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SOLVABILITY OF A BOUNDARY VALUE PROBLEM FOR A SECOND ORDER ELLIPTIC-DIFFERENTIAL OPERATOR EQUATION WITH SPECTRAL PARAMETER IN THE EQUATION AND BOUNDARY CONDITIONS

Abstract

In the paper, in a separable Hilbert H we study the following boundary value problem

$$\lambda u(x) - u''(x) + Au(x) = f(x), \; x \in [0,1], \quad (1)$$

$$\alpha u'(0) + \lambda u(1) = f_1,$$

$$u(0) = f_2, \quad (2)$$

where λ is a spectral parameter; A is a linear closed operator with everywhere dense in H domain of definition and a resolvent, decreasing as $|\lambda|^{-1}$ under large $|\lambda|$ at some angles containing a positive semi-axis; $\alpha \neq 0$ is any fixed complex number. Sufficient conditions for the solvability of problems $(1)-(2)$ in $L_p((0,1);H)$ $(p>1)$ are found.

1. Introduction

Boundary value problems on a finite segment for elliptic, second order differential-operator equations with a spectral parameter in the equation and boundary conditions are considered in different aspects in the papers $[1-6]$. In all these papers, the boundary conditions are separated.

In the present paper, a boundary value problem is studied for a second order elliptic differential-operator equation in the case when one and the same spectral parameter is contained in the equation and also in one of boundary conditions, moreover, the boundary conditions are not separated.

Thus, in the given paper, in a separable Hilbert space H we consider the following boundary value problem

$$L(\lambda, D)\, u := \lambda u(x) - u''(x) + Au(x) = f(x), \quad x \in [0,1], \quad (1.1)$$

$$L_1(\lambda)\, u := \alpha u'(0) + \lambda u(1) = f_1,$$

$$L_2 u := u(0) = f_2, \quad (1.2)$$

where λ is a spectral parameter; A is a linear closed operator with everywhere dense in H domain of definition and a resolvent, decreasing as $|\lambda|^{-1}$ under sufficiently large $|\lambda|$ at some angles containing a positive semi-axis; $\alpha \neq 0$ is any fixed complex number.

Sufficient conditions for the solvability of problem $(1.1),(1.2)$ are found, some coercive estimates (with respect to u and λ) for the solution of problem $(1.1),(1.2)$ in $L_p((0,1);H)$ $(p>1)$ are established.

Notice that a boundary value problem for equation (1.1) with irregular boundary conditions (without a spectral parameter in the boundary conditions) was studied in [7, section 5.5].

Let E_0 and E_1 be two Banach spaces continuously embedded into the Banach space E: $E_0\subset E, E_1\subset E$. Such two spaces is called interpolation pair $\{E_0,E_1\}$.

Consider a Banach space $E_0+E_1:=\{u: u\in E, \exists u_j\in E_j\ ,\ j=0,1$, where

$$u=u_0+u_1, \|u\|_{E_0+E_1}:=\inf_{u=u_0+u_1,\ u_{j\in E_j}}\left(\|u_0\|_E+\|u_1\|_E\right)\}.$$

According to statement 1.3.1 from [8], the functional

$$K(t,u):=\inf_{u=u_0+u_1,\ u_{j\in E_j}}\left(\|u_0\|_E+t\|u_1\|_E\right),\ u\in E_0+E_1$$

is continuous on $(0,+\infty)$ with respect to t, and the following estimate

$$\min\{1,t\}\|u\|_{E_0+E_1}\le K(t,u)\le\max\{1,t\}\|u\|_{E_0+E_1}$$

is true.

By the K method, an interpolational space for $\{E_0,E_1\}$ is determined as

$$(E_0,E_1)_{\theta,p}:=u: u\in E_0+E_1,\ \ \|u\|_{(E_0,E_1)_{\theta,p}}:=$$

$$=\left.\left(\int_0^\infty t^{-1-\theta p}K^p(t,u)\,dt\right)^{1/p}<\infty\right\},\quad 0<\theta<1, 1\le p<\infty$$

Let E and F be Banach spaces. The set $E\dot{+}F$ of all the vectors of the form (u,v), where $u\in E$ and $v\in F$ with ordinary linear operations by the coordinates and the norm

$$\|(u,v)\|_{E\dot{+}F}:=\|u\|_{E_1}+\|v\|_F$$

is a Banach space and is said to be a direct sum of Banach spaces E and F.

Let A be a linear closed operator in a separable Hilbert space H with domain of definition $D(A)$. $D(A)$ turns into the Hilbert space $H(A)$ with respect to the norm

$$\|u\|_{H(A)}:=\left(\|u\|_H^2+\|Au\|_H^2\right)^{1/2}.$$

Let E_1 and E be Banach spaces. By $B(E_1,E)$ we denote a Banach space of all bounded operators acting from E_1 to E, with ordinary operator norm. In the special case, $B(E):=B(E,E)$.

