**Elshan N. MAMEDOV**

# ON COMPLETENESS OF A PART OF EIGEN AND ADJOINED ELEMENTS OF SECOND ORDER OPERATOR BUNDLES

**Abstract**

*In the paper we receive conditions providing the completeness of the eigen and adjoined vectors responding to eigen-values from some half-planes, contained in the left half-plane.*

Let's consider in separable Hilbert space $H$ the operator bundle

$$P(\lambda) = -\lambda^2 E + \lambda A_1 + A_2 + A^2, \tag{1}$$

where $E$ is a unit operator, $A_1,\ A_2,\ A$ are linear, in general, unlimited operators. Later on we'll assume, that operator coefficients $A_1,\ A_2$ and $A$ satisfy the following conditions (see [1]):

1) $A$ is a positive-definite, self-adjoint operator with lower bound of spectrum $\mu_0 > 0$ i.e.

$$\inf_{\mu\in\sigma(A)} \mu = \mu_0 > 0;$$

2) $A^{-1}$ is a completely continuous operator, i.e
$A^{-1} \in \sigma_\infty(H)$;
3) Operators $B_j = A_j A^{-j}$ $(j = 1, 2)$ are restricted $H$, i.e
$B_j = A_j A^{-j} \in L(H)$, $(j = 1, 2)$;
4) Operator $E + A_2A^{-2} = E + B_2$ is boundedly reversible in $H$, i.e.
$(E + B_2)^{-1} \in L(H)$.

As is known, in this case the spectrum of the operator bundle is discrete, i.e. a point of the spectrum consists of isolated eigen-values, they are finite-to-one and have a unique limit point at infinity [2]. (See for definitions of eigen-values, eigen and adjoined vectors of bundle (1) for ex. The papers [1,2]).

Denote by

$$P_0(\lambda) = -\lambda^2 E + A^2,$$

$$P_1(\lambda) = \lambda A_1 + A_2$$

and

$$P(\lambda) = P_0(\lambda) + P_1(\lambda).$$

It is obvious, that the operator bundle

$$P_0(\lambda) = -\lambda^2 E + A^2$$

has a spectrum from the left half-plane in half-plane

$$\Pi(-\mu_0) = \{\lambda : \operatorname{Re}\lambda \le -\mu_0\}.$$

In the papers [1-7] the completeness of the eigen and adjoined vectors, responding to eigen-values from the left half-plane $\Pi_- = \{\lambda : \operatorname{Re}\lambda < 0\}$ is investigated. Later on using this that aud regular solvability of boundary value problem of the homogeneous equation the completeness of elementary solutions of the given homogeneous equation [3] is proved.

Note, that for the ground of the completeness of elementary solutions in the weight spaces it is necessary to prove the completeness of a part of eigen and adjoined vectors, responding to eigen-values from half-plane

$$\Pi(-\gamma) = \{\lambda : \operatorname{Re}\lambda < -\gamma \le 0\}, \quad \gamma \in [0, \mu_0).$$

To this end we investigate the resolvents of the operator bundle $P(\lambda)$ in the band

$$\Pi(\mu_0; \gamma) = \{\lambda : -\gamma \le \operatorname{Re}\lambda \le 0\}, \quad \gamma \in [0, \mu_0).$$

The following theorem is true.

**Theorem 1.** *Let $\gamma \in [0, \mu_0)$, and conditions 1), 3) be fulfilled, morever there holds the inequality*

$$C_1(\gamma; \mu_0)\|B_1\| + C_0(\gamma; \mu_0)\|B_0\| \le 1, \tag{2}$$

*where*

$$C_1(\gamma; \mu_0) = \frac{1}{2}\frac{\mu_0^2 + \gamma^2}{\mu_0^2 - \gamma^2}; \tag{3}$$

$$C_0(\gamma; \mu_0) = \frac{\mu_0^2}{\mu_0^2 - \gamma^2}. \tag{4}$$

*Then in the band $\Pi(\mu_0, \gamma)$ the operator bundle $P(\lambda)$ is boundedly reversible and there hold the estimates*

$$\left\|A^2 P^{-1}(\lambda)\right\| + \left\|\lambda A P^{-1}(\lambda)\right\| + \left\|A^2 P^{-1}(\lambda)\right\| \le const. \tag{5}$$

