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ON BEHAVIOR OF SOLUTIONS OF NONLINEAR HYPERBOLIC EQUATIONS IN NONSMOOTH DOMAIN

Abstract

We investigate nonlinear parabolic equation in nonsmooth domain. Smooth solution in corresponding spaces is obtained.

By now the theory of general boundary-value problems for hyperbolic equations and systems in cylindrical domains with smooth data are well developed. Many equations and systems of equations of mathematical physics lead to hyperbolic equations, for example, system of equations of gas dynamics are lead to nonlinear hyperbolic equations. Hyperbolic equations and systems in domains with nonsmooth boundary have been investigated not enough.

In connection with existed results in this direction, note the works of Raisman H. [1], Ibuki K. [2], Kraus L., Levine L. [3], Peter A.S. [4], Ocher S. [5], Sorason L. [6] and others.

Consider a mixed problem for hyperbolic equation:

$$u_u - \sum_{y=1}^{n} \frac{d}{dx_j} \left(a_{ij} \left(x, t, u, u_x \right) u_{x_i} \right) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(b_i \left(x, t \right) u \right) + b \left(x, t \right) u = f \left(x, t \right) \tag{1}$$

in cylinder $Q_T = \Omega \times [0, T]$, $\Omega \subset R_x^n$ is a bounded domain, with initial conditions and boundary conditions:

$$u|_{t=0} = 0, \quad u_t|_{t=0} = 0, \quad u|_{\partial\Omega x[0,T]} = 0 \tag{2}$$

We'll consider generalized solution $u \left(x_1, t \right)$ from $W_2^1 \left(Q_T \right)$, where $W_2^1 \left(Q_T \right)$ is a Sobolev space.

Our aim is to investigate a smoothness of the generalized solution of mixed problem (1), (2) in arbitrary domains with piecewise-smooth boundary.

For coefficients we suppose fulfilment of the following conditions:

$$a_{ij} \left(x, t, u, p \right) \in C^1 \left(Q_T \right) \tag{3}$$

$$\left(a_{ij} \left(x, t, u, p \right) - a_{ij} \left(x, t, u, q \right) \right) \left(p_i - q_i \right) \geq v \left| p - q \right|^2 , \tag{4}$$

for any $p, q \in R^n$, ν is a positive constant.

$$\left| \frac{\partial a_{ij} \left(x, t, u, p \right)}{\partial p_j} \right| \leq C_0, \quad \left| \frac{\partial a_{ij}}{\partial t} \right| \leq C_1, \qquad C_0 < \infty, \quad C_1 < \infty \tag{5}$$

and the functions $b_i(x,t)$, $b(x,t)$ are real and smooth in $\bar{Q}_T = \bar{\Omega} \times [0,T]$. Besides, the following condition:

$$a_{ij}(x,t,u,p)\,\xi_i\xi_j \geq \nu\,|\xi|^2 \quad \text{for all } p,\xi \in R^n \tag{6}$$

is fulfilled.

The function $u(x,t) \in W_2^1(Q_T)$, which satisfies the integral identity:

$$\int_{Q_T} \left[u_t\varphi_t - \sum_{i,j=1}^{n} a_{ij}(x,t,u,u_x)\,u_{x_i}\varphi_{x_j} - \right.$$

$$\left. - \sum_{i,j=1}^{n} b_i(x,t)\,u\varphi_{x_i} + b(x,t)\,u\varphi + f\varphi \right] dt = 0 \tag{7}$$

for any function $\varphi(x,t) \in W_2^1(Q_T)$, $\varphi|_{t=T} = 0$, $\varphi|_{\partial Q_T} = 0$ is said to be general solution of problem (1)-(2).

If the boundary is smooth, then for solution of problem (1), (2) the following theorem on smoothness of solution is true.

Theorem 1. *Let $f \in C^\infty$, $\partial\Omega \subset C^2$ and $\frac{\partial^s f}{\partial t^s}|_{t=0} = 0$, $0 \leq s \leq k-1$ in some vicinity of $\partial\Omega$. Then, the generalized solution $u(x,t)$ of problem (1), (2) belongs to $W_2^k(Q_T)$, $k \geq 3$. For solution $u(x,t)$ the following estimation is true*

$$\|u(x,t)\|_{W_2^k(Q_T)} \leq C\,\|f\|_{W_2^{k-1}(Q_T)} \tag{8}$$

For the proof of the theorem we use Galerkin's method, corresponding ideas from [7], estimations from [8].