By $L_p((0,1);H)$ $(1<p<\infty)$ we denote a Banach space (for $p=2$ a Hilbert space) of functions $x \to u(x) : [0,1] \to H$, strongly measurable and summable in the p - th power, with the norm

$$\|u\|_{L_p((0,1);H)} := \left(\int_0^1 \|u(x)\|_H^p \, dx \right)^{1/p} < \infty.$$

By $W_p^l((0,1);H)$, $(1<p<\infty,\ 0 \le l$ are integers) we denote a Banach space of functions $u(x)$ with values in H that have l - th order generalized derivatives of on $(0,1)$, with the norm

$$\|u\|_{W_p^l((0,1);H)} := \sum_{k=0}^{l} \left(\int_0^1 \left\|u^{(k)}(x)\right\|_H^p dx \right)^{1/p}.$$

The space

$$W_p^2((0,1);H(A),H) := \left\{ u : u \in L_p((0,1);H(A)),\, u'' \in L_p((0,1);H) \right\}$$

with the norm

$$\|u\|_{W_p^2((0,1);H(A),H)} := \|u\|_{L_p((0,1);H(A))} + \left\|u''\right\|_{L_p((0,1);H)}$$

is a Banach space (for more general spaces see [8, lemma 1.8.1] and also [7, section 1.7.7]).

2. Homogeneous equation

At first we consider the following boundary value problem in the Hilbert space H.

$$L(\lambda, D)\, u := \lambda u(x) - u''(x) + Au(x) = 0, x \in [0,1], \tag{2.1}$$

$$L_1(\lambda)\, u := \alpha u'(0) + \lambda u(1) = f_1,$$

$$L_2 u := u(0) = f_2. \tag{2.2}$$

Theorem 1. *Let the following conditions be fulfilled:*

(1) *A is a linear closed operator with a dense domain of definition in the Hilbert space H and*

$$\|R(\lambda, A)\| \le C(1+|\lambda|)^{-1},\ \ |\arg\lambda| \ge \pi - \varphi,\ 0 \le \varphi < \pi,$$

where $R(\lambda, A) := (\lambda I - A)^{-1}$ is a resolvent of the operator A and A is invertible.

(2) *$\alpha \neq 0$ is a complex number.*

Then, for $f_k \in \left(H\left(A^2\right), H\right)_{\frac{3}{4}-\frac{k}{4}+\frac{1}{4p}, p}, p \in (1,\infty)$ and for sufficiently large $|\lambda|$ from the angle $|\arg\lambda| \le \varphi$ the problem (2.1) − (2.2) has a unique solution from $W_p^2((0,1);H(A),H)$, and the following estimates hold.

$$|\lambda| \|u\|_{L_p((0,1);H)} + \left\|u''\right\|_{L_p((0,1);H)} + \|Au\|_{L_p((0,1);H)} \le$$

$$\leq C\sum_{k=1}^{2}\frac{1}{|\lambda|^{1/k}}\left(\|f_k\|_{(H(A^2),H)_{\frac{3}{4}-\frac{k}{4}+\frac{1}{4p}p}}+|\lambda|^{\frac{k+1}{2}-\frac{1}{2p}}\|f_k\|_H\right), \qquad (2.3)$$

$$|\lambda|^{1/2}\|u\|_{L_p((0,1);H)}+\|u'\|_{L_p((0,1);H)}+\left\|A^{1/2}u\right\|_{L_p((0,1);H)}\leq$$

$$\leq C\sum_{k=1}^{2}\frac{1}{|\lambda|^{1/k}}\left(\|f_k\|_{(H(A),H)_{1-\frac{k}{2}+\frac{1}{2p}p}}+|\lambda|^{\frac{k}{2}-\frac{1}{2p}}\|f_k\|_H\right). \qquad (2.4)$$

Proof. By [7,lemma 5.3.2/1] for $|\arg\lambda|\leq\varphi<\pi$ an arbitrary solution of equation (2.1) belonging to $W_p^2((0,1);H(A),H)$ is of the form:

$$u(x)=e^{-x(A+\lambda I)^{1/2}}g_1+e^{-(1-x)(A+\lambda I)^{1/2}}g_2, \qquad (2.5)$$

where $g_k\in(H(A),H)_{\frac{1}{2p}p}, k=1,2.$

It will be necessary the function $u(x)$ of the form (2.5) satisfy conditions (2.2). Then, for g_1 and g_2 we get the following system

$$\begin{cases}\left[-\alpha(A+\lambda I)^{1/2}+\lambda e^{-(A+\lambda I)^{1/2}}\right]g_1+\left[\alpha(A+\lambda I)^{1/2}e^{-(A+\lambda I)^{1/2}}+\lambda\right]g_2=f_1\\ g_1+e^{-(A+\lambda I)^{1/2}}g_2=f_2\end{cases} \qquad (2.6)$$

From the second equation of system (2.6) we have

$$g_1=f_2-e^{-(A+\lambda I)^{1/2}}g_2. \qquad (2.7)$$

Considering (2.7) in the first equation of system (2.6), we get the equation for g_2

$$\left[\alpha(A+\lambda I)^{1/2}+\lambda e^{-(A+\lambda I)^{1/2}}\right]\left(f_2-e^{-(A+\lambda I)^{1/2}}g_2\right)+$$
$$+\left[\alpha(A+\lambda I)^{1/2}e^{-(A+\lambda I)^{1/2}}+\lambda\right]g_2=f_1.$$

Hence we find g_2

$$g_2=\frac{1}{\lambda}\left[I+\frac{2\alpha}{\lambda}(A+\lambda I)^{1/2}e^{-(A+\lambda I)^{1/2}}-e^{-2(A+\lambda I)^{1/2}}\right]^{-1}\times$$
$$\times\left[f_1+\left(\alpha(A+\lambda I)^{1/2}-\lambda e^{-(A+\lambda I)^{1/2}}\right)f_2\right].$$

