Migdad I. ISMAILOV

ON SPECTRAL REPRESENTATION AND OPERATOR CALCULATION OF A CLASS OF OPERATORS IN BANACH SPACE

Abstract

In the paper we study operator calculation of a class of piecewise-analytic functions used in constructing spectral representation of some class of bounded operators in Banach space.

Let's consider a rectifiable, open Lordan curve l with ends at the points a and b. The curve l with orientation from a to b we denote by l^+, the contrary one by $-l^-$. Let $D^+ (D^-)$ be a domain right from l^+ (left from l^-), whose boundary contains $l^+ (l^-)$.

We'll consider the classes: $A_l^+ \left(A_l^-\right)$ is a class of functions analytic in $D^+ (D^-)$ continuous up to $l^+ (l^-)$, A is a class of functions analytic in some vicinity of the curve $l, C(\Gamma; b)$ is a class of piecewise-continuous in the curve $\Gamma = l^+ \cup l^-$ functions having may be I genus discontinuity at the point b, $L_p(\Gamma)$ is a class of functions summable of p-th power $(1 \leq p < +\infty)$.

Let B be a Banach space, $T \in L(B)$ be a bounded operator in $B, R_\lambda(T)$ is a resolvent of the operator $T, \sigma(T)$ is a spectrum of the operator T. Cite some notation and facts from [3] and the paper [2] to appear.

Definition 1. *A sequence of rectifiable, Jordan curves $\{l_n^\pm\}_{n\geq 0} \subset D^\pm$ having common ends with curve $l^\pm$, is said to be admissible, if*

$$\forall\ n \in N\ \ \overline{D_{0,n}}^{\ \pm} \subset D_{0,n+1}^\pm,\ \ D_{0,n}{}^{\pm} = int\left(l_0^\pm \cup l_n^\pm\right),$$

$$\forall\ z \in D_0{}^{\pm} \exists\ n_z \in N \forall n \in N \quad n \geq n_z\ \ z \in D_{0,n}{}^{\pm},\ D_0{}^{\pm} = int\left(l_0^\pm \cup l^\pm\right).$$

Definition 2. *We'll say that the operator-function T_λ analytic in a domain $D^\pm$ belongs to the class $BE_p^\pm(l^\pm)$ if for any admissible sequence $\{l_n^\pm\}_{n\geq 0} \subset D^\pm$:*

$$\sup_n \int\limits_{l_n^\pm} \left|x^\pm T_\lambda x\right|^p |d\lambda| \leq M^\pm{}_{x,x}{}^* < +\infty,\ \ \forall x \in B,\ \forall x^* \in B^*,$$

where $M^\pm{}_{x,x}{}^*$ is a constant depending only on x, x^*.

Definition 3. *$E_p{}^{\pm}(l^\pm)$ is a class of functions $f^\pm(\lambda)$ satisfying the condition:*

$$\sup_n \int\limits_{l_n^\pm} \left|f^t(\lambda)\right|^p |d\lambda| \leq M < +\infty,$$

for any admissible sequence $\{l_n{}^{\pm}\}_{n\geq 0} \subset D^\pm$.

Statement 1. Let $T \in L(C(l),\ L(B)),\ B = B^{**}$. Then there exists a unique family of operators such that $\{T_\lambda\}_{\lambda\in l} \subset L(B)$

$$Tf = \int\limits_l f(\lambda)\, dT_\lambda\ \ \forall f \in C(l),$$

Statement 2. Let $T \in L(L_p(l), L(B))$, $B = B^{**}$ or $L(B) = L(B)^{**}$. Then there exists a unique family of operators $\{T_\lambda\}_{\lambda\in l} \subset L(B)$ such that

$$Tf = \int_l f(\lambda) T_\lambda \, d\lambda, \ \forall f \in L_p(l),$$

moreover

$$\int_l \|T_\lambda\|^q |d\lambda| \le \|T\|^q, \ \frac{1}{p} + \frac{1}{q} = 1, \ 1 < p < +\infty, \text{ i.e. } T_\lambda \in L_p(l, L(B)).$$