Estimate solutions and their derivatives by t in norm $W_2^1(Q_T)$. Domain Ω has nonsmooth boundary. Approximate domain Ω by smooth domains Ω_ε such that $\Omega_\varepsilon \subset \Omega$, $\varepsilon > 0$, $\partial\Omega_\varepsilon \in C^\infty$, $\lim\limits_{\varepsilon\to\infty} \Omega_\varepsilon = \Omega$.

Consider $Q_\varepsilon^T = \Omega_\varepsilon \times [0,T]$. Let u_ε be a generalized solution from $W_2^1\left(Q_\varepsilon^T\right)$ of problem (1), (2) in Q_ε^T.

Lemma 1. *If $f(x,t) \in C^\infty\left(\bar{Q}_T\right)$, then $u_\varepsilon(x,t)$ satisfies the estimation*

$$\|u_\varepsilon(x,t)\|_{W_2^1(Q_\varepsilon^T)} \leq C_1\,\|f\|_{L_2(Q_\varepsilon^T)} \tag{9}$$

where C_1 is a constant, independent of ε.

Proof. Both parts of equation (1) multiply by $u_{\varepsilon t}$ and integrate by Q_ε^T and integrate by parts. If $\frac{\partial^k f}{\partial t^k}|_{t=0} = 0$, $\forall k \geq 0$ holds, then solution $u_\varepsilon(x,t)$ is classical and therefore we can do it. After integration by parts, using conditions (3)-(6) we get the required estimation (9). If condition $\frac{\partial^k f}{\partial t^k}|_{t=0} = 0$, $\forall k \geq 0$ is not fulfilled, we can consider the function

$$f_\eta = 0 \left| \begin{array}{lll} f(x,t) & at & t > \eta \\ 0 & at & t < \eta \\ 0 & all & Q_\varepsilon^T \end{array} \right. \qquad \text{for any } \eta > 0$$

Now, let averaging of the function f_η with averaging radius $\eta/2$ be a function $g_\eta(x,t)$ and it satisfy condition $\frac{\partial^k g_\eta}{\partial t^k}|_{t=0} = 0$, $\forall k \geq 0$. Then, for solution of problem (1), (2) u_ε^η, where f replaced by g_η, we have estimation (9). Directing $\eta \to 0$, we get the required result.

Lemma 2. *Let $f \in C^\infty(\bar{Q}_T)$, then $u_\varepsilon(x,t)$ satisfies estimation*

$$\|u_{\varepsilon t}(x,t)\|_{W_2^1(Q_\varepsilon^t)} \leq c_2\left(\|f(x,t)\|_{L_2(Q_\varepsilon^t)} + \|f_t\|_{L_2(Q_\varepsilon^t)}\right) \tag{10}$$

where the constant C_2 is independent of ε.

Proof. Let's differentiate both parts of equation (1) by t and multiply by $u_{\varepsilon tt}$. Integrate by parts. But, if $\frac{\partial^k f}{\partial t^k}|_{t=0} = 0$, the solution $u_\varepsilon(x,t)$ is classical and one can integrate by parts and further, using conditions on coefficients (3)-(6) we get the required estimation (10). If condition $\frac{\partial^k f}{\partial t^k}|_{t=0} = 0$ is fulfilled, passing to averaged function one can get estimation (10). Further, directing $\eta \to 0$, we get the required result.

Lemma 3. *If $f \in C^\infty(\bar{Q}_T)$, then problem (1), (2) has a generalized solution $u(x,t)$ in Q_T and the following estimations are true*

$$\|u(x,t)\|_{W_2^1(Q_T)} \leq C_3\|f\|_{L_2(Q_T)} \tag{11}$$

$$\|u_t(x,t)\|_{W_2^1(Q_T)} \leq C_4\left(\|f(x,t)\|_{L_2(Q_T)} + \|f_t(x,t)\|_{L_2(Q_T)}\right) \tag{12}$$

with the constants C_3, C_4.

Proof. If we consider problem (1), (2) in Q_ε^T, then solution of problem $u_\varepsilon(x,t)$ satisfies inequalities (9), (10). Then sequence $u_\varepsilon(x,t)$ has a weak limit in $W_2^1(Q_T)$ and let as $\varepsilon \to 0$ $u_0(x,t)$ will be a weak limit $u_\varepsilon(x,t)$ in $W_2^1(Q_T)$.

Function $u_0(x,t)$ will be a generalized solution of problem (1), (2), in Q_T and satisfies inequalities (11), (12).