**Proof.** It is obvious that spectrum of the operator bundle $P_0(\lambda) = -\lambda^2 E + A^2$ from in left half plane is contained in the half-plane

$$\Pi = \{\lambda : \operatorname{Re}\lambda \le -\mu_0\}.$$

Therefore operator bundle $P_0(\lambda)$ is reversible in the band $\Pi(\mu_0, \gamma)$, for any $\gamma \in [0, \mu_0)$. Then for $\lambda \in \Pi(\mu_0, \gamma)$ we have:

$$P(\lambda) = P_0(\lambda) + P_1(\lambda) = \left(E + P_1(\lambda) P_0^{-1}(\lambda)\right) P_0(\lambda). \tag{6}$$

For any $\lambda \in \Pi(\mu_0, \gamma)$ $(\lambda = \gamma + i\xi,\ \xi \in R = (-\infty, \infty))$ the estimates are true:

$$\left\|P_1(\lambda) P_0^{-1}(\lambda)\right\| \le \left\|\lambda A_1 P_0^{-1}(\lambda)\right\| + \left\|A_2 P_0^{-1}(\lambda)\right\| \le$$

$$\le \|B_1\| \cdot \left\|\lambda A P_0^{-1}(\lambda)\right\| + \|B_2\| \cdot \left\|A^2 P_0^{-1}(\lambda)\right\|. \tag{7}$$

Since, for $\lambda = \gamma + i\xi,\ \xi \in R = (-\infty, \infty)$

$$\left\|\lambda A P_0^{-1}(\lambda)\right\| = \left\|(\gamma + i\xi) A P_0^{-1}(i\xi + \gamma)\right\| = \|(i\xi + \gamma) \times =$$

$$\times A\left(-(i\xi + \gamma)^2 + A^2\right)^{-1}\Big\| = \left\|\left(\xi^2 + \gamma^2\right)^{1/2} A\left(-(i\xi + \gamma)^2 + A^2\right)^{-1}\right\|. \tag{8}$$

From the expansion of the self-adjoint operator it follows, that

$$\left\| \left(\xi^2+\gamma^2\right)^{1/2} A \left(-(i\xi+\gamma)^2+A^2\right)^{-1} \right\| =$$

$$= \sup_{\mu\in\sigma(A)} \left| \left(\xi^2+\gamma^2\right)^{1/2} \mu \left(-(i\xi+\gamma)^2+\mu^2\right)^{-1} \right| = \sup_{\mu\in\sigma(A)} \left| \left(\xi^2+\gamma^2\right)^{1/2} \times \right.$$

$$\left. \times \mu \left(-(i\xi+\gamma)^2+\mu^2\right)^{-1} \right| = \sup_{\mu\in\sigma(A)} \frac{\left(\xi^2+\gamma^2\right)^{1/2}\mu}{\left(\xi^2+\mu^2-\gamma^2\right)^2} \le \sup_{\mu\in\sigma(A)} \frac{\left(\xi^2+\gamma^2\right)^{1/2}\mu}{\xi^2+\mu^2-\gamma^2} \le$$

$$\le \sup_{\mu\ge\mu_0} \frac{1}{2}\frac{\xi^2+\gamma^2+\mu^2}{\xi^2+\mu^2-\gamma^2} = \sup_{\mu\ge\mu_0} \frac{1}{2}\left(1+\frac{2\gamma^2}{\xi^2+\mu^2-\gamma^2}\right) \le$$

$$\le \sup_{\mu\ge\mu_0} \frac{1}{2}\left(1+\frac{2\gamma^2}{\mu^2-\gamma^2}\right) \le \frac{1}{2}\left(1+\frac{2\gamma^2}{\mu_0^2-\gamma^2}\right) \le \frac{1}{2}\frac{\mu_0^2+\gamma^2}{\mu_0^2-\gamma^2} \tag{9}$$

