Transactions of NAS of Azerbaijan, 2010, vol. XXX, No 4, pp. 75-88.

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ESTIMATIONS OF THE SMOOTHNESS MODULES OF CONVOLUTION OF TWO PERIODIC FUNCTIONS BY MEANS OF THEIR BEST APPROXIMATIONS IN $L_p(\mathbb{T})$ (THE CASE OF DIFFERENT METRICS)

Abstract

In the paper the upper estimations of smoothess modules $\omega_k\left(h^{(s)};\delta\right)_\gamma$ *of derivative* $h^{(s)}$ *of order* $s(h^{(0)}\equiv h)$ *of the convolution* $h=f*g$ *of two* 2π *periodic functions* $f\in L_p(\mathbb{T})$ *and* $g\in L_q(\mathbb{T})$ *are obtained by means of expression containing the product* $E_{n-1}(f)_p\,E_{n-1}(g)_q$ *of the best approximations of these functions in the metrics of* $L_p(\mathbb{T})$ *and* $L_q(\mathbb{T})$ *respectively, where* $k\in\mathbb{N}$, $s\in\mathbb{Z}_+$, $p,q\in[1,\infty)$, $1/r=1/p+1/q-1>0$, $\gamma\in(r,\infty]$, $\mathbb{T}=(-\pi,\pi]$. *It is proved in the case* $p,q\in(1,\infty)$ *that the obtained estimations are exact in the sense of order on classes of convolutions with given majorants of sequences of the best approximations of* f *and* g *under some regularity of these majorants.*

In what follows we use the following notation.

- $L_p(\mathbb{T})$, $1\le p<\infty$, is the space of all measurable 2π periodic functions $f:\mathbb{R}\to\mathbb{C}$ with finite L_p-norm $\|f\|_p=\left((2\pi)^{-1}\int_{\mathbb{T}}|f(x)|^p\,dx\right)^{1/p}<\infty$.
- $C(\mathbb{T})\equiv L_\infty(\mathbb{T})$ is the space of all continuous 2π periodic functions with uniform norm $\|f\|_\infty\equiv\max\{|f(x)|:x\in\mathbb{T}\}$.
- $W_p^s(\mathbb{T})$, $s\in\mathbb{N}$, $p\in[1,\infty)$, is the class of functions $f\in L_p(\mathbb{T})$ having an absolutely continuous derivative of order $s-1$ and $f^{(s)}\in L_p(\mathbb{T})$.
- $C^s(\mathbb{T})\equiv W_\infty^s(\mathbb{T})$, $s\in\mathbb{N}$, is the class of functions $f\in C(\mathbb{T})$ having an ordinary derivative $f^{(s)}\in C(\mathbb{T})$.
- $E_n(f)_p$ is the best approximation of a function f in the metric of $L_p(\mathbb{T})$ by the trigonometric polynomials of order $\le n\in\mathbb{Z}_+$.
- $S_n(f;\cdot)$ is the partial sum of order $n\in\mathbb{Z}_+$ of the Fourier-Lebesgue series of a function $f\in L_1(\mathbb{T}):S_n(f;x)=\sum_{|\nu|=0}^{n}c_\nu(f)\,e^{i\nu x}$, $x\in\mathbb{T}$.
- $\omega_k(f;\delta)_p$ is the smoothness module of order k of a function $f\in L_p(\mathbb{T})$:

  $\omega_k(f;\delta)_p=\sup\left\{\left\|\Delta_t^k f\right\|_p:t\in\mathbb{R},\ |t|\le\delta\right\}$, $k\in\mathbb{N}$, $\delta\in[0,\infty)$, where $\Delta_t^k f(x)=\sum_{\nu=0}^{k}(-1)^{k-\nu}\binom{k}{\nu}f\ (x+\nu t)$, $x\in\mathbb{R}$.
- M_0 is the class of all sequences $\lambda=\{\lambda_n\}_{n=1}^\infty\subset\mathbb{R}$ such that $0<\lambda_n\downarrow 0\,(n\uparrow\infty)$.
- $E_p[\lambda]=\{f\in L_p(\mathbb{T}):E_{n-1}(f)_p\le\lambda_n,\ n\in\mathbb{N}\}$ for $p\in[1,\infty]$ and $\lambda\in M_0$.

The convolution $h = f*g$ of $f \in L_1(\mathbb{T})$ and $g \in L_1(\mathbb{T})$ is defined by the formula: $h(x) = (f*g)(x) = (1/2\pi)\int_{\mathbb{T}} f(x-y)\,g(y)\,dy$; it is known (see f.e. [1], v.1, § 2.1, pp.64-65, [2], v.1, § 3.1, pp.65-66) that the function h is defined almost everywhere, 2π periodic, measurable and $\|h\|_1 \le \|f\|_1 \|g\|_1$ (whence it follows in particular that $h = f*g \in L_1(\mathbb{T})$). The last statement is a particular case of the following result known as the W.Young's inequality (see, f.e. [1], v.1, Theorem (1.15), pp.67-68; [2], v.2, Theorem 13.6.1, pp.176-177; [2], v.1, Theorem 3.1.4, p.70, Theorem 3.1.6, p.72). Given $p \in [1,\infty]$, let $p' = p/(p-1)$ be the exponent conjugate to p. As usual, we assume that $p' = 1$ for $p = \infty$ and $p' = \infty$ for $p = 1$. If $p, q \in [1,\infty]$ and $1/r = 1/p + 1/q - 1 \ge 0$, then $r = pq/(p+q-pq)$ and $r \in [1,\infty)$ for $1/r > 0$ and $r = \infty$ for $1/r = 0$ (in this case $1/p + 1/q = 1$, so that $q = p'$).

