quad 0 < y < \sqrt{\mu_k}, \quad (11) \]

respectively.

Let’s prove the absence of solutions of equations (10) and (11) on the interval \( (0, \sqrt{\mu_k}) \). Rewrite equation (10) in the form

\[ y \ sh \frac{by}{2} + \left( \mu_k - y^2 \right) ch \frac{by}{2} = 0, \quad 0 < y < \sqrt{\mu_k}. \quad (12) \]

Let’s consider the function \( \Phi_k(y) = y \ sh \frac{by}{2} + \left( \mu_k - y^2 \right) ch \frac{by}{2}, \quad 0 < y < \sqrt{\mu_k} \). Obviously, at each \( k \) and for all \( y \in (0, \sqrt{\mu_k}) \) \( \Phi_k(y) > 0 \). Therefore, equation (12), consequently equation (10) has no solutions on the interval \( k \) for any \( (0, \sqrt{\mu_k}) \).

In a similar way it is shown that equation (11) has no solutions on the interval \( (0, \sqrt{\mu_k}) \) for any \( k \). Thus, equation (7) has no zeros in the case when \( \lambda < \mu_k \). Now let’s study that solutions of equation (7) that are greater than \( \lambda > \mu_k \). In this case equations (8) and (9) are equivalent to the equations

\[ e^{ibz} \left( z^2 + iz + \mu_k \right) + \left( -z^2 + iz - \mu_k \right) = 0, \quad z \in (0, +\infty), \quad (13) \]
respectively, where \( z = \sqrt{\lambda - \mu_k} \).

The left hand side of equation (13) is a quasipolynomial. Applying Langer’s theory on finding the asymptotics of zeros of a quasipolynomial [4, p.427] we find the asymptotics of (13). Equation (13) is equivalent to the condition

\[
e^{ibz} = 1 + O\left(\frac{1}{z}\right),
\]

The zeros of equation (15) at sufficiently great \( z \) are close (i.e. approximately equal) to the zeros of the equation

\[
e^{ibz} = 1.
\]

The zeros of the last equation are of the form:

\[
z_n = \frac{2\pi n}{b}, \quad n = 1, 2, \ldots.
\]

Hence for the zeros of equation (13) and hereby for the zeros of equation (8) we get the asymptotic formulae

\[
\lambda_{n,k}^{(1)} = \mu_k + \xi_n, \quad k = 1, 2, \ldots; \quad n = N, \ N + 1, \ldots
\]

where \( \xi_n \sim \frac{\pi^2}{b^2} \ (2n)^2 \), \( N \) is a natural number.

Equation (14) is equivalent to the equation

\[
e^{ibz} = -1 + O\left(\frac{1}{z}\right).
\]

The zeros of equation (17) are found in a similar way. They are of the form:

\[
\lambda_{n,k}^{(2)} \sim \mu_k + \frac{\pi^2}{b^2} \ (2n+1)^2, \quad k = 1, 2, \ldots; \quad n = N, \ N + 1, \ldots
\]

From (16) and (18) we get the asymptotic formula for the eigen-values of problem (1)-(2):

\[
\lambda_{n,k} \sim \mu_k + \frac{\pi^2}{b^2} n^2, \quad k = 1, 2, \ldots; \quad n = N, \ N + 1, \ldots
\]

Theorem 1 is proved.

**Corollary 1.** Let the conditions of theorem 1 be fulfilled.

Let the eigen-values of the operator \( A \) arranged increase order satisfy the condition

\[
\mu_k (A) \sim a k^\alpha \left( \lim_{k \to \infty} \frac{\mu_k (A)}{k^\alpha} = a, 0 < a, \alpha < \infty \right)
\]

then the eigen-values of problem (1)-(2) have the following asymptotics
\[ \lambda_m \sim dm^{\frac{2\alpha}{\alpha + \beta}}, \]

where \(0 < d < \infty\) are constants.

The proof of Corollary 1 follows from the following statement that is in the paper [5] (see also [6]).

**Lemma.** Let be given the two sequences \(\{\mu_k\}\) and \(\{\nu_n\}\) such that \(\mu_k \sim ak^\alpha, \nu_n = cn^\beta, 0 < a, c, \alpha, \beta < \infty; k = 1, 2, \ldots; n = 1, 2, \ldots\) Compose the sum \(\mu_k + \nu_n\) with all possible \(k\) and \(n\). We numerate the obtained numbers by increase and denote by \(\lambda_m\).

Then the sequences \(\{\lambda_m\}\) have the asymptotics \(\lambda_m \sim dm^\delta\), where

\[ \delta = \frac{\alpha \beta}{\alpha + \beta}, \quad d = \left(\frac{\alpha}{2\gamma}\right)^\delta a^\frac{\alpha}{\alpha + \beta} c^\frac{\beta}{\alpha + \beta}, \quad \gamma = \int_0^\frac{\pi}{2} \sin^\frac{\alpha}{2} t \cos^\frac{\beta}{2} t dt. \]

**Example.** In the rectangle \([0, b] \times [0, 1]\), \((0 < b < \infty)\) consider the eigen-value problem

\[ -\frac{\partial^2 v(x, y)}{\partial x^2} + \frac{\partial^4 v(x, y)}{\partial y^4} + \omega v(x, y) = \lambda v(x, y), \quad (19) \]
\[ \frac{\partial v(0, y)}{\partial x} - \lambda v(0, y) = 0, \quad (20) \]
\[ \frac{\partial v(b, y)}{\partial x} + \lambda v(b, y) = 0, \quad (21) \]

where \(\omega > 0\) is a number.

The problem (19)-(20) is led to the form (1)-(2) where \(u(x, \cdot)\) is a vector-function with values in a Hilbert space \(H = L_2(0, 1)\). The operator \(A\) is determined as follows:

\[ D(A) = W_2^4 \left( (0, 1); u^{(j)}(0) = u^{(j)}(1), j = 0, 3 \right), \quad Au = \frac{\partial^4 u}{\partial y^4} + \omega u, \quad (22) \]

Obviously, the operator \(A\) determined by formula (27) is self-adjoint and for sufficiently great \(\omega > 0\) is positive-definite, a \(A^{-1}\) is completely continuous in \(L_2(0, 1)\).

Simple calculations show that the eigen-values of the operator \(A\) are of the form: \(\mu_k(A) = 16k^4 + \omega\). Then on the basis of corollary 1 the eigen-values of the problem (19)-(21) have the asymptotics: \(\lambda_m \sim \text{const } m^\frac{4}{3}\).

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**References**


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