Theorem 1. *Let $T \in L(B)$, $R_\lambda(T) \in BE_1{}^+(l^+) \cap BE_1{}^-(l^-)$, $B = B^{**}$. Then there exists a unique family of operators $\{E_\lambda\}_{\lambda\in\Gamma} \subset L(B)$ such that*

$$f(T) = \int_\Gamma f(\lambda)\, dE_\lambda, \ \forall f \in C(\Gamma; b),$$

moreover, if $\sigma(T) \subset l$, then

$$I = \int_\Gamma dE_\lambda, \ \ T = \int_\Gamma \lambda dE_\lambda.$$

Proof. Let $f^\pm \in A_l^\pm$ with analyticity domain $D_f{}^\pm$. Denote by $D^\pm = D_f{}^\pm \cap D_T{}^\pm$, where $D_T{}^\pm$is an analyticity domain of $R_\lambda(T)$. Let's consider a sequence of integrals:

$$I_n{}^\pm = -\frac{1}{2\pi i}\int_{l_n^\pm} f^\pm(\lambda) R_\lambda(T)\, d\lambda, \ n \in N,$$

where $\{l_n{}^\pm\}_{n\ge 0} \subset D^\pm$ is an arbitrary admissible sequence.

Clearly, for any $n, m \in N$, $I_n{}^\pm = I_m{}^\pm$. Denote their common value by $f^\pm(T)$. Consequently,

$$f^\pm(T) = -\frac{1}{2\pi i}\lim_{n\to\infty}\int_{l_n^\pm} f^\pm(\lambda) R_\lambda(T)\, d\lambda.$$

Take $\forall x \in B$, $\forall x^* \in B^*$ and consider

$$x^* f^\pm(T)\, x = -\frac{1}{2\pi i}\lim_{n\to\infty}\int_{l_n^\pm} f^\pm(\lambda)\, x^* R_\lambda(T)\, x d\lambda.$$

We have

$$\left|x^* f^\pm(T)\, x\right| \le \frac{1}{2\pi}\lim_{n\to\infty} \max_{l_n^\pm}\left|f^\pm(\lambda)\right| \int_{l_n^\pm} |x^* R_\lambda(T)\, x|\, |d\lambda| \le$$

$$\le \frac{\|f^\pm\|_{C(l^\pm)}}{2\pi}\sup_n \int_{l_n^\pm} |x^* R_\lambda(T)\, x|\, |d\lambda| \le \frac{\|f^\pm\|_{C(l^\pm)}}{2\pi} M^\pm{}_{x,x}{}^* =$$

$$= \left\|f^{\pm}\right\|_{C(l^{\pm})} N^{\pm}_{\ \ x,x^{\ \ *}} < +\infty.$$

Let's consider the operator

$$F^{\pm}(T) = \frac{1}{\|f^{\pm}\|_{C(l^{\pm})}} f^{\pm}(T),$$

then

$$\left|x^{*} F^{\pm}(T) x\right| \le N^{\pm}_{\ \ x,x^{\ \ *}} < +\infty.$$

By virtue of reflexivity of B we can identify the element $F^{\pm}(T)x \in B$ by the element $F^{\pm}(T)x \in B^{**}$. From the uniform boundedness principle, for fixed x we have

$$\left\|F^{\pm}(T)x\right\| \le N^{\pm}_{x} < +\infty,$$

where $N^{\pm}_{x}$ is a constant that depends only on x. Since the operators $F^{\pm}(T)$ are bounded for any $x \in B$, then by uniform boundedness principle

$$\left\|F^{\pm}(T)\right\| \le N^{\pm} < +\infty,$$

where $N^{\pm}$ is a constant.