Theorem A. *Let $p, q \in [1,\infty]$, $f \in L_p(\mathbb{T})$ and $g \in L_q(\mathbb{T})$, $h = f*g$, $1/r = 1/p + 1/q - 1 \ge 0$. Then*

- *If $1/r > 0$ then h belongs to $L_r(\mathbb{T})$ and $\|h\|_r \le \|f\|_p \|g\|_q$.*
- *If $1/r = 0$ then h belongs to $C(\mathbb{T}) \equiv L_\infty(\mathbb{T})$ and $\|h\|_\infty \le \|f\|_p \cdot \|g\|_{p'}$.*

Recall that the Fourier coefficients $c_n(h)$ of $h = f*g$ of two arbitrary functions $f \in L_1(\mathbb{T})$ and $g \in L_1(\mathbb{T})$ are calculated by the formula (see [1], v.1, Theorem (1.5), p.64; [2], v.1, p.66, formula (3.1.5)) $c_n(h) = c_n(f*g) = c_n(f) \cdot c_n(g)$ for every $n \in \mathbb{Z}$.

We use also the following obvious inequalities (see f.e. [3], Lemma 1, pp. 18-19): let $f \in L_p(\mathbb{T})$, $p \in [1,\infty]$, $k \in \mathbb{N}$ and $f = Re\, f + i Im\, f$; then

(i) $max\{E_n(Re\, f)_p, E_n(Im\, f)_p\} \le E_n(f)_p \le$
$$\le E_n(Re\, f)_p + E_n(Im\, f)_p \le 2E_n(f)_p,\ n \in \mathbb{Z}_+.$$

(ii) $max\{\omega_k(Re\, f;\delta)_p, \omega_k(Im\, f;\delta)_p\} \le \omega_k(f;\delta)_p \le$
$$\le \omega_k(Re\, f;\delta)_p + \omega_k(Im\, f;\delta)_p \le 2\omega_k(f;\delta)_p,\quad \delta \in [0,\infty).$$

The following statement be so called the inverse theorem of the approximation theory of 2π periodic functions in different metrics of $L_p(\mathbb{T})$.

Theorem B. *Let $1 \le p < q \le \infty$, $f \in L_p(\mathbb{T})$, $\tau = \tau(q) = q$ for $q < \infty$ and $\tau(\infty) = 1$, $s \in \mathbb{Z}_+$, $k \in \mathbb{N}$, $\sigma = s + 1/p - 1/q$ and*

$$\sum_{n=1}^{\infty} n^{\tau\sigma-1} E_{n-1}^{\tau}(f)_p < \infty. \tag{1}$$

Then $f \in W_q^s(\mathbb{T})$ (more precisely, f almost everywhere equal to some function from $W_q^s(\mathbb{T})$ for $q < \infty$ and $C^s(\mathbb{T})$ for $q = \infty$) and the following estimation holds:

$$\omega_k\left(f^{(s)};\pi/n\right)_q \le C_1(k,s,p,q)\left\{\left(\sum_{\nu=n+1}^{\infty} \nu^{\tau\sigma-1} E_{\nu-1}^{\tau}(f)_p\right)^{1/\tau} + \right.$$
$$\left. + n^{-k}\left(\sum_{\nu=1}^{n} \nu^{\tau(k+\sigma)-1} E_{\nu-1}^{\tau}(f)_p\right)^{1/\tau}\right\},\qquad n \in \mathbb{N}, \tag{2}$$

where $C_1(k,s,p,q)$ is a positive constant depending only on parameters k, s, p and q.

Theorem B was proved by A.A.Konyushkov [4], Theorem 2, pp.56-57, in the case $s=0, q=\infty$, and by A.F.Timan [5], Theorem 6.4.1, p.378, in the case $s\in\mathbb{Z}_+, q=\infty$ (more precisely, in these cited works was given weak version of formulated theorem with exponent $\tau=\tau(q)=1<q$ for all $q\in(1,\infty)$).

The implication (1)$\Longrightarrow f\in W_q^s(\mathbb{T})$ was proved by P.L.Ul'yanov [6], Theorem 4, p.121, inequality (4.2), for $s=0, q<\infty$ (see also [7], Remark 6, pp. 671-672, inequalities (3.6$'$); [8], Theorem 4, p.1045, inequality (8); [9], pp. 1251-1253; [10], Theorem A, pp. 62-65) and by M.F.Timan [10], Theorem 8, p.73, for $s\in\mathbb{Z}_+, q<\infty$.

The inequality (2) was proved by the author [11], Proposition 2.7, pp.27-41, in the case $s\in\mathbb{Z}_+$ and $1\le p<q\le 2, s\in\mathbb{Z}_+$ and $p=1, 2<q<\infty, s\in\mathbb{Z}_+$ and $1\le p<q=\infty$; [12], Proposition 1, (2), p.49-50 (see also [13], Proposition 1, (3), pp. 4-5) in the case $s\in\mathbb{Z}_+, q\le 2$; [14], Theorem 1, pp. 57-61 (see also [15], Proposition 1, pp. 3-9) in the case $s\in\mathbb{Z}_+, 2<q<\infty$.