Since by [7,lemma 5.4.2/6] for sufficiently large $|\lambda|$ from the angle $|\arg\lambda|\leq\varphi$

$$\left\|e^{-2(A+\lambda I)^{1/2}}-\frac{2\alpha}{\lambda}(A+\lambda I)^{1/2}e^{-(A+\lambda I)^{1/2}}\right\|_{B(H)}\leq e^{-\omega|\lambda|^{1/2}}<q<1$$

then, by Neuann's identity we have

$$\left[I-\left(e^{-2(A+\lambda I)^{1/2}}-\frac{2\alpha}{\lambda}(A+\lambda I)^{1/2}e^{-(A+\lambda I)^{1/2}}\right)\right]^{-1}:=I+S(\lambda),$$

where

$$S(\lambda)=\sum_{k=1}^{\infty}\left[e^{-2(A+\lambda I)^{1/2}}-\frac{2\alpha}{\lambda}(A+\lambda I)^{1/2}e^{-(A+\lambda I)^{1/2}}\right]^{k} \quad (2.8)$$

It is obvious that the series in the right hand side of (2.8) converges by the norm of the space of operator in H.

Then, for the representation g_2 we get

$$g_2=\left(\frac{1}{\lambda}+R_{21}(\lambda)\right)f_1+\left(\frac{\alpha}{\lambda}(A+\lambda I)^{1/2}+R_{22}(\lambda)\right)f_2, \quad (2.9)$$

where

$$R_{21}(\lambda):=\frac{1}{\lambda}S(\lambda);\ R_{22}(\lambda):=\frac{\alpha}{\lambda}(A+\lambda I)^{1/2}S(\lambda)-e^{-(A+\lambda I)^{1/2}}(I+S(\lambda)).$$

Considering (2.9) in(2.7), we get

$$g_1=R_{11}(\lambda)f_1+(I+R_{12}(\lambda))f_2, \quad (2.10)$$

where

$$R_{11}(\lambda):=-e^{-(A+\lambda I)^{1/2}}\left(\frac{1}{\lambda}+R_{21}(\lambda)\right),$$

$$R_{12}(\lambda):=-e^{-(A+\lambda I)^{1/2}}\left(\frac{\alpha}{\lambda}(A+\lambda I)^{1/2}+R_{22}(\lambda)\right).$$

By [7, lemma 5.4.2/6] ,

$$\|R_{jk}(\lambda)\|_{B(H)}\to 0,\quad |\arg\lambda|\le\varphi,\ \ |\lambda|\to\infty.$$

Considering (2.9) and (2.10) in (2.5), we get

$$u(x)=e^{-x(A+\lambda I)^{1/2}}\left[R_{11}(\lambda)f_1+(I+R_{12}(\lambda))f_2\right]+$$
$$+e^{-(1-x)(A+\lambda I)^{1/2}}\left[\left(\frac{1}{\lambda}+R_{21}(\lambda)\right)f_1+\left(\frac{\alpha}{\lambda}(A+\lambda I)^{1/2}+R_{22}(\lambda)\right)f_2\right]. \quad (2.11)$$

For sufficiently large $|\lambda|$ from the angle $|\arg\lambda|\le\varphi$ we have

$$|\lambda|\,\|u\|_{L_p((0,1);H)}+\left\|u''\right\|_{L_p((0,1);H)}+\|Au\|_{L_p((0,1);H)}\le$$
$$\le\left\{|\lambda|\left[\left(\int_0^1\left\|e^{-x(A+\lambda I)^{1/2}}R_{11}(\lambda)f_1\right\|_H^p dx\right)^{1/p}+\left(\int_0^1\left\|e^{-x(A+\lambda I)^{1/2}}f_2\right\|_H^p dx\right)^{1/p}+\right.\right.$$
$$+\left(\int_0^1\left\|e^{-x(A+\lambda I)^{1/2}}R_{12}(\lambda)f_2\right\|_H^p dx\right)^{1/p}+\left(\int_0^1\left\|e^{-(1-x)(A+\lambda I)^{1/2}}\frac{1}{\lambda}f_1\right\|_H^p dx\right)^{1/p}+$$
$$+\left(\int_0^1\left\|e^{-(1-x)(A+\lambda I)^{1/2}}R_{21}(\lambda)f_1\right\|_H^p dx\right)^{1/p}+$$

$$+\left(\int_0^1 \left\| e^{-(1-x)(A+\lambda I)^{1/2}} \frac{\alpha}{\lambda} (A+\lambda I)^{1/2} f_2 \right\|_H^p dx\right)^{1/p} +$$

$$\left. + \left(\int_0^1 \left\| e^{-(1-x)(A+\lambda I)^{1/2}} R_{22}(\lambda) f_2 \right\|_H^p dx\right)^{1/p} \right] +$$

$$+\left(1+\left\|A(A+\lambda I)^{-1}\right\|\right)\left[\left(\left(\int_0^1 \left\|(A+\lambda I) e^{-x(A+\lambda I)^{1/2}} R_{11}(\lambda) f_1\right\|_H^p dx\right)^{1/p} +\right.\right.$$

