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STABILITY OF OPTIMAL CONTROL PROBLEM FOR SYSTEMS WITH NON-SEPARATED BOUNDARY CONDITIONS

Abstract

In the paper we consider an optimal control problem for systems with non-separated boundary conditions under inexact input data. Convergence of an optimal control problem with inexact data to initial problem by functional and gradient is proved under some conditions.

In modelling, development of methods and numerical solution of applied problems of optimal control, as in other fields of computing mathematics there arises a problem on determination of closeness of two mathematical models one of which is considered as a perturbed one with respect to other one. Therewith it is necessary to know a priori if the considered problem is stable with respect to perturbation, and have estimations of deviation convergence rate of solutions.

In many applied problems that are unstable and non-correct proximity of information on the problem and its input data gives negative answer to these convergences. Therefore, special approximations with use of the regularization methods are worked out [1]. At the same time the problems may be correct, stable at which proximity of information in the initial problem and its input data doesn't influence essentially on closeness of approximate optimal elements to the solution of the initial problem. From the computing point of view the solution methods of such problems are less laborious in comparison with unstable problems. Convergence of approximate optimal controls to the set of optimal controls allows not to conduct regularization, and thereby numerical method is simplified, volume and time of calculations in solving applied problems is shortened. Therefore, in working out numerical methods for solving this or other concrete problem it is necessary to know a priori to which class does this problem belong and how does it behave itself with respect to perturbations and has convergence estimate.

In the paper we consider a linear-quadratic of optimal control problem with non-separated boundary conditions for inexact input data.

Let the functional

$$J(u) = \alpha \left\| x(t_0, u) - y \right\|^2 + \beta \left\| x(T, u) - z \right\|^2, \tag{1}$$

be minimized, where $x(t,u)$ is determined from the conditions

$$\dot{x}(t) = A(t)\, x(t) + B(t)\, u(t) + f(t), \quad t_0 \le t \le T; \tag{2}$$

$$D_1 x(t_0) + D_2\, x(T) = C \tag{3}$$

$$u = u(\cdot) \in U \subseteq L_2^r[t_0, T]. \tag{4}$$

Here, it is assumed that $A(t)$, $B(t)$, $f(t)$ are the given piece-wise continuous matrices of the function of order $n \times n$, $n \times r$, $n \times 1$, respectively, α and β are non-negative numbers; time moments t_0, T and the points $y, z, c \in E^n$ are given; D_1 and D_2 dimensional $n \times n$ are constant matrices, moreover $\det(D_1 + D_2) \neq 0$. U is a convex closed set. We can show that for

$$A_{\max}(T - t_0)\left((D_1 + D_2)^{-1} D_2 + 1\right) < 1,$$

where $A_{\max} = \max\limits_{[t_0,T]} |A(t)|$ the boundary value problem (2)-(4) has a unique solution for each fixed $u \in U$.

In the paper [2] it is shown that the functional (1) is differentiable under restrictions (2)-(4) and its gradient is of the form:

$$J'(u) = B^*(t)\psi(t,u) \in L_2^r[t_0,T], \tag{5}$$

where $\psi(t,u)$ is the solution of the integral equation

$$\begin{aligned}\psi(t) = -2\alpha\left(x(t_0,u) - y\right)^*(D_1 + D_2)^{-1} D_2 + \\ +2\beta\left(x(T,u) - z\right)^*(D_1 + D_2)^{-1} D_1 - \\ -\int_{t_0}^{T} A^*(t)\psi(t)\,dt\,(D_1 + D_2)^{-1} D_1 + \\ +\int_{t_0}^{t} A^*(\tau)\psi(\tau)\,d\tau,\ \ t_0 \le t \le T.\end{aligned} \tag{6}$$

Remark. We can write the integral equation (6) in the following equivalent form:

$$\begin{gathered}\dot{\psi}(t) = A^*(t)\psi(t), \quad t_0 \le t \le T, \\ D_1^*(D_1^* + D_2^*)^{-1}\psi(T) + D_2^*(D_1^* + D_2^*)^{-1}\psi(t_0) = \\ = 2D_2^*(D_1^* + D_2^*)^{-1}\alpha\left(x(t_0,u) - y\right) + 2D_1^*(D_1^* + D_2^*)^{-1}\beta\left(x(T,u) - z\right).\end{gathered}$$

The conditions imposed the on input data of the problem provide boundedness of the solution of problem (2)-(3) and (6) i.e.

$$\|x(t,u)\| \le C_0\ , \quad \|\psi(t,u)\| \le C_1\ ,\ t_0 \le t \le T.$$

Here and below, by $C_0\ , C_1\ , C_2\ , ...$ we denote positive constants independent of $t \in [t_0, T]\,,\ u = u(t) \in L_2^r[t_0, T]$.