We note also that in the case $s=0, 1<p<q<\infty$ the inequality (2) was formulated without proof by M.B.Sikhov [16], Theorem 1, p. 46, inequality (2).

The estimation (2) is exact in the sense of order on the class $E_p[\lambda]$ for all values $1\le p<q\le\infty$, namely

$$\sup\left\{\omega_k\left(f^{(s)};\pi/n\right)_q : f\in E_p[\lambda]\right\}\asymp$$

$$\asymp\left(\sum_{\nu=n+1}^{\infty}\nu^{\tau\sigma-1}\lambda_\nu^\tau\right)^{1/\tau}+n^{-k}\left(\sum_{\nu=1}^{n}\nu^{\tau(k+\sigma)-1}\lambda_\nu^\tau\right)^{1/\tau},\qquad n\in\mathbb{N}. \tag{3}$$

under condition that $\sum\limits_{n=1}^{\infty}n^{\tau\sigma-1}\lambda_n^\tau<\infty\iff E_p[\lambda]\subset W_q^s(\mathbb{T})$. The sufficiency of denote condition follows from implication (1) $\Longrightarrow f\in W_q^s(\mathbb{T})$ (see Theorem B). The necessity in the case $s=0$ was proved by N.T.Temirqaliev [17], Theorem 2, pp. 840-841, for $p=1, q<\infty$, V.I.Kolyada [18], Theorems 3 and 4, pp. 212-215, for $1\le p<q\le\infty$, M.F.Timan [19], Theorem 1, pp. 76-79, for $1\le p<q<\infty$ (see also [9], p.1253; [10], Theorem 6, pp. 70-72), author [11], p.135, Theorem 3, point (3.1), in the case $s\in\mathbb{Z}_+, 1\le p<q\le\infty$, [12], Remark after theorem on the page 49 (see also [13], point (1) of theorem on the page 3), the case $s\in\mathbb{Z}_+, 1\le p<q\le 2$, [14], Theorem 2, p.61 (see also [15], the point (1) of theorem on the page 3), the case $r\in\mathbb{Z}_+, 2<q<\infty$.

The upper estimation in (3) immediately follows from inequality (2). The lower estimation in (3) is realized by means of individual functions in $E_p[\lambda]$; more precisely, for every $p\in[1,\infty)$ and for arbitrary $\lambda\in M_0$ there exists a function $f_0(\cdot;p;\lambda)\in L_p(\mathbb{T})$ with $E_{n-1}(f_0)\le\lambda_n$, $n\in\mathbb{N}$, such that

(i) $f_0\in W_q^s(\mathbb{T})\Leftrightarrow\sum\limits_{n=1}^{\infty}n^{\tau\sigma-1}\lambda_n^\tau<\infty$;

(ii) if the series in (i) converge, then $\omega_k\left(f_0^{(s)};\pi/n\right)_q\ge$

$$\ge C_2(k,s,p,q)\left\{\left(\sum_{\nu=n+1}^{\infty}\nu^{\tau\sigma-1}\lambda_\nu^\tau\right)^{1/\tau}+n^{-k}\left(\sum_{\nu=1}^{n}\nu^{\tau(k+\sigma)-1}\lambda_\nu^\tau\right)^{1/\tau}\right\},\qquad n\in\mathbb{N}.$$

The statement (i) and estimation (ii) was proved by the author [11], Lemma 3.13, p.98, for $s\in\mathbb{Z}_+, 1\le p<q\le 2$, Lemma 3.14, p.101, for $s\in\mathbb{Z}_+, 1\le p<q<\infty$ and $q>2$; [12]; Lemma 2, pp. 54-56 (see also [13], Lemma 3, pp.7-9), for

$s \in \mathbb{Z}_+, 1 \le p < q \le 2$; [14], Lemma 3, pp.62-63 (see also [15], Lemma 1, pp. 12-14), for $s \in \mathbb{Z}_+, 1 \le p < q < \infty$ and $q > 2$; [20], Lemma 5, pp.57-60, for $1 \le p < q = \infty$.

Theorem 1. *Let* $p, q \in [1,\infty)$, $1/r = 1/p + 1/q - 1 > 0$, $\gamma \in (r, \infty]$, $k \in \mathbb{N}, s \in \mathbb{Z}_+, \sigma = s + 1/r - 1/\gamma, \tau = \tau(\gamma) = \gamma$ for $\gamma < \infty$ and $\tau(\infty) = 1$, $f \in L_p(\mathbb{T})$, $g \in L_q(\mathbb{T})$, $h = f * g$ and

$$\sum_{n=1}^{\infty} n^{\tau\sigma-1} E_{n-1}^{\tau}(f)_p E_{n-1}^{\tau}(g)_q < \infty. \tag{4}$$