$$+\left(\int_0^1 \left\|(A+\lambda I) e^{-x(A+\lambda I)^{1/2}} f_2\right\|_H^p dx\right)^{1/p} +$$

$$+\left(\int_0^1 \left\|(A+\lambda I) e^{-x(A+\lambda I)^{1/2}} R_{12}(\lambda) f_2\right\|_H^p dx\right)^{1/p} +$$

$$+\left(\int_0^1 \left\|(A+\lambda I) e^{-(1-x)(A+\lambda I)^{1/2}} \frac{1}{\lambda} f_1\right\|_H^p dx\right)^{1/p} +$$

$$\left(\int_0^1 \left\|(A+\lambda I) e^{-(1-x)(A+\lambda I)^{1/2}} R_{21}(\lambda) f_1\right\|_H^p dx\right)^{1/p} +$$

$$+\left(\int_0^1 \left\|\frac{\alpha}{\lambda}(A+\lambda I)^{3/2} e^{-(1-x)(A+\lambda I)^{1/2}} f_2\right\|_H^p dx\right)^{1/p} +$$

$$\left.\left.+\left(\int_0^1 \left\|(A+\lambda I) e^{-(1-x)(A+\lambda I)^{1/2}} R_{22}(x) f_2\right\|_H^p dx\right)^{1/p}\right]\right\}. \qquad (2.12)$$

Using [7, lemma 5.4.2/6 and theorem 5.4.2./1], we estimate some integrals participating in the right hand side of inequality (2.12).

$$\left(\int_0^1 \left\|\frac{\alpha}{\lambda}(A+\lambda I)^{3/2} e^{-(1-x)(A+\lambda I)^{1/2}} f_2\right\|_H^p dx\right)^{1/p} \le$$

$$\le \frac{C}{|\lambda|}\left(\|f_2\|_{(H(A^2),H)_{\frac{1}{4}+\frac{1}{4p},p}} + |\lambda|^{\frac{3}{2}-\frac{1}{2p}} \|f_2\|_H\right);$$

$$\left(\int_0^1 \left\|(A+\lambda I) e^{-x(A+\lambda I)^{1/2}} f_2\right\|_H^p dx\right)^{1/p} =$$

$$= \left(\int_0^1 \left\| (A+\lambda I)^{-\frac{1}{2}} (A+\lambda I)^{3/2} e^{-x(A+\lambda I)^{1/2}} f_2 \right\|_H^p dx \right)^{1/p} \leq$$

$$\leq \left\| (A+\lambda I)^{-\frac{1}{2}} \right\|_{B(H)} \left(\int_0^1 \left\| (A+\lambda I)^{3/2} e^{-x(A+\lambda I)^{1/2}} f_2 \right\|_H^p dx \right)^{1/p} \leq$$

$$\leq \frac{C}{|\lambda|^{1/2}} \left(\|f_2\|_{(H(A^2),H)_{\frac{1}{4}+\frac{1}{4p},p}} + |\lambda|^{\frac{3}{2}-\frac{1}{2p}} \|f_2\|_H \right);$$

$$\left(\int_0^1 \left\| \frac{1}{\lambda} (A+\lambda I) e^{-(1-x)(A+\lambda I)^{1/2}} f_1 \right\|_H^p dx \right)^{1/p} \leq$$

$$\leq \frac{C}{|\lambda|} \left(\|f_1\|_{(H(A^2),H)_{\frac{1}{2}+\frac{1}{4p},p}} + |\lambda|^{1-\frac{1}{2p}} \|f_1\|_H \right);$$

$$|\lambda| \left(\int_0^1 \left\| e^{-(1-x)(A+\lambda I)^{1/2}} R_{21}(\lambda) f_1 \right\|_H^p dx \right)^{1/p} \leq$$

$$\leq C |\lambda| \left\| (A+\lambda I)^{-1} \right\| \|R_{21}(\lambda)\|_{B(H)} \times$$

$$\times \left(\int_0^1 \left\| \frac{1}{\lambda} (A+\lambda I)^{1/2} e^{-(1-x)(A+\lambda I)^{1/2}} f_1 \right\|_H^p dx \right)^{1/p} \leq$$

$$\leq \frac{C}{|\lambda|} \left(\|f_1\|_{(H(A^2),H)_{\frac{1}{2}+\frac{1}{4p},p}} + |\lambda|^{1-\frac{1}{2p}} \|f_1\|_H \right).$$

The remaining integrals in the right hand side of inequality (2.12) are estimated in the same way.

By [7, lemma 5.4.2/6], for sufficiently large $|\lambda|$ and $|\arg \lambda| \leq \varphi$ from (2.10) we have

$$|\lambda|^{1/2} \|u\|_{L_p((0,1);H)} + \|u'\|_{L_p((0,1);H)} + \left\| A^{1/2} u \right\|_{L_p((0,1);H)} \leq$$

$$\leq C \left\{ |\lambda|^{1/2} \left(\int_0^1 \left\| e^{-x(A+\lambda I)^{1/2}} R_{11}(\lambda) f_1 \right\|_H^p dx \right)^{1/p} + \right.$$

$$+ \left(\int_0^1 \left\| e^{-x(A+\lambda I)^{1/2}} f_2 \right\|_H^p dx \right)^{1/p} + \left(\int_0^1 \left\| e^{-x(A+\lambda I)^{1/2}} R_{12}(\lambda) f_2 \right\|_H^p dx \right)^{1/p} +$$