Along with problem (1)-(4) we consider the optimal control pronlem:

$$J_k(u) = \alpha_k\left\|x_k(t_0,u) - y_k\right\|^2 + \beta_k\left\|x_k(T,u) - z_k\right\|^2 \to \min \tag{7}$$

under restrictions

$$\dot{x}_k(t) = A_k(t)x_k(t) + B_k(t)u(t) + f_k(t), \quad t_0 \le t \le T; \tag{8}$$

$$D_1 x_k(t_0) + D_2\ x_k(T) = C_k \tag{9}$$

Here we assume that the matrices $A_k(t)$, $B_k(t)$, $f_k(t)$, D_1, D_2, the points y_k, z_k, c_k and the numbers α_k, β_k are approximations of the corresponding matrices $A(t)$, $B(t)$, $f(t)$, D_1, D_2, the points y, z, c and the numbers α, β, moreover

$$\max\left\{\sup_{t_0\le t\le T}\|A_k(t)-A(t)\|,\ \sup_{t_0\le t\le T}\|B_k(t)-B(t)\|,\right.$$

$$\sup_{t_0\le t\le T}\|f_k(t)-f(t)\|, \|y_k-y\|, \|c_k-c\|,$$

$$\left.\|z_k-z\|, |\alpha_k-\alpha|, |\beta_k-\beta|\right\}\le\eta_k,\quad k=1,2,3,... \tag{10}$$

$$\lim_{k\to\infty}\eta_k=0.$$

If the inequality

$$(A_{\max}+\eta_k)(T-t_0)\left[\left[(D_1+D_2)^{-1}D_2\right]+1\right]<1,$$

is fulfilled we can show that problem (8)-(9) has a unique solution for each fixed $u\in U$. Obviously, the difference $\Delta x_k(t)=x_k(t,u)-x(t,u)$ satisfies the conditions

$$\Delta\dot{x}(t)=A_k(t)\Delta x_k(t)+(A_k(t)-A(t))x(t)+$$

$$+(B_k(t)-B(t))u(t)+f_k(t)-f(t),\quad t_0\le t\le T; \tag{11}$$

$$D_1^k\Delta x_k(t_0)+D_2^k\ \Delta x_k(T)=C_k-C_0 \tag{12}$$

we can represent the boundary value problem (11), (12) in the form

$$\Delta x_k(t)\equiv x_k(t,u)-x(t,u)=(D_1+D_2)^{-1}(C_k-C)-$$

$$-(D_1+D_2)^{-1}D_2\int_{t_0}^{T}\left[A_k(t)\Delta x_k(t)+(A_k(t)-A(t))x(t)+\right.$$

$$\left.(B_k(t)-B(t))u(t)+(f_k(t)-f(t))\right]dt+$$

$$+\int_{t_0}^{t}\left[A_k(\tau)\Delta x_k(\tau)+(A_k(\tau)-A(\tau))x(\tau)+\right.$$

$$\left.+(B_k(\tau)-B(\tau))u(\tau)+(f_k(\tau)-f(\tau))\right]d\tau.$$

Here we pass to the norm, consider the conditions (10) and have:

$$|\Delta x_k(t)|\le\left|(D_1+D_2)^{-1}\right|\eta_k+\left|(D_1+D_2)^{-1}D_2\right|\times$$

$$\left\{(A_{\max}+\eta_k)\int_{t_0}^{T}|\Delta x_k(t)|\,dt+\eta_k\int_{t_0}^{T}|\Delta x_k(t)|\,dt+\right.$$

$$+\eta_k \int_{t_0}^{T} |u(t)|\, dt + \eta_k (T - t_0) \Bigg\} + (A_{\max} + \eta_k) \int_{t_0}^{t} |\Delta x_k(t)|\, dt +$$

$$+\eta_k \int_{t_0}^{t} |x(\tau)|\, d\tau + \eta_k \int_{t_0}^{t} |u(\tau)|\, d\tau + \eta_k (T - t_0).$$