Thus, from inequality (8) with regard to (9) it follows, that

$$\left\|\lambda A P_0^{-1}(\lambda)\right\| \le \frac{1}{2}\frac{\mu_0^2+\gamma^2}{\mu_0^2-\gamma^2} = C_1(\gamma;\mu_0). \tag{10}$$

Later on analogously we obtain for $\lambda = \gamma + i\xi \quad (\gamma\in[0,\mu_0), \xi\in R)$ that the following estimate is true:

$$\left\|A^2 P_0^{-1}(\lambda)\right\| \le \left\|A^2 P_0^{-1}(i\xi+\gamma)\right\| = \sup_{\mu\notin\sigma(A)} \left| \mu^2 \left(-(i\xi+\gamma)^2+\mu^2\right)^{-1} \right| \le$$

$$\le \sup_{\mu\in\sigma(A)} \frac{\mu^2}{\xi^2+\mu^2-\gamma^2} \le \sup_{\mu\ge\mu_0} \frac{\mu^2}{\mu^2-\gamma^2} = \frac{\mu_0^2}{\mu_0^2-\gamma^2} = C_0(\gamma;\mu_0) \tag{11}$$

Thus, from inequality (7) with regard to inequality (10) and (11) we obtain, that at $\lambda\in\Pi(\gamma,\mu_0)$ norm $\left\|P_1(\lambda) P_0^{-1}(\lambda)\right\| < 1$. In fact,

$$\left\|P_1(\lambda) P_0^{-1}(\lambda)\right\| \le \left(\|B_1\| C_1(\gamma;\mu_0) + \|B_2\| C_2(\gamma;\mu_0)\right) = \alpha(\gamma;\mu_0) < 1. \tag{12}$$

Then operator $E + P_1(\lambda) P_0^{-1}$ is boundedly reversible in band $\Pi(\mu_0,\gamma)$ and from equality (6) we obtain, that in this band

$$P^{-1}(\lambda) = P_0^{-1}(\lambda)\left(E + P_1(\lambda) P_0^{-1}(\lambda)\right)^{-1}. \tag{13}$$

To prove the theorem we'll beforehand estimate the norm $\left\|\lambda^2 P_0^{-1}(\lambda)\right\|$ for $\lambda\in\Pi(\mu_0,\gamma)$. Let $\lambda = \gamma + i\xi,\ \gamma\in[0,\mu_0),\ \xi\in R = (-\infty,\infty)$, then

$$\left\|\lambda^2 P_0^{-1}(\lambda)\right\| = \left\| \left(\xi^2+\gamma^2\right)\left(-(i\xi+\gamma)^2+A^2\right)^{-1} \right\| =$$

$$= \sup_{\mu\in\sigma(A)} \left| \left(\xi^2+\gamma^2\right)\left(-\left(-\xi^2+\gamma^2+2i\xi\gamma\right)\mu^2\right)^{-1} \right| \le \sup_{\mu\in\sigma(A)} \frac{\xi^2+\gamma^2}{\xi^2+\mu^2-\gamma^2} \le$$

$$\le \frac{\xi^2+\gamma^2}{\xi^2+\mu_0^2-\gamma^2} = \frac{\xi^2-\gamma^2}{\xi^2+\mu_0^2-\gamma^2} + \frac{2\gamma^2}{\xi^2+\mu_0^2-\gamma^2} \le$$

$$\leq 1+\frac{2\gamma^2}{\mu_0^2-\gamma^2}=\frac{\mu_0^2+\gamma^2}{\mu_0^2-\gamma^2}=C_2\left(\gamma;\mu_0\right) \tag{14}$$