$$+ \left(\int_0^1 \left\| \frac{1}{\lambda} e^{-(1-x)(A+\lambda I)^{1/2}} f_1 \right\|_H^p dx \right)^{1/p} +$$

$$+\left(\int_0^1 \left\|e^{-(1-x)(A+\lambda I)^{1/2}} R_{21}(\lambda) f_1\right\|_H^p dx\right)^{1/p}+$$

$$+\left(\int_0^1 \left\|\frac{\alpha}{\lambda}(A+\lambda I)^{1/2} e^{-(1-x)(A+\lambda I)^{1/2}} f_2\right\|_H^p dx\right)^{1/p}+$$

$$\left.+\left(\int_0^1 \left\|e^{-(1-x)(A+\lambda I)^{1/2}} R_{22}(\lambda) f_2\right\|_H^p dx\right)^{1/p}\right]+$$

$$+\left(1+\left\|A^{1/2}(A+\lambda I)^{-1/2}\right\|\right)\left[\left(\int_0^1 \left\|(A+\lambda I)^{1/2} e^{-x(A+\lambda I)^{1/2}} R_{11}(\lambda) f_1\right\|_H^p dx\right)^{1/p}+\right.$$

$$+\left(\int_0^1 \left\|(A+\lambda I)^{1/2} e^{-x(A+\lambda I)^{1/2}} f_2\right\|_H^p dx\right)^{1/p}+$$

$$+\left(\int_0^1 \left\|(A+\lambda I)^{1/2} e^{-x(A+\lambda I)^{1/2}} R_{21}(\lambda) f_2\right\|_H^p dx\right)^{1/p}+$$

$$+\left(\int_0^1 \left\|\frac{1}{\lambda}(A+\lambda I)^{1/2} e^{-(1-x)(A+\lambda I)^{1/2}} f_1\right\|_H^p dx\right)^{1/p}+$$

$$+\left(\int_0^1 \left\|(A+\lambda I)^{1/2} e^{-(1-x)(A+\lambda I)^{1/2}} R_{21}(\lambda) f_1\right\|_H^p dx\right)^{1/p}+$$

$$+\left(\int_0^1 \left\|\frac{\alpha}{\lambda}(A+\lambda I) e^{-(1-x)(A+\lambda I)^{1/2}} f_2\right\|_H^p dx\right)^{1/p}+$$

$$\left.\left.+\left(\int_0^1 \left\|(A+\lambda I)^{1/2} e^{-(1-x)(A+\lambda I)^{1/2}} R_{22}(x) f_2\right\|_H^p dx\right)^{1/p}\right]\right\}. \qquad (2.13)$$

Using [7, lemma 5.4.2/6 and theorem 5.4.2/1] , we estimate some integrals in the right hand side of inequality (2.13),

$$\left(\int_0^1 \left\|\frac{\alpha}{\lambda}(A+\lambda I) e^{-(1-x)(A+\lambda I)^{1/2}} f_2\right\|_H^p dx\right)^{1/p} \leq$$

$$\leq \frac{C}{|\lambda|}\left(\|f_2\|_{(H(A),H)_{\frac{1}{2p},p}}+|\lambda|^{1-\frac{1}{2p}}\|f_2\|_H\right);$$

$$\left(\int_0^1 \left\|(A+\lambda I)^{1/2}\, e^{-x(A+\lambda I)^{1/2}} f_2\right\|_H^p dx\right)^{1/p} \leq$$

$$\leq C\left\|(A+\lambda I)^{-\frac{1}{2}}\right\|_{B(H)} \left(\int_0^1 \left\|(A+\lambda I)\, e^{-x(A+\lambda I)^{1/2}} f_2\right\|_H^p dx\right)^{1/p} \leq$$

$$\leq \frac{C}{|\lambda|^{1/2}}\left(\|f_2\|_{(H(A),H)_{\frac{1}{2p},p}} + |\lambda|^{1-\frac{1}{2p}}\|f_2\|_H\right);$$

$$\left(\int_0^1 \left\|\frac{1}{\lambda}(A+\lambda I)^{1/2}\, e^{-(1-x)(A+\lambda I)^{1/2}} f_1\right\|_H^p dx\right)^{1/p} \leq$$

$$\leq \frac{C}{|\lambda|}\left(\|f_1\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}} + |\lambda|^{\frac{1}{2}-\frac{1}{2p}}\|f_1\|_H\right);$$

$$|\lambda|^{1/2}\left(\int_0^1 \left\|\frac{1}{\lambda} e^{-(1-x)(A+\lambda I)^{1/2}} f_1\right\|_H^p dx\right)^{1/p} \leq$$

$$\leq \frac{C}{|\lambda|^{1/2}}\left\|(A+\lambda I)^{-1/2}\right\|_{B(H)} \left(\int_0^1 \left\|(A+\lambda I)^{1/2}\, e^{-(1-x)(A+\lambda I)^{1/2}} f_1\right\|_H^p dx\right)^{1/p} \leq$$

$$\leq \frac{C}{|\lambda|}\left(\|f_1\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}} + |\lambda|^{\frac{1}{2}-\frac{1}{2p}}\|f_1\|_H\right).$$

The remaining integrals in the right hand side of (2.13) are estimated in the similar way.