Strengthening the right hand side of the last inequality and passing to the norm, we get

$$\max_{[t_0,T]} |\Delta x_k(t)| \le (A_{\max} + \eta_k)(T - t_0) \times$$

$$\times \left[\left| (D_1 + D_2)^{-1} D_2 \right| + 1 \right] \max_{[t_0,T]} |\Delta x_k(t)| +$$

$$(D_1 + D_2)^{-1} \eta_k + C_0 (T - t_0) \eta_k \left[1 + \left| (D_1 + D_2)^{-1} D_2 \right| \right] +$$

$$+\eta_k R (T - t_0)^{1/2} \left[\left[1 + \left| (D_1 + D_2)^{-1} D_2 \right| \right] + \right.$$

$$\left. +\eta_k (T - t_0)\, 1 + \left| (D_1 + D_2)^{-1} D_2 \right| \right].$$

In view of $\eta_k \to 0$ as $k \to \infty$ it follows that there exists such a number k_0 that for $k > k_0$

$$(A_{\max} + \eta_k)(T - t_0) \left[\left| (D_1 + D_2)^{-1} D_2 \right| + 1 \right] < 1$$

so

$$\max |\Delta x_k(t)| \le [1 - (A_{\max} + \eta_k)(T - t_0) \times$$

$$\times \left[\left| (D_1 + D_2)^{-1} D_2 \right| + 1 \right] \Big]^{-1} \left| (D_1 + D_2)^{-1} \right| +$$

$$+ \left(1 + \left| (D_1 + D_2)^{-1} D_2 \right| \right) \left[C (T - t_0) + R (T - t_0)^{1/2} + (T - t_0) \right] \eta_k$$

or

$$\max |\Delta x_k(t)| \le C_2 \eta_k$$

As an approximation for the gradient $J'(u)$ we take

$$J'_k(u) = B^*_k(t)\, \psi_k(t, u) \in L_2[t_0, T] \tag{13}$$

where $\psi_k(t, u)$ is determined from the equality

$$\psi_k(t) = -2\alpha_k (x_k(t_0, u) - y_k)^* \left(D^k_1 + D^k_2 \right)^{-1} D^k_2 +$$

$$+2\beta_k (x_k(T, u) - z_k)^* \left(D^k_1 + D^k_2 \right)^{-1} D^k_1 -$$

$$- \int_{t_0}^{T} A^*_k(t)\, \psi_k(t)\, dt \left(D^k_1 + D^k_2 \right)^{-1} D^k_1 + \int_{t_0}^{t} A^*_k(\tau)\, \psi_k(\tau)\, d\tau.$$

Then for the difference $\Delta\psi_k(t) = \psi_k(t,u) - \psi(t,u)$ we can get the estimation. For the difference of conjugate equations we have

$$\Delta\psi_k(t) \equiv \psi_k(t,u) - \psi(t,u) =$$
$$= -2\left[\alpha_k(x_k(t_0) - y_k)^* - \alpha(x(t_0) - y)^*\right](D_1 + D_2)^{-1} D_2 +$$
$$+2\left[\beta_k(x_k(T) - z_k)^* - \beta(x(T) - z)^*\right](D_1 + D_2)^{-1} D_1 -$$
$$-\left[\int_{t_0}^{T} A_k^*(t)\psi_k(t)\,dt - \int_{t_0}^{T} A^*(t)\psi_k(t)\,dt\right](D_1 + D_2)^{-1} D_1 +$$
$$+\int_{t_0}^{t} A_k^*(\tau)\psi_k(\tau)\,d\tau - \int_{t_0}^{t} A^*(\tau)\psi_k(\tau)\,d\tau.$$