Lemma 4. *If $f(x,t)$, $f_t(x,t) \in L_2(Q_T)$, then problem (1), (2) has a generalized solution $u(x,t) \in W_2^1(Q_T)$ and the following inequality is true*

$$\|u(x,t)\|_{W_2^1(Q_T)} \leq C_5\|f(x,t)\|_{L_2(Q_T)} \tag{13}$$

$$\|u_t(x,t)\|_{W_2^1(Q_T)} \leq C_6\left(\|f(x,t)\|_{L_2(Q_T)} + \|f_t(x,t)\|_{L_2(Q_T)}\right) \tag{14}$$

with the constants C_5, C_6.

Proof. Let's take averaged right hand side of equation (1) f_h. Let u_h be a solution of problem (1), (2) with right hand side f_h. From the estimations (11), (12) it follows convergence of u_h to the limit u, for each of them inequalities (13), (14) are fulfilled. Take section of cylinder Q_T with hyperplane $t = const$. Denote this section by Ω_T.

Lemma 5. *Let $f(x,t)$, $f_t(x,t) \in L_2(Q_T)$. Then for almost all t generalized solution $u(x,t)$ of problem (1), (2) satisfies the identity*

$$\int_{\Omega_t} \left[\sum_{i,j=1}^{n} a_{ij}(x,t,u,u_x) u_{x_i} \psi_{x_j} - \sum_{i=1}^{n} b_i(x,t) u\psi - (f - u_{tt}) \psi \right] dx = 0 \qquad (15)$$

For any function $\psi(x) \in \overset{\circ}{W}{}_2^1(\Omega_t)$, where $\overset{\circ}{W}{}_2^1(\Omega_t)$ is a closure of functions from C^∞, whose supports belong to Ω_t, by norm $W_2^1(\Omega_t)$.

Hence, we get the following

Corollary. *Almost for all $t \in [0,T]$ $u(x,t)$ is a generalized solution from $W_2^1(\Omega_t)$ of the elliptic equation*

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{ij}(x,t,u,u_x) u_{x_i}) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(b_i(x,t) u) + b(x,t) u = f - u_{tt} \qquad (16)$$

in domain Ω_t, $u|_{\partial\Omega_t} = 0$.

Suppose, that a boundary of domain Ω consists of two $(n-1)$-dimensional infinitely smooth surfaces which intersect along some $(n-2)$-dimensional infinitely smooth manifold $\alpha \in C^\infty$ at non-zeroangles $\gamma(p)$. Let's fix $p \in \alpha \times [0,T]$ and reduce operator $\sum_{i,j=1}^{n} a_{ij}(p) \frac{\partial^2}{\partial x_i \times_j}$ to the canonical form. After that, angle $\gamma(p)$ turns to some angle $\omega(p)$. Note, that since results are local, they are true and in case, when $\partial\Omega$ consists of finite number of surfaces C^∞, which pairwise intersect along manifolds of class C^∞.

Note, that in the small neighborhood U_p of each point p of the manifold $S = \alpha \times [0,T]$ one can introduce a local system of coordinates by the following special way. If neighborhood U_p is enough small, the there exists infinitely differentiable transformation, turning domain $\Omega \cap U_p$ to the angle. Therefore, without loosing generality in the neighborhood of point t one can consider, that boundary surface has the form $0 < \sqrt{x_1^2 + x_2^2} = r < \infty$, $0 < \varphi < \beta = const$ is a polar angle in the plane (x_1, x_2), $|x_i| < \infty$, $i = 3, .., n$.

Theorem 2. *Let $\omega < \pi$, $f(x,t)$, $f_t(x,t) \in L_2(Q_T)$. Then $u(x,t) \in W_2^2(Q_T)$ and the following inequality is true:*

$$\|u(x,t)\|_{W_2^2(Q_T)} \le C_7 \left(\|f(x,t)\|_{L_2(Q_T)} + \|f_t(x,t)\|_{L_2(Q_T)} \right) \qquad (17)$$

Proof. Rewrite equation (1) in the following form:

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{ij}(x,t,u,u_x) u_{x_i}) + \sum_{i=1}^{n} \frac{\partial}{\partial x_j}(b_i(x,t) u) + b(x,t) u = f - u_{tt} = F \qquad (18)$$