3. Inhomogeneous equations

Now, let's consider a boundary value problem for an inhomogeneous equation with a parameter

$$L(\lambda, D)\, u := \lambda u(x) - u''(x) + Au(x) = f(x), \quad x \in [0,1], \tag{3.1}$$

$$L_1(\lambda)\, u := \alpha u'(0) + \lambda u(1) = f_1,$$

$$L_2 u := u(0) = f_2. \tag{3.2}$$

Theorem 2. *Let the following conditions be fulfilled:*

(1) *A is a linear closed densely defined operator in the separable Hilbert space H and*

$$\|R(\lambda, A)\| \leq C(1+|\lambda|)^{-1}, \ \ |\arg\lambda| \geq \pi - \varphi,$$

for each fixed $0 \leq \varphi < \pi$, and A is invertible;

(2) *$\alpha \neq$ is any fixed complex number.*

Then for $f \in L_p\left((0,1);H\left(A^{1/2}\right)\right)$, $f_k \in \left(H\left(A^2\right),H\right)_{\frac{3}{4}-\frac{k}{4}+\frac{1}{4p},p}$ *and sufficiently large* $|\lambda|$ *from the angle* $|\arg\lambda| \le \varphi$ *the problem* $(3.1)-(3.2)$ *has a unique solution from* $W_p^2\left((0,1);H(A),H\right)$ *and the following non-coercive estimate holds for its solution*

$$|\lambda|^{1/2}\|u\|_{L_p((0,1);H)} + \|u'\|_{L_p((0,1);H)} + \left\|A^{1/2}u\right\|_{L_p((0,1);H)} \le$$

$$\le C\left[\|f\|_{L_p((0,1);H)} + \sum_{k=1}^{2}|\lambda|^{-1/k}\left(\|f_k\|_{(H(A),H)_{1-\frac{k}{2}+\frac{1}{2p},p}} + |\lambda|^{\frac{k}{2}-\frac{1}{2p}}\|f_k\|_H\right)\right]. \quad (3.3)$$

Proof. The uniqueness follows from theorem 1. Define $\widetilde{f}(x) := f(x)$, if $x \in [0,1]$ and $\widetilde{f}(x) = 0$, if $x \notin [0,1]$. We represent the solution of problem $(3.1)-(3.2)$ in the form of the sum $u(x) = u_1(x) + u_2(x)$, where $u_1(x)$ is the contraction on $[0,1]$ of the solution of equation

$$L(\lambda, D)\,\widetilde{u}_1(x) = \widetilde{f}(x), \quad x \in \mathbb{R} = (-\infty, +\infty) \quad (3.4)$$

and $u_2(x)$ is the solution of the problem

$$L(\lambda, D)\,u_2 = 0, \quad L_k u_2 = f_k - L_k u_1, \quad k = 1,2. \quad (3.5)$$

It is proved [7, theorem 5.5.3] that for $|\arg\lambda| \le \varphi$ for $\widetilde{u}_1(x)$ it holds the estimate

$$\|\widetilde{u}_1\|_{W_p^2\left(\mathbb{R};\mathbb{H}\left(A^{3/2}\right),H\left(A^{1/2}\right)\right)} \le C\left\|\widetilde{f}\right\|_{L_p\left(\mathbb{R};\mathbb{H}\left(A^{1/2}\right)\right)}.$$

Hence, it follows that

$$u_1 \in W_p^2\left((0,1);H\left(A^{3/2}\right),H\left(A^{1/2}\right)\right) \subset W_p^2\left((0,1);H(A),H\right).$$

By [7, theorem 1.7.7/1] we have

$$u_1'(x_0) \in \left(H\left(A^{3/2}\right),H\left(A^{1/2}\right)\right)_{\frac{1}{2}+\frac{1}{2p},p},$$

$$u_1(x_0) \in \left(H\left(A^{3/2}\right),H\left(A^{1/2}\right)\right)_{\frac{1}{2p},p} \quad \forall x_0 \in [0,1].$$

According to [7, lemma 1.7.3/1, 1.7.3/6 and 1.7.3/5], for $k = 0,1$ we have

$$\left(H\left(A^{3/2}\right),H\left(A^{1/2}\right)\right)_{\frac{1+kp}{2p},p} = \left(H\left(A^{1/2}\right),H\left(A^{3/2}\right)\right)_{1-\frac{1+kp}{2p},p} =$$

$$= \left(H,H\left(A^{3/2}\right)\right)_{1-\frac{1+kp}{3p},p} = \left(H,H\left(A^2\right)\right)_{\frac{3}{4}-\frac{1+kp}{4p},p} = \left(H\left(A^2\right),H\right)_{\frac{1}{4}+\frac{1+kp}{4p},p}.$$

Consequently,

$$u_1'(x_0) \in \left(H\left(A^2\right),H\right)_{\frac{1}{2}+\frac{1}{4p},p}, \quad u_1(x_0) \in \left(H\left(A^2\right),H\right)_{\frac{1}{4}+\frac{1}{4p},p}.$$

Since $\left(H\left(A^2\right),H\right)_{\frac{1}{4}+\frac{1}{4p},p}\subset\left(H\left(A^2\right),H\right)_{\frac{1}{2}+\frac{1}{4p},p}$, therefore

$$L_1(\lambda)u_1\in\left(H\left(A^2\right),H\right)_{\frac{1}{2}+\frac{1}{4p},p},\quad L_2u_1\in\left(H\left(A^2\right),H\right)_{\frac{1}{4}+\frac{1}{4p},p}.$$