Then $h \in W_\gamma^s(\mathbb{T})$ and the following estimation holds:

$$\omega_k\left(h^{(s)}; \pi/n\right)_\gamma \le C_3(k,s,r,\gamma)\left\{\left(\sum_{\nu=n+1}^{\infty} \nu^{\tau\sigma-1} E_{\nu-1}^{\tau}(f)_p E_{\nu-1}^{\tau}(g)_q\right)^{1/\tau} + \right.$$

$$\left. + n^{-k}\left(\sum_{\nu=1}^{n} \nu^{\tau(k+\sigma)-1} E_{\nu-1}^{\tau}(f)_p E_{\nu-1}^{\tau}(g)_q\right)^{1/\tau}\right\}, \ n \in \mathbb{N}. \tag{5}$$

Proof. Since $f \in L_p(\mathbb{T})$ and $g \in L_q(\mathbb{T})$ we have that $h \in L_r(\mathbb{T})$ for $1/r > 0$ ($\Longrightarrow r \in [1,\infty)$) by Theorem A. We need the following estimation (see [21], the inequality (2) in the proof of Theorem 1, p.41)

$$E_{n-1}(f * g)_r \le E_{n-1}(f)_p \cdot E_{n-1}(g)_q, \ \ n \in \mathbb{N}, \ \ r \in [1,\infty]. \tag{6}$$

Taking into account (4) and by inequality (6) we have that

$$\sum_{n=1}^{\infty} n^{\tau\sigma-1} E_{n-1}^{\tau}(h)_r \le \sum_{n=1}^{\infty} n^{\tau\sigma-1} E_{n-1}^{\tau}(f)_p E_{n-1}^{\tau}(g)_q < \infty,$$

whence it follows that (1) hold for h. Therefore $h \in W_\gamma^s(\mathbb{T})$ by Theorem B and applying the inequalities (2) for h and (6), we obtain (5). Theorem 1 is proved.

For further exposition we need preliminary lemmas.

Lemma 1. *Let* $1 < \gamma \le 2, s \in \mathbb{Z}_+, k \in \mathbb{N}, \psi \in W_\gamma^s(\mathbb{T})$ and have the Fourier series $\psi(x) \sim \sum_{n\in\mathbb{Z}} c_n(\psi) e^{inx}$, $x \in \mathbb{T}$. Then

(i) $n^{-k}\left(\sum\limits_{\nu=1}^{n} \nu^{\gamma k+\gamma-2} |c_\nu(\psi)|^\gamma\right)^{1/\gamma} \le C_4(k,\gamma)\omega_k(\psi;\pi/n)_\gamma, n \in \mathbb{N}$;

(ii) $\left(\sum\limits_{n=1}^{\infty} n^{\gamma s+\gamma-2} |c_n(\psi)|^\gamma\right)^{1/\gamma} \le C_5(\gamma)\left\|\psi^{(s)}\right\|_\gamma$;

(iii) $\left(\sum\limits_{\nu=n+1}^{\infty} \nu^{\gamma s+\gamma-2} |c_\nu(\psi)|^\gamma\right)^{1/\gamma} \le C_6(k,\gamma)\omega_k\left(\psi^{(s)};\pi/n\right)_\gamma, \quad n \in \mathbb{N}$.

Lemma 1 was proved in [3], Lemma 2 (point (i)) and in [22], Lemma 1 (points (ii) and (iii)).

Lemma 2. *Let* $s \in \mathbb{Z}_+, k \in \mathbb{N}, \psi \in C^s(\mathbb{T})$ and have the Fourier series $\psi(x) \sim \sum_{n=1}^{\infty} c_n(\psi) e^{inx}$, $x \in \mathbb{T}$, with $c_n(\psi) > 0$ for every $n \in \mathbb{N}$. Then

(i) $n^{-æ}\sum\limits_{\nu=1}^{n} \nu^{æ} c_\nu(\psi) \le 2^{-k}\omega_k(\operatorname{Re}\psi;\pi/n)_\infty, \quad n \in \mathbb{N}$,

where $æ = k + (1-(-1)^k)/2 = \{k$ for even $k; k+1$ for odd $k\}$.

(ii) $n^{-æ}\sum\limits_{\nu=1}^{n} \nu^{æ} c_\nu(\psi) \le 2^{-(k+1)}\pi\omega_k(\operatorname{Im}\psi;\pi/n)_\infty, \quad n \in \mathbb{N}$,

where $æ = k + (1+(-1)^k)/2 = \{k+1$ for even $k; k$ for odd $k\}$.

(iii) $\sum_{n=1}^{\infty} n^s c_n(\psi) \leq \begin{cases} \left\|\operatorname{Re}\psi^{(s)}\right\|_\infty & \text{for } s=0,2,4,\cdots; \\ \left\|\operatorname{Im}\psi^{(s)}\right\|_\infty & \text{for } s=1,3,\cdots. \end{cases}$

(iv) $\sum_{\nu=n+1}^{\infty} \nu^s c_\nu(\psi) \leq 2^{k+2}C_7(k) \begin{cases} \omega_k\left(\operatorname{Re}\psi^{(s)};\pi/n\right)_\infty & \text{for } s=0,2,4,\cdots; \\ \omega_k\left(\operatorname{Im}\psi^{(s)};\pi/n\right)_\infty & \text{for } s=1,3,\cdots. \end{cases}$

Lemma 2 was proved in [3], Lemma 4 (points (i) and (ii)) and in [22], Lemma 3 (points (iii) and (iv)).