As is proved in lemma 4, $u_t(x,t) \in \dot{W}_2^1(Q_T)$. Therefore, $u_{tt}(x,t) \in L_2(Q_t)$. Since $f(x,t)$ by condition from $L_2(Q_t)$, then $F(x,t)$ also belongs to $L_2(Q_t)$. Again consider section Ω_t of cylinder Q_t by hyperplane $t = const$. Almost for all t the right hand side of (18) $F(x,t)$ by Foubini's theorem belongs to $L_2(Q_t)$. Then, by lemma 5 $u(x,t)$ is a generalized solution of elliptical equation (16) from space $W_2^1(\Omega_t)$ almost for all t. Considering, that $F \in L_2(\Omega_t)$, one can apply theorem on smoothness of solution of Diriclet's problem for second order elliptical equation in domain with nonsmooth boundary [9], at $k = 0$, $\alpha = 0$ and get that $u \in W_2^2(\Omega_t)$. The following estimation is true:

$$\|u(x,t)\|^2_{W_2^2(\Omega_T)} \leq C_7 \left(\|F\|^2_{L_2(\Omega_T)} + \|u\|^2_{L_2(\Omega_T)} \right) \leq$$

$$\leq C_7 \left(\|f\|^2_{L_2(\Omega_T)} + \|u_{tt}\|^2_{L_2(\Omega_T)} + \|u\|^2_{L_2(\Omega_T)} \right) \tag{19}$$

Now integrating by t from 0 to T both parts of inequality (19) and taking (13), (14) into account, we get, that $u(x,t) \in W_2^2(Q_t)$. And estimation (17) is true.

The theorem is proved.

Theorem 3. *Let* $f(x,t)$, $f_t(x,t) \in W_2^2(Q_t)$, $f(x,t) \equiv 0$ *in the neighborhood of set* $\{t = 0,\ x \in \partial\Omega\}$, $\omega < \dfrac{\pi}{k+1}$. *Then* $u(x,t) \in W_2^{k+2}(Q_t)$ *and the following estimation is true:*

$$\|u_t(x,t)\|_{W_2^{k+2}(Q_T)} \leq C \left(\|f(x,t)\|_{W_2^k(Q_T)} + \|f_t(x,t)\|_{W_2^k(Q_T)} \right)$$

Proof. At $k = 0$ the result follows from theorem 2. In general case we have to make induction and use the above mentioned lemmas.

The theorem is proved.

Let's introduce the following denotation. $x = (x_1, x_2)$, $z = (x_3, ..., x_n) = (z_1, ..., z_{n-2})$, $dx = dx_1, ..., dx_2$, $dzdz_1, .., dz_{n-2}$ $\dfrac{\partial}{\partial x}$ is a derivative of $\dfrac{\partial}{\partial x_i}$, $1 \leq i \leq 2$, $\dfrac{\partial}{\partial z}$ is a derivative $\dfrac{\partial}{\partial z_i}$, $1 \leq i \leq n-2$, $W_2^k(Q_T)$ is a Sobolev's space with the norm

$$\|u\|_{W_2^k(Q_T)} = \left(\iint\limits_{Q_T} \sum_{\substack{s=0 \\ s_0+s_1+s_2=s}}^{k} \left| \frac{\partial^s u}{\partial t^{s_0} \partial x^{s_1} \partial z^{s_2}} \right|^2 dxdzdt \right)^{1/2}$$

$\overset{\circ}{V}{}_2^k(Q_T)$ is space of function, having the generalized derivatives by all variables up to order k inside of Q_T with the norm

$$\|u\|_{\overset{\circ}{V}{}_2^1(Q_T)} = \left(\iint\limits_{Q_T} \sum_{\substack{s=0 \\ s_0+s_1+s_2=s}}^{k} \rho^{2s-2k} \left| \frac{\partial^s u}{\partial t^{s_0} \partial x^{s_1} \partial z^{s_2}} \right|^2 dxdzdt \right)^{1/2}$$

where the function $\rho(x,t) \in C^\infty(Q_T)$ is positive everywhere except the manifold S and coincides with $r(x)$ in the neighborhood of S.

Let for various pairs (m_1,s_1), (m_2,s_2) ($m_i>0,\ s_i\geq 0$ are entire) such that $\dfrac{\pi m_i}{\omega}+S_i-k-1<0$, it is required

$$\frac{\pi m_1}{\omega}+s_1\neq\frac{\pi m_2}{\omega}+s_2 \tag{20}$$

Theorem 4. *Suppose that* $\dfrac{\partial^i f(x,t,z)}{\partial t^{i_1}\partial z^{i_2}}\in \overset{\circ}{V}{}_2^k(Q_T)$, $0\leq i\leq l$, $f=0$ *in the neighborhood of set* $\{(x,t):\partial\Omega,\ \ t=0\}$, $0<\omega<2\pi$, $\omega\neq\pi$, $k+1\neq\dfrac{m\pi}{\omega}$, $m>0$ *is entire. Condition (20) is fulfilled. Then, generalized solution of problem (1), (2) is represented in the form*

$$\sigma(x,t)\,u(x,z,t)=$$
$$=\sum_j\sigma(x,z)\,r^{\frac{m\pi}{\omega}+s}\ln^q r\cdot C_j(z,t)\,\Phi_j(\varphi,z,t)+\sigma(x,z)\,\nu(x,z,t) \tag{21}$$