By [7,theorem 5.4.4], for $|\arg\lambda|\leq\varphi, |\lambda|\to\infty$ and for $f\in L_p((0,1);H)$ the problem (3.4) has a unique solution from $W_p^2((0,1);H(A),H)$ and it holds the following estimate

$$|\lambda|\,\|u_1\|_{L_p((0,1);H)}+\|u_1\|_{W_p^2((0,1);H(A),H)}\leq C\,\|f\|_{L_p((0,1);H)}. \tag{3.6}$$

Using theorem 1, for sufficiently large $|\lambda|$ from the angle $|\arg\lambda|\leq\varphi$ for the solution of problem (3.5) we have

$$|\lambda|^{1/2}\|u_2\|_{L_p((0,1);H)}+\left\|u_2'\right\|_{L_p((0,1);H)}+\left\|A^{1/2}u_2\right\|_{L_p((0,1);H)}\leq$$

$$\leq\frac{C}{|\lambda|}\left(\|f_1-L_1(\lambda)u_1\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}}+|\lambda|^{\frac{1}{2}-\frac{1}{2p}}\|f_1-L_1(\lambda)u_1\|_H\right)+$$

$$+\frac{C}{|\lambda|^{1/2}}\left(\|f_2-L_2u_1\|_{(H(A),H)_{\frac{1}{2p},p}}+|\lambda|^{1-\frac{1}{2p}}\|f_2-L_2u_1\|_H\right)\leq$$

$$\leq C\left(\frac{1}{|\lambda|}\|f_1\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}}+\frac{1}{|\lambda|^{1/2}}\|f_2\|_{(H(A),H)_{\frac{1}{2p},p}}+\right.$$

$$+|\lambda|^{-\frac{1}{2}-\frac{1}{2p}}\|f_1\|_H+|\lambda|^{\frac{1}{2}-\frac{1}{2p}}\|f_2\|_H+\frac{1}{|\lambda|}\left\|u_1'(0)\right\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}}+$$

$$+\|u_1(1)\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}}+\frac{1}{|\lambda|^{1/2}}\|u_1(0)\|_{(H(A),H)_{\frac{1}{2p},p}}+$$

$$\left.+|\lambda|^{-\frac{1}{2}-\frac{1}{2p}}\left\|u_1'(0)\right\|_H+|\lambda|^{\frac{1}{2}-\frac{1}{2p}}\|u_1(1)\|_H+|\lambda|^{\frac{1}{2}-\frac{1}{2p}}\|u_1(0)\|_H\right). \tag{3.7}$$

By [7. theorem 1.7.7./1] and estimate (3.6) we have

$$\frac{1}{|\lambda|}\left\|u_1'(0)\right\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}}\leq\frac{1}{|\lambda|}\|u_1\|_{W_p^2((0,1);H(A),H)}\leq\frac{C}{|\lambda|}\|f\|_{L_p((0,1);H)};$$

$$\|u_1(1)\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}}\leq C\,\|u_1(1)\|_{(H(A),H)_{\frac{1}{2p},p}}\leq$$

$$\leq C\,\|u_1\|_{W_p^2((0,1);H(A),H)}\leq C\,\|f\|_{L_p((0,1);H)};$$

$$\frac{1}{|\lambda|^{1/2}}\|u_1(0)\|_{(H(A),H)_{\frac{1}{2p},p}}\leq\frac{C}{|\lambda|^{1/2}}\|u_1\|_{W_p^2((0,1);H(A),H)}\leq\frac{C}{|\lambda|^{1/2}}\|f\|_{L_p((0,1);H)}.$$

Summing up these inequalities, for sufficiently large $|\lambda|$ from the angle $|\arg\lambda|\leq\varphi$ we get

$$\frac{1}{|\lambda|}\left\|u_1'(0)\right\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}}+\|u_1(1)\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}}+$$

$$+\frac{1}{|\lambda|^{1/2}}\|u_1(0)\|_{(H(A),H)_{\frac{1}{2p},p}}\leq C\,\|f\|_{L_p((0,1);H)}.$$

By [7, theorem 1.7.7./2] for $\mu \in \mathbb{C},\ u \in W_p^2((0,1);H)$

$$|\mu|^{2-s} \left\| u^{(s)}(x_0) \right\|_H \le C \left(|\mu|^{\frac{1}{p}} \|u\|_{W_p^2((0,1);H)} + |\mu|^{2+\frac{1}{p}} \|u\|_{L_p((0,1);H)} \right),\ s=0,1. \quad (3.9)$$

Dividing (3.9) into $|\mu|^{1/p}$ and denoting $\lambda = \mu^2$ for $\lambda \in \mathbb{C}$, $|\lambda| \to \infty$, $u \in W_p^2((0,1);H)$, we have

$$|\lambda|^{1-\frac{s}{2}-\frac{1}{2p}} \left\| u^{(s)}(x_0) \right\|_H \le C \left(\|u\|_{W_p^2((0,1);H)} + |\lambda| \|u\|_{L_p((0,1);H)} \right) \le$$
$$\le C \|f\|_{L_p((0,1);H)}, \quad s=0,1. \quad (3.10)$$