Here we conduct same groupings and get

$$\Delta\psi_k(t) \equiv -2\left[\alpha_k(x_k(t_0) - x(t_0))^* - (y_k - y)^*\right.$$
$$+(\alpha_k - \alpha)(x(t_0) - y)^*\left.\right](D_1 + D_2)^{-1} D_2 + 2\left[\beta_k(x_k(T) - x(T))^* -\right.$$
$$\left.-\beta(z_k - z)^* + (\beta_k - \beta)(x(T) - y)\right](D_1 + D_2)^{-1} D_1 -$$
$$-\left[\int_{t_0}^{T} A_k^*(t)\psi_k(t)\,dt - \int_{t_0}^{T} (A_k(t) - A(t))^*\psi(t)\,dt\right](D_1 + D_2)^{-1} D_1 +$$
$$+\int_{t_0}^{t} A_k^*(\tau)\psi_k(\tau)\,d\tau - \int_{t_0}^{t} (A_k(\tau) - A(\tau))^*\psi(\tau)\,d\tau.$$

We pass to the norm and strengthen the right hand side of the inequality. Then

$$|\Delta\psi_r(t)| \le 2\left[|\alpha_k|\left(|x_k(t_0) - x(t_0)| - |y_k - y_0| +\right.\right.$$
$$\left.\left.+|\alpha_k - \alpha|\right)(|x(t_0)| - |y|)\right]\left|(D_1 + D_2)^{-1} D_2\right| +$$
$$+2\left[|\beta_k|\left(|x_k(T) - x(T)| + |z_k - z|\right) +\right.$$
$$\left.+|\beta_k - \beta|\left(|x(T)| - |y|\right)\right]\left|(D_1 + D_2)^{-1} D_1\right| +$$
$$+\left(\int_{t_0}^{T} |A_k(t)|\,|\Delta\psi_k(t)|\,dt + \int_{t_0}^{T} |A_k(t) - A(t)|\,|\psi(t)|\,dt\right)\left|(D_1 + D_2)^{-1} D_1\right| +$$
$$+\int_{t_0}^{T} |A_k(t)|\,|\Delta\psi_k(t)|\,dt - \int_{t_0}^{T} |A_k(t) - A(t)|\,|\psi(t)|\,dt.$$

Considering conditions (10) we have

$$|\Delta\psi_k(t)| \le 2\left((|\alpha| + \eta_k)(C_2\eta_k + \eta_k) + \eta_k(C_0 + |y|)\right)\left((D_1 + D_2)^{-1} D_2\right) +$$

$$+2\left(\left(|\beta|+\eta_k\right)\left(C_2\eta_k+\eta_k\right)+\eta_k\left(C_0+|z|\right)\right)\left|\left(D_1+D_2\right)^{-1}D_1\right|+$$

$$+\left[\left(A_{\max}+\eta_k\right)\int_{t_0}^{T}\left|\Delta\psi_k\left(t\right)\right|dt+\eta_k\int_{t_0}^{T}\left|\psi\left(t\right)\right|dt\right]\left|\left(D_1+D_2\right)^{-1}D_1\right|+$$

$$+\left(A_{\max}+\eta_k\right)\int_{t_0}^{T}\left|\Delta\psi_k\left(t\right)\right|dt+\eta_k\int_{t_0}^{T}\left|\psi\left(t\right)\right|dt.$$

Hence

$$\left(1-\left(A_{\max}+\eta_k\right)\left(T-t_0\right)\left(\left|\left(D_1+D_2\right)^{-1}D_2\right|+1\right)\right)\max\left|\psi\left(t\right)\right|\leq$$

$$\leq\left[2\left(\left(|\alpha|+\eta_k\right)\left(C_2+1\right)+\left(C_0+|y|\right)\right)\left|\left(D_1+D_2\right)^{-1}D_2\right|+\right.$$

$$\left.+2\left(\left(|\beta|+\eta_k\right)\left(C_2+1\right)+\left(C_0+|z|\right)\right)\left|\left(D_1+D_2\right)^{-1}D_1\right|\right]\eta_k+$$

$$+\left(C_1\left(T-t_0\right)\left(\left|\left(D_1+D_2\right)^{-1}D_1\right|+1\right)\right)\eta_k$$