where $\sigma(x,z)$ *is infinitely differentiable function,* $\operatorname{supp}\sigma(x,z)\subset V_p$ *and summation is introduced by multiindex* $J=(m,s,q,p)$, *moreover,* $\dfrac{m\pi}{\omega}+s<k+1$, $0\leq q\leq q_0(ms)$, $0<p\leq p_0(m,s)$, $m>0$, $s\geq 0$ *(*m,s,p,q *are entire). Here* $q_0(m,0)=0$, $P_0(m,0)=1$ *and for functions* $C_j(z,t)$, $\nu(x,z,t)$ *the following estimation is true*

$$\sum_{j=0}^{l-1}\left\|\frac{\partial^j C_j}{\partial t^j\partial^{j_2}t}\right\|^2_{L_2(Q_T)}+\sum_{j=0}^{l-1}\sum_{i=0}^{k-2}\left\|r^{i-k-2}\frac{\partial^{j+i}C_j}{\partial t^{j_1}\partial^{j_2}\partial x^i}\right\|^2_{L_2(Q_T)}\leq$$
$$\leq C_{10}\sum_{j=0}^{l}\left\|\frac{\partial^i f}{\partial t^{i_1}\partial z^{l_2}}\right\|^2_{\overset{\circ}{V}{}_2^k(Q_p)} \tag{22}$$

where $Q_p=(\Omega\cap U_p)\times[0,T]$, C_{10} *is a constant dependent only on the domain* Q_T. *And the functions* $\Phi_j(\varphi,z,t)$ *are infinitely differentiable function by each of variables independent of solution.*

Proof. In section Ω_T of cylinder Q_T by hyperplane $t=const$ by lemma 5 $U(x,t)$ is a generalized solution of elliptical equation (16) from the space $W_2^1(\Omega_t)$ almost for all t. Taking into account the conditions of the theorem, one can apply theorem on smoothness of solution of Diriclet problem for second order elliptical equation in domain with nonsmooth boundary [3], we get $u(x,t)\in C^{1,\beta}(\Omega_t)$, $0<\beta<1$. Now, using methods [10], we'll get the required result. Note, that the similar result was obtained in [11] for linear equations.

The theorem is proved.

Remark. One can prove result of theorem of type 2 for the following class of evolutional equations as well

$$\frac{\partial^2 u}{\partial t^2}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(a_{ij}(t,x)\frac{\partial u}{\partial x_j}\right)+a(t,x)\,u-$$
$$-\sum_{i,j=1}^{n}\frac{\partial^2}{\partial t\partial x_j}\left(b_{ij}(t,x)\frac{\partial u}{\partial x_j}\right)+\frac{\partial}{\partial t}(b(t,x)\,u)=f(x,t) \tag{23}$$

where the real-valued functions $a_{ij}(t,x)$, $b_{ij}(t,x)$, $i=1,2,...,n$, $a(t,x)$ and $b(t,x)$ satisfy the conditions $a_{ij}(t,x)=a_{ji}(t,x)$, $\gamma^{-1}|\xi|^2 \le \sum\limits_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j \le \gamma|\xi|^2$, $\gamma>0$, $\xi\in R^n$, $\sum\limits_{i,j=1}^{n} \frac{\partial a_{ij}(t,x)}{\partial t}\xi_i\xi_j \ge 0$, $a_{ij}(t,x)\in C^{1,1}_{t,x}(Q_T)$, $a(t,x)\ge 0$, $\frac{\partial a(t,x)}{\partial t} \ge 0$, $a(t,x)\in C^{1,0}_{t,x}(Q_T)$, $\sum\limits_{i,j=1}^{n} b_{ij}(t,x)\xi_i\xi_j \ge 0$, $\sum\limits_{i,j=1}^{n} \frac{\partial b_{ij}(t,x)}{\partial t}\xi_i\xi_j \ge 0$, $b_{ij}(t,x)\in C^{1,1}_{t,x}(Q_T)$, $b(t,x)\ge 0$, $b(t,x)\in C^{1,0}_{t,x}(\bar{Q}_T)$

Besides, $f(x,t)\in L_2(Q_T)$.

Equation (23) is a generalization of equation of the type

$$\frac{\partial^2 u}{\partial t^2} = \eta\Delta\frac{\partial u}{\partial t} + \Delta u \tag{24}$$

where $\eta = const > 0$ is a parameter. Equation (24) describes propagation of perturbations in viscous media, sound propagation in viscous gas and other processes of the similar nature.

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