Lemma 3. *Let $\gamma \in (1,\infty), \psi \in L_\gamma(\mathbb{T})$ and have the Fourier series $\psi(x) \sim (1/2)a_0(\psi) + \sum_{n=1}^{\infty}(a_n(\psi)\cos nx + b_n(\psi)\sin nx), x \in \mathbb{T},$ where $a_0(\psi) \geq 0, a_n(\psi) \geq 0, b_n(\psi) \geq 0$ for every $n \in \mathbb{N}$. Then*

(i) $\sum_{\nu=n}^{2n}(a_\nu(\psi)+b_\nu(\psi)) \leq C_8(\gamma)n^{1/\gamma}E_n(\psi)_\gamma, \quad n \in \mathbb{N};$

Furthermore, if $a_n(\psi)\downarrow, b_n(\psi)\downarrow$ for $n\uparrow$, then

(ii) $(a_{2n}(\psi)+b_{2n}(\psi))\,n^{1-1/\gamma} \leq C_8(\gamma)E_n(\psi)_\gamma, \quad n \in \mathbb{N};$

(iii) $\left(\sum_{\nu=n+1}^{\infty}\nu^{\gamma-2}(a_\nu(\psi)+b_\nu(\psi))^\gamma\right)^{1/\gamma} \leq C_9(\gamma)E_{[(n+1)/2]}(\psi)_\gamma, \quad n \in \mathbb{N}.$

Lemma 3 was proved by A.A.Konyushkov [23], Theorem 5, inequalities (17) and (19), p.73; Theorem 6, inequality (20), p.74. In the inequality (iii) for $2<\gamma<\infty$, in general, dos not exchange $E_{[(n+1)/2]}(\psi)_\gamma$ by means $E_n(\psi)_\gamma$ (see [23], p.75); in the case $1<\gamma\leq 2$ it is possible without denote assumption $a_n(\psi)\downarrow, b_n(\psi)\downarrow$ $(n\uparrow)$ (see the proof (iii) of Lemma 1 for $s=0$).

Lemma 4. *Let $\gamma \in (1,\infty), l,k \in \mathbb{N}, s \in \mathbb{Z}_+, \psi \in W_\gamma^s(\mathbb{T}), \eta = \max\{2,\gamma\}$. Then $(n \in \mathbb{N})$*

(i) $n^{-k}\left(\sum_{\nu=1}^{n}\nu^{\eta(k+s)-1}\omega_l^\eta(\psi;\pi/\nu)_\gamma\right)^{1/\eta} \leq C_{10}(l,k+s,\gamma)\,\pi^s\omega_k\left(\psi^{(s)};\pi/n\right)_\gamma$

$(s \in \mathbb{Z}_+, l > k+s);$

(ii) $\left(\sum_{n=1}^{\infty}n^{\eta s-1}\omega_l^\eta(\psi;\pi/n)_\gamma\right)^{1/\eta} \leq C_{11}(l,s,\gamma)\left\|\psi^{(s)}\right\|_\gamma \quad (s \in \mathbb{N}, l > s);$

(iii) $\left(\sum_{\nu=n+1}^{\infty}\nu^{\eta s-1}\omega_l^\eta(\psi;\pi/\nu)_\gamma\right)^{1/\eta} \leq C_{12}(l,k,s,\gamma)\,\omega_k\left(\psi^{(s)};\pi/n\right)_\gamma$

$(s \in \mathbb{N}, l \geq k+s).$

Proof. We need the following known inequalities $(\theta = \min\{2,\gamma\}, \psi \in L_\gamma(\mathbb{T}))$

$$\omega_l(\psi;\pi/n)_\gamma \leq C_{13}(l,\gamma)\,n^{-l}\left(\sum_{\nu=1}^{n}\nu^{\theta l-1}E_{\nu-1}^\theta(\psi)_\gamma\right)^{1/\theta}, \quad n \in \mathbb{N}, \tag{7}$$

$$n^{-l}\left(\sum_{\nu=1}^{n}\nu^{\eta l-1}E_{\nu-1}^\eta(\psi)_\gamma\right)^{1/\eta} \leq C_{14}(l,\gamma)\,\omega_l(\psi;\pi/n)_\gamma, \quad n \in \mathbb{N}. \tag{8}$$

The inequality (7) was proved by S.B.Stechkin [24], p. 502, Lemma 1, for $l=1$, $\gamma=2$, and by M.F.Timan [25], Theorem 1, p. 126, inequalities (7), for $l \in \mathbb{N}$, $\gamma \in (1,\infty)$ (see also [5], §6.1.5; [26], §7.3, Theorem 3.4, p. 210, inequality (3.9)). The inequality (8) was proved by M.F. Timan [27], pp. 135-137.