Considering the condition

$$A_{\max}\left(T-t_0\right)\left(\left|\left(D_1+D_2\right)^{-1}D_2\right|+1\right)<1$$

we have

$$\max\left|\Delta\psi_k\left(t\right)\right|\leq\left[1-\left(A_{\max}+\eta_k\right)\left(T-t_0\right)\left(\left|\left(D_1+D_2\right)^{-1}D_2\right|+1\right)\right]^{-1}\times$$

$$\times\left[2\left(\left(|\alpha|+\eta_k\right)\left(C_2+1\right)+\left(C_0+|y|\right)\right)\left|\left(D_1+D_2\right)^{-1}D_2\right|+\right.$$

$$+2\left(\left(|\beta|+\eta_k\right)\left(C_2+1\right)+\left(C_0+|z|\right)\right)\left|\left(D_1+D_2\right)^{-1}D_1\right|+$$

$$\left.+C_1\left(T-t_0\right)\left(\left|\left(D_1+D_2\right)^{-1}D_1\right|+1\right)\right]\eta_k$$

or

$$\left|\Delta\psi_k\left(t\right)\right|\equiv\left|\psi_k\left(t,u\right)-\psi\left(t,u\right)\right|\leq C_3\eta_k \tag{14}$$

Now, let's estimate the differences $J_k\left(u\right)-J\left(u\right).$

$$J_k\left(u\right)-J\left(u\right)=\alpha_k\left\|x_k\left(t_0,u\right)-y_k\right\|^2+\beta_k\left\|x_k\left(T,u\right)-z_k\right\|^2-$$

$$-\alpha\left\|x\left(t_0,u\right)-y\right\|^2+\beta\left\|x\left(T,u\right)-z\right\|^2=$$

$$=\left(\alpha_k-\alpha\right)\left\|x_k\left(t_0,u\right)-y_k\right\|^2+\left(\beta_k-\beta\right)\left\|x_k\left(T,u\right)-z_k\right\|^2+$$

$$+\alpha\left(\left\|x_k\left(t_0,u\right)-y_k\right\|^2-\left\|x\left(t_0,u\right)-y\right\|^2\right)+$$

$$+\beta\left(\left\|x_k\left(T,u\right)-z_k\right\|^2-\left\|x\left(T,u\right)-z\right\|^2\right).$$

Hence we can easily get the following estimate

$$\left|J_k\left(u\right)-J\left(u\right)\right|\leq C_4\eta_k$$

We estimate $J_k'(u)-J'(u)$ in the norm $L_2[t_0,T]$ in a similar way. From corresponding formulae it is seen that

$$J_k'(u)-J'(u)=\int_{t_0}^{T}(B_k^*(t)\psi_k(t,u)-B_k^*(t)\psi(t,u)+$$

$$|+B_k^*(t)\psi(t,u)-B^*(t)\psi(t,u))\,dt=$$

$$=\int_{t_0}^{T}B_k^*(t)(\psi_k(t,u)-\psi(t,u))\,dt-\int_{t_0}^{T}(B_k^*(t)-B^*(t))\psi(t,u)\,dt.$$

Passing to the norm in the space $L_2^r[t_0,T]$, we have

$$\left\|J_k'(u)-J'(u)\right\|\leq\int_{t_0}^{T}|B_k^*(t)||\psi_k(t,u)-\psi(t,u)|\,dt+$$

$$+\int_{t_0}^{T}|B_k^*(t)-B^*(t)||\psi(t,u)|\,dt\leq$$

$$\leq C_3(B_{\max}+\eta_k)(T-t_0)\eta_k+C_1(T-t_0)\eta_k=C_5\eta_k$$

Using (5), (9)-(12) we have

$$\left\|J_k'(u)-J'(u)\right\|_{L_2[t_0,T]}=\left(\int_{t_0}^{T}|B_k^*(t)\psi_k(t,u)-B^*(t)\psi(t,u)|^2\,dt\right)^{1/2}\leq C_5\eta_k$$

Thus, we prove the

Theorem. *Let the above enumerated conditions be fulfilled. Then,*

$$\lim_{k\to\infty}\left\|J_k'(u)-J'(u)\right\|_{L_2[t_0,T]}=0,\qquad\lim_{k\to\infty}m_k=m$$

where $m_k=\min\limits_U J_k(u)$, $m=\min\limits_U J(u)$.

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