Consequently,

$$\left\|f^{\pm}(T)\right\| \le N^{\pm} \left\|f^{\pm}\right\|_{C(l^{\pm})} < +\infty, \ f^{\pm} \in A^{\pm}_{l}$$

Let $C^{\pm}(l^{\pm})$ be a manifold consisting of boundary values of functions $A^{\pm}_{l}$. Clearly, we can establish one to one correspondence between $C^{\pm}(l^{\pm})$ and $A^{\pm}_{l}$. Allowing for density of $C^{\pm}(l^{\pm})$ in $C(l^{\pm})$ we'll have $f^{\pm}(T) \in L(C(l^{\pm}), L(B))$. Then, from statement 1 there exists a unique family of operators $\{E^{\pm}_{\lambda}\}_{\lambda \in l} \subset L(B)$ such that

$$f^{\pm}(T) = \int_{l^{\pm}} f^{\pm}(\lambda)\, dE^{\pm}_{\lambda}, \ \forall f^{\pm} \in C\left(l^{\pm}\right).$$

Let

$$f(\lambda) = \begin{cases} f^{\pm}(\lambda), \lambda \in l^{\pm} \\ f^{-}(\lambda), \lambda \in l^{-} \end{cases}.$$

Thus, we can establish one-to-one correspondence between $A^{+}_{l} \times A^{-}_{l}$ in some dense in $C(\Gamma; b)$ manifold $C(l^{+}) \times C(l^{-})$. Define the operator $f(T) = f^{+}(T) + f^{-}(T)$.

Then

$$f(T) = \int_{l^{+}} f(\lambda)\, dE^{+}_{\lambda} + \int_{l^{-}} f(\lambda)\, dE^{-}_{\lambda} = \int_{\Gamma} f(\lambda)\, dE_{\lambda}, \quad \forall f \in C(\Gamma; b),$$

where $E_{\lambda} = E^{+}_{\lambda \cap l^{+}} + E^{-}_{\lambda \cap l^{-}}$, in particular, since the operators I and T correspond to the functions $f(\lambda) = 1, \ f(\lambda) = \lambda$ and $\sigma(T) \subset l$ then

$$I = \int_{\Gamma} dE_{\lambda}, \ T = \int_{\Gamma} \lambda dE_{\lambda}.$$

The theorem is proved.

Theorem 2. *In conditions of theorem 1 there exists a unique family of projectors* $\{E_\lambda\}_{\lambda\in l}\subset L(B)$ *such that*

$$f(T)=\int_l f(\lambda)\,dE_\lambda,\ \forall f\in A,$$

moreover, for $\sigma(T)\subset l$,

$$I=\int_l dE_\lambda,\ T=\int_l \lambda dE_\lambda.$$

Proof. Since A is a sub-algebra of $A_l^+\cap A_l^-$ then it follows from the above-proved one that there exists a unique family of operators $\{E_\lambda\}_{\lambda\in l}\subset L(B)$ such that

$$f(T)=\int_l f(\lambda)\,dE_\lambda,\ \forall f\in A,$$

in particular, for $\sigma(T)\subset l$,

$$I=\int_l dE_\lambda,\ T=\int_l \lambda dE_\lambda.$$

Let $\chi_\alpha(\lambda)$ be a characteristic function of the set $\alpha\subset\Gamma$. Following [1] and [2] we show that for any Borel $\alpha\subset\Gamma$ the mapping $\alpha\to E_\alpha$ where

$$E_\alpha=\int_\Gamma \chi_\alpha(\lambda)\,dE_\lambda,$$

is a spectral measure. It is easy to show that $(fg)(T)=f(T)\,g(T)$ for any $f,g\in A$. Clearly, for any $\alpha\subset\Gamma$ there will be found $f_n\in A$ such-that $f_n(\lambda)\to\chi_\alpha(\lambda)$, $|f_n|\le M$ then $f_n(T)\to E_\alpha$, but since $f_n^2(\lambda)\to\chi_\alpha(\lambda)$ then $f_n^2(T)\to E_\alpha$, however $f_n^2(T)=(f_n(T))^2\to E_\alpha^2$, so E_α is a projector. Take an arbitrary $\alpha,\ \beta\subset\Gamma$. Let f_n and g_n be sequences from A converging to $\chi_\alpha(\lambda)$ and $\chi_\beta(\lambda)$, respectively. Consequently, $f_n(T)\to E_\alpha, g_n(T)\to E_\beta$ and $f_n(T)\,g_n(T)\to E_\alpha E_g$. But $f_n(\lambda)\,g_n(\lambda)\to\chi_{\alpha\cap\beta}(\lambda)$, thereby $f_n(T)\,g_n(T)\to E_{\alpha\cap\beta}$, so $E_{\alpha\cap\beta}=E_\alpha E_\beta$. E_α is a denumerable additive, if a sequence of Borel subsets $\alpha_n\subset l$ joins to empty set, then $x^*E_{\alpha_n}x\to 0$, $\forall x\in B$, $\forall x^*\in B^*$. The theorem is proved.