First we proof the estimation ($m \in \mathbb{N}$, $m < l$)

$$\sum_{\nu=1}^{n} \nu^{\eta m-1} \omega_l^{\eta} (\psi; \pi/\nu)_{\gamma} \le C_{15} (l, m, \gamma) \sum_{\nu=1}^{n} \nu^{\eta m-1} E_{\nu-1}^{\eta} (\psi)_{\gamma}, \qquad n \in \mathbb{N}. \tag{9}$$

In virtue of inequality (7) we have that

$$\sum_{\nu=1}^{n} \nu^{\eta m-1} \omega_l^{\eta} (\psi; \pi/\nu)_{\gamma} \le (C_{13} (l, \gamma))^{\eta} \sum_{\nu=1}^{n} \nu^{-\eta (l-m)-1} \left(\sum_{\mu=1}^{\nu} \mu^{\theta l-1} E_{\mu-1}^{\theta} (\psi)_{\gamma} \right)^{\eta/\theta},$$

whence in the case $\gamma \ne 2$ by Hardy's inequality [28], p. 308, Theorem 346, we obtain that ($\eta/\theta > 1$, $\eta (l-m) + 1 > 1$)

$$\sum_{\nu=1}^{n} \nu^{\eta m-1} \omega_l^{\eta} (\psi; \pi/\nu)_{\gamma} \le (C_{13} (l, \gamma))^{\eta} C_{16} (l, m, \theta, \eta) \sum_{\nu=1}^{n} \nu^{\eta m-1} E_{\nu-1}^{\eta} (\psi)_{\gamma},$$

and in the case $\gamma = 2$ ($\Rightarrow$ $\eta = \theta = 2$) we have that

$$(C_{13} (l, 2))^{-2} \sum_{\nu=1}^{n} \nu^{2m-1} \omega_l^{2} (\psi; \pi/\nu)_{2} \le \sum_{\nu=1}^{n} \nu^{-2 (l-m)-1} \sum_{\mu=1}^{\nu} \mu^{2l-1} E_{\mu-1}^{2} (\psi)_{2} =$$

$$= \sum_{\mu=1}^{n} \mu^{2l-1} E_{\mu-1}^{2} (\psi)_{2} \sum_{\nu=\mu}^{n} \nu^{-2 (l-m)-1} \le \left(1 + \frac{1}{2 (l-m)} \right) \sum_{\mu=1}^{n} \mu^{2m-1} E_{\mu-1}^{2} (\psi)_{2}.$$

If we put $m = k + s < l$, $s \in \mathbb{Z}_+$, in (9), then by (8) and known inequality $\omega_{k+s} (\psi; \delta)_{\gamma} \le 2\delta^s \omega_k \left(\psi^{(s)}; \delta \right)_{\gamma}$ for $s \in \mathbb{N}$, we obtain that ($C_{15} = C_{15} (l, k+s, \gamma)$, $C_{14} = C_{14} (k+s, \gamma)$)

$$n^{-k} \left(\sum_{\nu=1}^{n} \nu^{\eta(k+s)-1} \omega_l^{\eta} (\psi; \pi/\nu)_{\gamma} \right)^{1/\eta} \le C_{15}^{1/\eta} n^{-k} \left(\sum_{\nu=1}^{n} \nu^{\eta(k+s)-1} E_{\nu-1}^{\eta} (\psi)_{\gamma} \right)^{1/\eta} \le$$

$$\le C_{15}^{1/\eta} C_{14} n^s \omega_{k+s} (\psi; \pi/n)_{\gamma} \le C_{15}^{1/\eta} \cdot C_{14} \cdot 2\pi^s \omega_k \left(\psi^{(s)}; \pi/n \right)_{\gamma},$$

whence if follows the estimation (i) with $C_{10}(l, k+s, \gamma) = 2\pi^s C_{15}^{1/\eta} C_{14}$.

Furthermore, putting $m = s < l$, $s \in \mathbb{N}$, in (9), by inequality (8) and known inequality $\omega_s (\psi; \delta)_{\gamma} \le 2\delta^s \left\| \psi^{(s)} \right\|_{\gamma}$, $\psi \in W_{\gamma}^{s} (\mathbb{T})$, we have that

$$\left(\sum_{\nu=1}^{n} \nu^{\eta s-1} \omega_l^{\eta} (\psi; \pi/\nu)_{\gamma} \right)^{1/\eta} \le (C_{15} (l, s, \gamma))^{1/\eta} \left(\sum_{\nu=1}^{n} \nu^{\eta s-1} E_{\nu-1}^{\eta} (\psi)_{\gamma} \right)^{1/\eta} \le$$

$$\le (C_{15} (l, s, \gamma))^{1/\eta} C_{14} (s, \gamma) n^s \omega_s (\psi; \pi/n)_{\gamma} \le C_{15}^{1/\eta} \cdot C_{14} \cdot 2\pi^s \left\| \psi^{(s)} \right\|_{\gamma},$$

whence it follows the estimation (ii) as $n \to \infty$.

We note that in virtue of (9) for $m = s \in \mathbb{N}$ the inequality (ii) follows also from the lower estimations of L_{γ} - norm $\left\| \psi^{(s)} \right\|_{\gamma}$ by means of expression containing

$E_n(\psi)_\gamma$, which was obtained by O.V.Besov in [29], p. 15, inequalities (5) and (7) (see also [30], p. 224).

For proof the estimation (iii) we use the following inequalities ($\theta=\min\{2,\gamma\}$, $\psi\in W_\gamma^s(\mathbb{T})$, $l\in\mathbb{N}$, $s\in\mathbb{Z}_+$, $l>s$)

$$\omega_l(\psi;\pi/n)_\gamma\le C_{17}(l,\gamma)\left(\sum_{\nu=n+1}^{\infty}\nu^{-\theta l-1}\left\|S_\nu^{(l)}(\psi;\,\cdot)\right\|_\gamma^\theta\right)^{1/\theta},\quad n\in\mathbb{N}, \tag{10}$$

$$\left(\sum_{\nu=n+1}^{\infty}\nu^{-(l-s)\eta-1}\left\|S_\nu^{(l)}(\psi;\,\cdot)\right\|_\gamma^\eta\right)^{1/\eta}\le C_{18}(l-s,\gamma)\,\omega_{l-s}\left(\psi^{(s)};\pi/n\right)_\gamma,\ n\in\mathbb{N}. \tag{11}$$

The inequality (10) was proved by V.V.Zhuk and Q.I. Natanson [31], see the proof of Theorem 2, inequality (6) on the p. 22. The inequality (11) was proved in [32], Theorem 2, inequality (22) on the p.9 (see also [33], inequality (17) on the p.8).