Then for $\lambda=\gamma+i\xi,\ \xi\in(-\infty,\infty)$ and $\gamma\in[0,\mu_0)$ the following inequality is true:

$$\left\|\lambda^2P^{-1}(\lambda)\right\|+\left\|\lambda AP^{-1}(\lambda)\right\|+\left\|A^2P^{-1}(\lambda)\right\|=$$

$$=\left\|\lambda^2P_0^{-1}(\lambda)\left(E+P_1(\lambda)P_0^{-1}(\lambda)\right)^{-1}\right\|+\left\|\lambda AP_0^{-1}(\lambda)\left(E+P_1(\lambda)P_0^{-1}(\lambda)\right)^{-1}\right\|+$$

$$+\left\|A^2P_0^{-1}(\lambda)\left(E+P_1(\lambda)P_0^{-1}(\lambda)\right)\right\|\leq$$

$$\leq\left(\left\|\lambda^2P_0^{-1}(\lambda)\right\|+\left\|\lambda AP_0^{-1}(\lambda)\right\|+\left\|A^2P_0^{-1}(\lambda)\right\|\right)\left\|\left(E+P_1(\lambda)P_0^{-1}(\lambda)\right)^{-1}\right\|\leq$$

$$\leq\left(C_2(\gamma;\mu_0)+C_1(\gamma;\mu_0)+C_0(\gamma;\mu_0)\right)\frac{1}{1-\alpha(\gamma;\mu_0)}=$$

$$=\frac{1}{1-\alpha(\gamma;\mu_0)}\frac{2\mu_0^2+\mu_0^2+\gamma^2+2\mu_0^2+2\gamma^2}{2\left(\mu_0-\gamma^2\right)}=\frac{5\mu_0^2+3\gamma^2}{1-\alpha(\gamma;\mu_0)}=const.$$

The theorem is proved.

From this theorem it follows

**Corollary 1.** *Let conditions 1), 3) be fulfilled and there exist the inequality*

$$\frac{1}{2}\left\|B_1\right\|+\left\|B_2\right\|<1 \tag{15}$$

*Then on an imaginary axis estimates (5) hold.*

The proof follows from the theorem for $\gamma=0$.

**Corollary 2.** *Let the conditions of corollary 1 be fulfilled . Then for sufficiently small $\theta>0$ in sectors*

$$S_{\pm\theta}=\left\{\lambda:\lambda=re^{\pm i(\pi/2+s)},r>0,\ 0\leq s\leq\theta\right\}$$

*estimates (5) hold.*

**Proof.** Let $\lambda=\pi/2+s,\ 0\leq s\leq\theta$. Then

$$P(\lambda)=P\left(i\xi e^{is}\right)=P(i\xi)+(i\xi)^2\left(e^{2is}-1\right)+i\xi\left(e^{2is}-1\right)A_1=$$

$$=\left(E+(i\xi)^2\times\left(e^{2is}-1\right)P^{-1}(i\xi)+i\xi\left(e^{2is}-1\right)A_1P^{-1}(i\xi)\right)P(i\xi)\equiv$$

$$\equiv\left(E+Q(\xi,s)\right)P(i\xi) \tag{16}$$

By corollary (1) $P(i\xi)$ is boundedly reversible on $H$ and estimates (5) hold for it.

As for small $\theta>0$

$$\left\|Q(\xi,s)\right\|\leq\left\|\xi^2P^{-1}(\xi)\left(e^{2is}-1\right)\right\|+\left\|i\xi A_1P^{-1}(i\xi)\left(e^{is}-1\right)\right\|\leq$$

$$\leq\left\|\xi^2P^{-1}(i\xi)\right\|2\sin s+\left\|B_1\right\|\left\|i\xi P^{-1}(i\xi)\right\|2\sin s/2\leq$$

$$\leq const\left(\sin\theta+\sin\theta/2\right)<\varepsilon_1<1$$

for small $\theta > 0$ . Then from (16) it follows, that there exists the operator $(E + Q(\xi, s))^{-1}$ and bounded for small $\theta > 0$ . That's why the there exists $P^{-1}(\lambda)$ and for $\lambda \in S_{\pm\theta}$ there hold estimates

$$\left\|\lambda^2 P^{-1}(\lambda)\right\| + \left\|\lambda A P^{-1}(\lambda)\right\| + \left\|A^2 P^{-1}(\lambda)\right\| \leq$$

$$\leq \left(\left\|A^2 P^{-1}(i\xi)\right\| + \left\|\xi A P^{-1}(i\xi)\right\| + \left\|\xi^2 P^{-1}(i\xi)\right\|\right) \left\|(E + Q(\xi, \sigma))^{-1}\right\| \leq const$$

Now using the results of [1-5] and corollary 1.2 we formulate a theorem on completeness of the system of egen and adjoined vectors, responding to eigen values from $\Pi(-\gamma)$.