Theorem 3. *Let* $T\in L(B)$, $R_\lambda(T)\in BE_q^+(l^+)\cap BE_q^-(l^-)$, $1<q<+\infty$, $B=B^{**}$. *Then there exists a unique family of operators* $\{T_\lambda\}_{\lambda\in\Gamma}\subset L(B)$, *such that*

$$f(T)=\int_\Gamma f(\lambda)\,T_\lambda d\lambda,\ \ \forall f\in L_p(\Gamma)\,,\ \frac{1}{p}+\frac{1}{q}=1,$$

moreover, if $\sigma(T)\subset l$, *then*

$$I=\int_\Gamma T_\lambda d\lambda,\ T=\int_\Gamma \lambda T_\lambda d\lambda.$$

Proof. Let $f^{\pm} \in E_p^{\pm}(l^{\pm})$. Define the operators $f^{\pm}(T)$ as in the proof of theorem 1.

$\forall x \in B,\ \forall x^* \in B^*$ estimate $x^* f^{\pm}(T)x$,

$$\left|x^* f^{\pm}(T)x\right| \leq \frac{1}{2\pi}\lim_{n\to\infty}\int_{l_n^{\pm}} \left|f^{\pm}(\lambda)\right| \left|x^* R_\lambda(T)x\right| |d\lambda| \leq$$

$$\leq \frac{1}{2\pi}\lim_{n\to\infty}\left(\int_{l_n^{\pm}} \left|f^{\pm}(\lambda)\right|^p |d\lambda|\right)^{\frac{1}{p}} \left(\int_{l_n^{\pm}} \left|x^* R_\lambda(T)x\right|^q |d\lambda|\right)^{\frac{1}{q}} \leq$$

$$\leq \left\|f^{\pm}\right\|_{L_p(l^{\pm})} N^{\pm}{}_{x,x}{}^{*} < +\infty.$$

Let's consider the operator

$$F^{\pm}(T) = \frac{1}{\left\|f^{\pm}\right\|_{L_p(l^{\pm})}} f^{\pm}(T),$$

then

$$\left|\times^* F^{\pm}(T)x\right| \leq N^{\pm}{}_{x,x}{}^{*} < +\infty.$$

By virtue of reflexivity of B we can identify the element $F^{\pm}(T)x \in B$ with the element $F^{\pm}(T)x \in B^{**}$. By uniform boundedness principle, for x we get

$$\left\|F^{\pm}(T)x\right\| \leq N_x^{\pm} < +\infty,$$

where $N_x^{\pm}$ is a constant that depends only on x. Since the operators $F^{\pm}(T)$ are bounded for any $x \in B$, by uniform boundedness principle

$$\left\|F^{\pm}(T)\right\| \leq N^{\pm} < +\infty,$$

where $N^{\pm}$ is a constant.