First we proof the estimation ($m\in\mathbb{N},\quad m<l$)

$$\sum_{\nu=n+1}^{\infty}\nu^{\eta m-1}\omega_l^\eta(\psi;\pi/\nu)_\gamma\le C_{19}(l,m,\gamma)\sum_{\nu=n+1}^{\infty}\nu^{-(l-m)\eta-1}\left\|S_\nu^{(l)}(\psi;\cdot)\right\|_\gamma^\eta,\ n\in\mathbb{N}. \tag{12}$$

Indeed, in the case $\gamma\ne2$ by inequality (10) and by Hardy's inequality [28], Theorem 346, p. 308, we have that ($\eta/\theta>1$, $1-\eta m<1$)

$$\sum_{\nu=n+1}^{\infty}\nu^{\eta m-1}\omega_l^\eta(\psi;\pi/\nu)_\gamma\le$$

$$\le (C_{17}(l,\gamma))^\eta\sum_{\nu=n+1}^{\infty}\nu^{\eta m-1}\left(\sum_{\mu=\nu+1}^{\infty}\mu^{-\theta l-1}\left\|S_\mu^{(l)}(\psi;\,\cdot)\right\|_\gamma^\theta\right)^{\eta/\theta}\le$$

$$\le (C_{17}(l,\gamma))^\eta C_{20}(m,\theta,\eta)\sum_{\nu=n+1}^{\infty}\nu^{-\eta(l-m)-1}\left\|S_\nu^{(l)}(\psi;\,\cdot)\right\|_\gamma^\eta;$$

in the case $\gamma=2$ ($\Rightarrow\ \eta=\theta=2$) we obtain that

$$(C_{17}(l,2))^{-2}\sum_{\nu=n+1}^{\infty}\nu^{2m-1}\omega_l^2(\psi;\pi/\nu)_2\le\sum_{\nu=n+1}^{\infty}\nu^{2m-1}\sum_{\mu=\nu+1}^{\infty}\mu^{-2l-1}\left\|S_\mu^{(l)}(\psi;\,\cdot)\right\|_2^2=$$

$$=\sum_{\mu=n+1}^{\infty}\mu^{-2l-1}\left\|S_\mu^{(l)}(\psi;\,\cdot)\right\|_2^2\sum_{\nu=n+1}^{\mu}\nu^{2m-1}\le\sum_{\mu=n+1}^{\infty}\mu^{-2(l-m)-1}\left\|S_\mu^{(l)}(\psi;\,\cdot)\right\|_2^2.$$

Furthermore, putting $m=s\in\mathbb{N}$, $l=k+s$, in (12) and applying the inequality (11), we obtain (iii) in the case $l=k+s$ ($C_{19}=C_{19}(k+s,s,\gamma)$)

$$\left(\sum_{\nu=n+1}^{\infty}\nu^{\eta s-1}\omega_{k+s}^\eta(\psi;\pi/\nu)_\gamma\right)^{1/\eta}\le C_{19}^{1/\eta}\left(\sum_{\nu=n+1}^{\infty}\nu^{-\eta k-1}\left\|S_\nu^{(k+s)}(\psi;\,\cdot)\right\|_\gamma^\eta\right)^{1/\eta}\le$$

$$\le C_{19}^{1/\eta}C_{18}(k,\gamma)\,\omega_k\left(\psi^{(s)};\pi/n\right)_\gamma,\qquad n\in\mathbb{N}.$$

In the case $l > k+s$ the estimation (iii) reduce to the case $l = k+s$ by known inequality $\omega_l(\psi;\delta)_\gamma \le 2\cdot 2^{l-(k+s)}\omega_{k+s}(\psi;\delta)_\gamma$.

Lemma 4 is proved.

Given $\alpha\in(0,\infty)$, let $M_0(\alpha)$ be the set of all sequences $\lambda=\{\lambda_n\}_{n=1}^\infty\in M_0$ such that $n^\alpha\lambda_n\downarrow(n\uparrow)$.