**Theorem.** *Let $\gamma \in [0, \mu_0)$ , the conditions 1)-3) inequality (2) and one of the following conditions be fulfilled*

*a) $A^{-1} \in \sigma_p \quad (0 < p \leq 1)$;*

*b) $A^{-1} \in \sigma_\infty(H)$, $\quad B_j \in \sigma_\infty \quad (j = 1, 2)$.*

Then the system of eigen and adjoined vectors responding to eigen values from half-plane $\Pi(-\gamma)$ is complete in the space $H_{3/2}$.

Here $H_{3/2}$ is a Hilbert space $H_{3/2} = D\left(A^{3/2}\right)$ , with scalar product $(x, y)_{3/2} = \left(A^{3/2}x, A^{3/2}y\right)$.

**Proof.** By theorem 1 at realization of inequality (2) operator bundle has no eigen values in band $\{\lambda : -\gamma \leq \operatorname{Re}\lambda \leq 0\}$. On the other hand, it is clear, that for any $\gamma \in [0, \mu_0)$ the numbers $C_1(\gamma; \mu_0) \geq C_0(0; \mu_0) = \frac{1}{2}$, $C_2(\gamma; \mu_0) \geq C_2(0; \mu_0) = 1$.

Then, it is clear, if operators $B_1$ and $B_2$ satisfy inequality (2), then they satisfy also inequality (15). And at realization of inequality (15) and one of the conditions a) or b), system of eigen and adjoined vectors, responding to eigen values from the left half-plane is complete in the space $H_{3/2}$ (see [3-5]).

As all eigen values of the bundle $P(\lambda)$ from the left half-plane are contained in the half-plane $\Pi(-\gamma)$, eigen and adjoined vectors from half-plane $\Pi(-\gamma)$ are complete in $H_{3/2}$. The theorem is proved.

## References

[1].Gasymov M.G. *To the theory polynomial operator bundles.* DAN SSSR, 1971, v.199, No747-750 (Russian)

[2].Gasymov M.G. *On multi completeness of a part of eigen and adjoined vectors of the polynomial operator bundles.*, In AN. Arm. SSR, series mathematics, 1971, v.6, No 2-3, pp.131-147 (Russian)

[3].Gasymov M.G., Mirzoyev S.S. *On solvability of boundary value problems for elliptical type second order operator-differential equations.* Differential uravn. 1992, v. 28, No 4, pp.651-661 (Russian)

[4].Mirzoyev S.S. *On multi completeness of a part of boundary vectors of polynomial operator bundles, responding to boundary value problems on semi-axis.* Functional analys I ego prilo, 1983, v.17, pp.84-85 (Russian)

[5].Mirzoyev S.S. *Problems of solvability theory of boundary value problems for operator-differential equations in Hilbert space and related spectral problems*, Thesis for Doctor's degree. Baku, 1993, 229p. (Russian)

[6].Shkalikov A.A. *Operator-differential equations on semi-axis and related spectral problems of polynomial operator bundles of operators.* UNN, 1989, v.39, No4, p.106. (Russian)

[7].Shkalikov A.A. *Elliptical equations in Hilbert space and related spectral problems.* Trudi of Petrovsky seminar, 1989, issue 14, pp.140-244. (Russian)

**Elshan N. Mamedov**
Nakhchivan State University
AZ7000, Nakhchivan, Azerbaijan

Received January 05.2006 ; Revised May 04.2006
Translated by Agayeva Z.A.