Consequently

$$\left\|f^{\pm}(T)\right\| \leq N^{\pm}\left\|f^{\pm}\right\|_{L_p(l^{\pm})} < +\infty, \quad f^{\pm} \in E_p^{\pm}\left(l^{\pm}\right).$$

Hence, allowing for density of $E_p^{\pm}(l^{\pm})$ in $L_p(l^{\pm})$ we have

$f^{\pm}(T) \in L(L_p(l^{\pm}),\ L(B))$. Then by statement 2 there exists a unique family of operators $\left\{T_\lambda^{\pm}\right\}_{\lambda\in l^{\pm}} \subset L(B)$ such that

$$f^{\pm}(T) = \int_{l^{\pm}} f^{\pm}(\lambda)T_\lambda^{\pm}d\lambda,\ \forall f^{\pm} \in L_p\left(l^{\pm}\right),\ \text{ i.e } f^{\pm}(T) \in L\left(L_p\left(l^{\pm}\right),\ L(B)\right).$$

Let

$$f(\lambda) = \begin{cases} f^{+}(\lambda), \lambda \in l^{+} \\ f^{-}(\lambda), \lambda \in l^{-} \end{cases},$$

then we can identify $E_p(l^+) \times E_p(l^-)$ with the manifold $L_p(l^+) \times L_p(l^-)$ dense in $L_p(\Gamma)$. Defining the operator $f(T) = f^+(T) + f^-(T)$ we have

$$f(T) = \int_\Gamma f(\lambda) T_\lambda d\lambda, \ \forall f \in L_p(\Gamma),$$

where $T_\lambda = T^+_{\lambda \cap l^+} + T^-_{\lambda \cap l^-}$, in particular, if $\sigma(T) \subset l$, then

$$I = \int_\Gamma T_\lambda d\lambda, \quad T = \int_\Gamma \lambda T_\lambda d\lambda.$$

Theorem 4. *Let all the conditions of theorem 3 be fulfilled. Then there exists a unique family of operators* $\{T_\lambda\}_{\lambda \in l} \in L_q(l, L(B))$ *such that*

$$f(T) = \int_l f(\lambda) T_\lambda d\lambda, \ \forall f \in L_p(l),$$

moreover, if $\sigma(T) \subset l$*, then*

$$I = \int_l T_\lambda d\lambda, \quad T = \int_l \lambda T_\lambda d\lambda,$$

for any Borel subset $\alpha \subset l$ *the operators*

$$E_\alpha = \int_\alpha T_\lambda d\lambda,$$

are the projectors in $L(B)$.

Proof. Since $L_p(l^+) \cap L_p(l^-) = L_p(l)$, then the existence of the indicated family of operators follows from theorem 3. Show that for any Borel subset $\alpha \subset l$ the operators E_α are the projectors in $L(B)$. Let $f \in A$. Then, similar to the proof of theorem 2 it is easy to show that there exists a unique family of projectors $\{E_\lambda\}_{\lambda \in l} \subset L(B)$, such that

$$f(T) = \int_l f(\lambda) dE_\lambda.$$

Since A is dense in $L_p(l)$ then

$$f(T) = \int_l f(\lambda) dE_\lambda, \quad f \in L_p(l),$$

consequently, considering $f(\lambda) = \chi_\alpha(\lambda)$ for any Borel $\alpha \subset \Gamma$, we have

$$E_\alpha = \int_\alpha T_\lambda d\lambda.$$

The theorem is proved.

Definition 4. *We'll say that the curve l belongs to the class $\Gamma^{\pm}_{\lambda-\mu}$, if there exists an admissible sequence $\{l^{\pm}_n\}_{n\geq 0}\subset D^{\pm}$ of piece-wise curves with common ends with curve $l^{\pm}$, such that there exist the functions $\nu^{\pm}_n(\lambda): l^{\pm}_n\to l^{\pm}$ analytic in $D^{\pm}$ for which there exists a uniform limit $\lim\limits_{n\to\infty}\nu^{\pm}_n(\lambda)=\lambda$ in $D^{\pm}$, moreover, the mapping $\nu^{\pm}_n(\lambda)$ is one to one and the functions $\mu^{\pm}_n(\lambda): l^{\pm}\to l^{\pm}_n$ inverse to $\nu^{\pm}_n(\lambda)$ satisfy the condition:*

$$\frac{\mu'^{\pm}_n(\lambda)-1}{dist\left(\mu^{\pm}_n(\lambda),l\right)}\to 0,$$

as $n\to\infty$ uniformly with respect to λ.