From (3.10) and (3.6) for $|\arg\lambda| \le \varphi$, $|\lambda| \to \infty$ we have

$$|\lambda|^{1-\frac{s}{2}-\frac{1}{2p}} \left\| u_1^{(s)}(x_0) \right\|_H \le C \left(\|u_1\|_{W_p^2((0,1);H(A),H)} + |\lambda| \|u_1\|_{L_p((0,1);H)} \right) \le$$
$$\le C \|f\|_{L_p((0,1);H)}, \quad s=0,1. \quad (3.11)$$

By (3.11) for $|\arg\lambda| \le \varphi$, $|\lambda| \to \infty$ we have

$$|\lambda|^{-\frac{1}{2}-\frac{1}{2p}} \left\| u_1'(0) \right\|_H = |\lambda|^{-1} |\lambda|^{-\frac{1}{2}-\frac{1}{2p}} \left\| u_1'(0) \right\|_H \le C |\lambda|^{-1} \|f\|_{L_p((0,1);H)};$$

$$|\lambda|^{\frac{1}{2}-\frac{1}{2p}} \|u_1(x_0)\|_H = |\lambda|^{-1/2} |\lambda|^{1-\frac{1}{2p}} \|u_1(x_0)\|_H \le C |\lambda|^{-1/2} \|f\|_{L_p((0,1);H)}.$$

Summing up these inequalities, for $|\arg\lambda| \le \varphi$, $|\lambda| \to \infty$ we get

$$|\lambda|^{-\frac{1}{2}-\frac{1}{2p}} \left\| u_1'(0) \right\|_H + |\lambda|^{\frac{1}{2}-\frac{1}{2p}} \|u_1(1)\|_H + |\lambda|^{\frac{1}{2}-\frac{1}{2p}} \|u_1(0)\| \le$$
$$\le \frac{C}{|\lambda|} \|f\|_{L_p((0,1);H)} + \frac{C}{|\lambda|^{1/2}} \|f\|_{L_p((0,1);H)} \le \frac{C}{|\lambda|^{1/2}} \|f\|_{L_p((0,1);H)}. \quad (3.12)$$

Considering estimates (3.8) and (3.12) in (3.7), for $|\arg\lambda| \le \varphi$, $|\lambda| \to \infty$ we have

$$|\lambda|^{\frac{1}{2}} \|u_2\|_{L_p((0,1);H)} + \left\| u_2' \right\|_{L_p((0,1);H)} + \left\| A^{1/2} u \right\|_{L_p((0,1);H)} \le$$
$$\le C \left(\|f\|_{L_p((0,1);H)} + \sum_{k=1}^{2} |\lambda|^{-1/k} \left(\|f_k\|_{(H(A),H)_{1-\frac{k}{2}+\frac{1}{2p},p}} + |\lambda|^{\frac{k}{2}-\frac{1}{2p}} \|f_k\|_H \right) \right). \quad (3.13)$$

Show that for sufficiently large $|\lambda|$ from the angle $|\arg\lambda| \le \varphi$ it holds the estimate

$$|\lambda|^{1/2} \|u_1\|_{L_p((0,1);H)} + \left\| u_1' \right\|_{L_p((0,1);H)} + \left\| A^{1/2} u_1 \right\|_{L_p((0,1);H)} \le C \|f\|_{L_p((0,1);H)}. \quad (3.14)$$

By [9, theorem 5.1.2/1.7], the operator $u(x) \to u'(x)$ from $W_p^2((0,1);H(A),H)$ in $W_p^2\left((0,1);H\left(A^{1/2}\right),H\right)$ is bounded, i.e. for any $u \in W_p^2((0,1);H(A),H)$ it holds the estimate

$$\left\| u' \right\|_{W_p^1\left((0,1);H\left(A^{1/2}\right),H\right)} \le C \|u\|_{W_p^2((0,1);H(A),H)}, \quad \exists C > 0.$$

Hence, for any $u_1 \in W_p^2\left((0,1);H(A),H\right)$ it holds the inequality

$$\left\|u_1'\right\|_{L_p\left((0,1);H\left(A^{1/2}\right)\right)} \le C\left\|u_1\right\|_{W_p^2((0,1);H(A),H)}. \tag{3.15}$$

On the other hand, for any $u_1 \in W_p^2\left((0,1);H(A),H\right)$

$$\left\|A^{1/2}u_1\right\|_{L_p((0,1);H)} = \left\|A^{-1/2}Au_1\right\|_{L_p((0,1);H)} \le \left\|A^{-1/2}\right\|_{B(H)}\left\|Au_1\right\|_{L_p((0,1);H)} \le$$

$$\le C\left\|u_1\right\|_{L_p((0,1);H(A))} \le C\left\|u_1\right\|_{W_p^2((0,1);H(A),H)}. \tag{3.16}$$

Obviously, for sufficiently large $|\lambda|$ from the angle $|\arg\lambda| \le \varphi$

$$|\lambda|^{1/2}\left\|u_1\right\|_{L_p((0,1);H)} \le |\lambda|\left\|u_1\right\|_{L_p((0,1);H)} \tag{3.17}$$

Summing up the estimates (3.15) – (3.17) and considering estimate (3.6) for sufficiently large $|\lambda|$ from the angle $|\arg\lambda| \le \varphi$ we get estimate (3.14).

Further, (3.3) follows from (3.13) and (3.14).

Theorem 2 is proved.

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