Let $\Gamma_{\lambda-\mu}=\Gamma^{+}_{\lambda-\mu}\cap\Gamma^{-}_{\lambda-\mu}$. By $E_p(\Gamma_{\lambda-\mu})\,(BE_p(\Gamma_{\lambda-\mu}))$ we denote a class of operators T for which

$$\sup_n\int\limits_l\left\|R_{\mu^{+}_n(\lambda)}(T)-R_{\mu^{-}_n(l)}(T)\right\|^p|d\lambda|<+\infty,$$

$$\left(\sup_n\int\limits_l\left|x^*\left(R_{\mu^{+}_n(\lambda)}(T)-R_{\mu^{-}_n(\lambda)}(T)\right)x\right|^p|d\lambda|\leq\right.$$

$$\left.\leq M_{x,x}{}^*<+\infty,\quad\forall x\in B,\quad\forall x^*\in B^*\right),$$

for any admissible sequence $\{l^{\pm}_n\}_{n\geq 0}\subset D^{\pm}$.

Theorem 5. *Let $l\in\Gamma_{\lambda-\mu}T\in E_q(\Gamma_{\lambda-\mu})$, $1<q<+\infty$, $B=B^{**}$. Then there exists a unique family of operators such that $\{T_\lambda\}_{\lambda\in l}\subset L(B)$ moreover, for*

$$f(T)=\int\limits_l f(\lambda)T_\lambda d\lambda,\quad\forall f\in L_p(l),\ \frac{1}{p}+\frac{1}{q}=1,\quad 1<q<+\infty,$$

and for $\sigma(T)\subset l$,

$$I=\int\limits_l T_\lambda d\lambda,\quad T=\int\limits_l\lambda T_\lambda d\lambda$$

and for *any Borel $\sigma\subset\Gamma$ the operators*

$$E_\alpha=\int\limits_\alpha T_\lambda d\lambda$$

are the projectors in $L(B)$.

Proof. Let $f\in E^{+}_p(l^{+})\cap E^{-}_p(l^{-})$. Consider the operator

$$f^{\pm}(T)=-\frac{1}{2\pi i}\lim_{n\to\infty}\int\limits_{l^{\pm}_n}f\left(\nu^{\pm}_n(\lambda)\right)R_\lambda(T)\,d\lambda.$$

Having denoted $f(T)=f^{+}(T)+f^{-}(T)$ well have

$$f(T)=-\frac{1}{2\pi i}\lim_{n\to\infty}\left(\int\limits_{l^+}f(\lambda)\,\mu'_n{}^{+}(\lambda)\,R_{\mu^{+}_n(\lambda)}(T)\,d\lambda+\right.$$

$$+\int_{l^-} f(\lambda)\,{\mu'}_n^-(\lambda)\,R_{\mu_n^-(\lambda)}(T)\,d\lambda\Bigg)$$

then

$$f(T)=-\frac{1}{2\pi i}\lim_{n\to\infty}\int_l f(\lambda)\left({\mu'}_n^+(\lambda)\,R_{\mu_n^+(\lambda)}(T)-{\mu'}_n^-(\lambda)\,R_{\mu_n^-(\lambda)}(T)\right)d\lambda=$$

$$=-\frac{1}{2\pi i}\lim_{n\to\infty}\int_l f(\lambda)\left({\mu'}_n^+(\lambda)\left(R_{\mu_n^+(\lambda)}(T)-R_{\mu_n^-(\lambda)}(T)\right)+\right.$$

$$+{\mu'}_n^-(\lambda)\left(R_{\mu_n^+(\lambda)}(T)-R_{\mu_n^-(\lambda)}(T)\right)+$$

$$\left.+{\mu'}_n^+(\lambda)\,R_{\mu_n^-(\lambda)}(T)-{\mu'}_n^-(\lambda)\,R_{\mu_n^+(\lambda)}(T)\right)d\lambda=$$

$$=-\frac{1}{2\pi i}\lim_{n\to\infty}\int_l f(\lambda)\left(\left({\mu'}_n^+(\lambda)+{\mu'}_n^-(\lambda)\right)\left(R_{\mu_n^+(\lambda)}(T)-R_{\mu_n^-(\lambda)}(T)\right)-\right.$$

$$-\left({\mu'}_n^-(\lambda)-1\right)R_{\mu_n^+(\lambda)}(T)+\left({\mu'}_n^+(\lambda)-1\right)R_{\mu_n^-(\lambda)}(T)-$$

$$\left.-\left(R_{\mu_n^+(\lambda)}(T)-R_{\mu_n^-(\lambda)}(T)\right)\right)d\lambda=$$

$$=-\frac{1}{2\pi i}\lim_{n\to\infty}\int_l f(\lambda)\left(\left({\mu'}_n^+(\lambda)+{\mu'}_n^-(\lambda)-1\right)\times\right.$$

$$\times\left(R_{\mu_n^+(\lambda)}+(T)-R_{\mu_n^-(\lambda)}(T)\right)d\lambda-$$

$$-\frac{1}{2\pi i}\lim_{n\to\infty}\int_l f(\lambda)\left(\left({\mu'}_n^-(\lambda)-1\right)R_{\mu_n^+(\lambda)}(T)+\left(\mu_n^+(\lambda)'-1\right)R_{\mu_n^-(\lambda)}(T)\right)d\lambda.$$

Since $E_p^\pm(l^\pm)$ is dense in $L_p(l^\pm)$, $L_p(l)=L_p(l^+)\cap L_p(l^-)$,
then $\forall f\in L_p(l)$ we have

$$\|f(T)\|\le\frac{1}{2\pi}\lim_{n\to\infty}\int_l|f(\lambda)|\left|{\mu'}_n^+(\lambda)+{\mu'}_n^-(\lambda)-1\right|\times$$

$$\times\left\|R_{\mu_n^+(\lambda)}+(T)-R_{\mu_n^-(\lambda)}(T)\right\||d\lambda|\le$$

$$\le\frac{1}{2\pi i}\max_l\left|{\mu'}_n^+(\lambda)+{\mu'}_n^-(\lambda)-1\right|\left(\int_l|f(\lambda)|^p|d\lambda|\right)^{\frac{1}{p}}\times$$

$$\times\left(\sup_n\int_l\left\|R_{\mu_n^+(\lambda)}+(T)-R_{\mu_n^-(\lambda)}(T)\right\|^q|d\lambda|\right)^{\frac{1}{q}}\le\|f(T)\|_{L_p(l)}M<+\infty.$$

so $f(T)\in L(L_p(l),L(B))$. It remains to apply statement 2 and the procedure of above-proved theorems. The theorem is proved.

Theorem 6. *Let* $T \in L(B)$ *with resolvent* $R_\lambda(T) \in BE_1(\Gamma_{\lambda-\mu}), B = B^{**}$. *Then there exists a unique family of projectors* $\{E_\lambda\}_{\lambda \in l} \subset L(B)$ *such that*

$$f(T) = \int_l f(\lambda)\, dE_\lambda, \quad \forall f \in A,$$

moreover, if $\sigma(T) \subset l$, *then*

$$I = \int_l dE_\lambda, \quad T = \int_l \lambda dE_\lambda.$$

Proof. Let $f \in A$. Define the operator $f(T)$ by the formula:

$$f(T) = -\frac{1}{2\pi i} \lim_{n\to\infty} \int_l f(\lambda) \left(\mu'{}_n{}^{+}(\lambda) R_{\mu_n^+(\lambda)}(T) - \mu'{}_n{}^{-}(\lambda) R_{\mu_n^-(\lambda)}(T)\right) d\lambda,$$

Then for $\forall x \in B$, $\forall x^* \in B^*$ we have

$$x^* f(T) x = -\frac{1}{2\pi i} \lim_{n\to\infty} \int_l f(\lambda) \left(\mu'{}_n{}^{+}(\lambda) - \mu'{}_n{}^{-}(\lambda) - 1\right) \times$$

$$\times x^* \left(R_{\mu_n^+(\lambda)}(T) - R_{\mu_n^-(\lambda)}(T)\right) x d\lambda -$$

$$-\frac{1}{2\pi i} \lim_{n\to\infty} \int_l f(\lambda) \left(\left(\mu'{}_n{}^{-}(\lambda) - 1\right) x R_{\mu_n^+(\lambda)}(T) x + \right.$$

$$\left. + \left(\mu'{}_n{}^{+}(\lambda) - 1\right) x^* R_{\mu_n^-(\lambda)}(T) x\right) d\lambda.$$

Estimate:

$$|x^* f(T) x| \le \frac{1}{2\pi i} \lim_{n\to\infty} \int_l |f(\lambda)| \left|\mu'{}_n{}^{+}(\lambda) + \mu'{}_n{}^{-}(\lambda) - 1\right| \times$$

$$\times \left|x^* \left(R_{\mu_n^+(\lambda)}(T) - R_{\mu_n^-(\lambda)}(T)\right) x\right| \|d\lambda\| +$$

$$+\frac{1}{2\pi i} \lim_{n\to\infty} \int_l |f(\lambda)| \left(\left|\mu'{}_n{}^{-}(\lambda) - 1\right| \left|x^* R_{\mu_n^+(\lambda)}(T) x\right| + \right.$$

$$\left. + \left|\mu'{}_n{}^{+}(\lambda) - 1\right| \left|x^* R_{\mu_n^-(\lambda)}(T) x\right|\right) |d\lambda| \le$$

$$\le \frac{1}{2\pi} \max_l \left|\mu'{}_n{}^{+}(\lambda) + \mu'{}_n{}^{-}(\lambda) - 1\right| \|f\|_{C(l)} \times$$

$$\times \sup_n \int_l \left|x^* \left(R_{\mu_n^+(\lambda)}(T) - R_{\mu_n^-(\lambda)}(T)\right) x\right| |d\lambda| \le \|f\|_{C(l)} M_{x,x^*} < +\infty.$$

Further, the proof is similar to the proof of theorem 2.

Theorem 7. *Let* $l \in \Gamma_{\lambda-\mu} T \in L(B)$, $R_\lambda(T) \in BE_q(\Gamma_{\lambda-\mu})$, $B = B^{**}$. *Then there exists a unique family of operators* $\{T_\lambda\}_{\lambda\in l} \subset L(B)$ *such that*

$$f(T) = \int_l f(\lambda) T_\lambda d\lambda, \ \forall f \in L_p(l), \quad \frac{1}{p} + \frac{1}{q} = 1, \ 1 < q < +\infty,$$

moreover, for $\sigma(T) \subset l$

$$I = \int_l T_\lambda d\lambda, \quad T = \int_l \lambda T_\lambda d\lambda.$$

and for any Borel $\alpha \subset \Gamma$ *the operators*

$$E_\alpha = \int_\alpha T_\lambda d\lambda$$

are the projectors in $L(B)$.

Proof. Let $f \in E_p^+(l^+) \cap E_p^-(l^-)$. Defining $f(T)$ as in the proof of theorem 5 and allowing for density of $E_p^+(l^+) \cap E_p^-(l^-)$ in $L_p(l)$, it is easy to show that $f(T) \in L(L_p(l), L(B))$. It remains to apply theorem 5. The theorem is proved.

Remark: Note that for piece-wise smooth curve l with ends at the point a and b representing a graph of a function as an example for admissible sequence $\{l_n^\pm\}_{n\geq 1} \subset D^\pm$ we can take:

$$l_n^\pm : \lambda \pm a_n, \ \lambda \in l,$$

where $\{a_n\}_{n\geq 1}$ is a sequence of complex numbers monotonically converging to zero, moreover

$$\mu_n{}^\pm(\lambda) = \lambda \pm a_n, \ \text{and} \ \mu'_n{}^\pm(\lambda) = 1.$$

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Migdad I. Ismailov
Baku State University
23, Z.I.Khalilov str., AZ1148, Baku, Azerbaijan
Tel.: (99455) 716 33 73 (mob.)

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