Lemma 5. *Let* $p,q\in(1,\infty)$, $1/r=1/p+1/q-1>0$ $(\Rightarrow r\in(1,\infty))$, $\gamma\in(r,\infty]$, $k\in\mathbb{N}$, $s\in\mathbb{Z}_+$, $\sigma=s+1/r-1/\gamma$, $\tau=\tau(\gamma)=\gamma$ *for* $\gamma<\infty$ *and* $\tau(\infty)=1$, $\lambda=\{\lambda_n\}_{n=1}^\infty\in M_0(\alpha)$ *and* $\varepsilon=\{\varepsilon_n\}_{n=1}^\infty\in M_0(\beta)$ *for some* α, $\beta\in(0,\infty)$. *Then there are functions* $f_0(\,\cdot\,;p\,;\lambda)\in L_p(\mathbb{T})$ *and* $g_0(\,\cdot\,;q\,;\varepsilon)\in L_q(\mathbb{T})$ *such that*

(i) $E_{n-1}(f_0)_p\le C_{21}(p,\alpha)\lambda_n$, $E_{n-1}(g)_q\le C_{21}(q,\beta)\varepsilon_n$, $n\in\mathbb{N}$;

(ii) $h_0=f_0*g_0\in W_\gamma^s(\mathbb{T})$ $\Leftrightarrow$ $\sum\limits_{n=1}^\infty n^{\tau\sigma-1}\lambda_n^\tau\varepsilon_n^\tau<\infty$;

(iii) if the series in (ii) converge, then

$$\left(\sum_{\nu=n+1}^\infty \nu^{\tau\sigma-1}\lambda_\nu^\tau\varepsilon_\nu^\tau\right)^{1/\tau}+n^{-k}\left(\sum_{\nu=1}^n \nu^{\tau(k+\sigma)-1}\lambda_\nu^\tau\varepsilon_\nu^\tau\right)^{1/\tau}\le$$

$$\le C_{22}(k,\ s,\ r,\ \tau)\,\omega_k\left(h_0^{(s)};\pi/n\right)_\gamma,\quad n\in\mathbb{N}.$$

Proof. For $p,\ q\in(1,\infty)$ $(p'=p/(p-1),\quad q'=q/(q-1))$, let

$$f_0(x;\ p;\ \lambda)=\sum_{n=1}^\infty n^{-1/p'}\lambda_n e^{inx},\quad g_0(x;\ q;\ \varepsilon)=\sum_{n=1}^\infty n^{-1/q'}\varepsilon_n e^{inx},\quad x\in\mathbb{T}.$$

Since $\lambda\in M_0(\alpha)$ and $\varepsilon\in M_0(\beta)$, in virtue of Lemma 1 [34] we have $f_0\in L_p(\mathbb{T})$, $E_{n-1}(f_0)_p\le C_{21}(p,\alpha)\lambda_n$ and $g_0\in L_q(\mathbb{T})$, $E_{n-1}(g_0)_q\le C_{21}(q,\beta)\varepsilon_n$, $n\in\mathbb{N}$.

If the series in (ii) converge, then by (i) we have that

$$\sum_{n=1}^\infty n^{\tau\sigma-1}E_{n-1}^\tau(f_0)_p\,E_{n-1}^\tau(g_0)_q\le\left(C_{21}(p,\alpha)\,C_{21}(q,\beta)\right)^\tau\sum_{n=1}^\infty n^{\tau\sigma-1}\,\lambda_n^\tau\,\varepsilon_n^\tau<\infty,$$

whence $h_0=f_0*g_0\in W_\gamma^s(\mathbb{T})$ by Theorem 1.

For further exposition of proof we consider by itself the cases: $\gamma\le2$, $2<\gamma<\infty$ and $\gamma=\infty$.

First we consider the case $\gamma\le2$. If $h_0\in W_\gamma^s(\mathbb{T})$, then taking into account $c_n(h_0)=c_n(f_0)\cdot c_n(g_0)=n^{-(1/p'+1/q')}\lambda_n\varepsilon_n$ and $\gamma\sigma-1=\gamma s+\gamma/r-2=\gamma s+\gamma-2+\gamma(1/p+1/q-2)=\gamma s+\gamma-2-\gamma(1/p'+1/q')$, we have by (ii) of Lemma 1 that

$$\left(\sum_{n=1}^\infty n^{\gamma\sigma-1}\,\lambda_n^\gamma\,\varepsilon_n^\gamma\right)^{1/\gamma}=\left(\sum_{n=1}^\infty n^{\gamma s+\gamma-2}\left|c_n(h_0)\right|^\gamma\right)^{1/\gamma}\le C_{23}(\gamma)\left\|h_0^{(s)}\right\|_\gamma.$$

Further, applying the inequalities (i) and (iii) of Lemma 1 for $h_0\in W_\gamma^s(\mathbb{T})$, we obtain that

$$\left(\sum_{\nu=n+1}^\infty \nu^{\gamma\sigma-1}\,\lambda_\nu^\gamma\,\varepsilon_\nu^\gamma\right)^{1/\gamma}+n^{-k}\left(\sum_{\nu=1}^n \nu^{\gamma(k+\sigma)-1}\lambda_\nu^\gamma\,\varepsilon_\nu^\gamma\right)^{1/\gamma}=$$

$$=\left(\sum_{\nu=n+1}^\infty \nu^{\gamma s+\gamma-2}\,\left|c_\nu(h_0)\right|^\gamma\right)^{1/\gamma}+n^{-k}\left(\sum_{\nu=1}^n \nu^{\gamma(k+s)+\gamma-2}\left|c_\nu(h_0)\right|^\gamma\right)^{1/\gamma}\le$$

$$\leq C_{24}(k,\gamma)\,\omega_k\left(h_0^{(s)};\pi/n\right)_\gamma + n^s C_{25}(k+s,\gamma)\,\omega_{k+s}(h_0;\pi/n)_\gamma \leq$$

$$\leq \left(C_{24}(k,\gamma)+\pi^s C_{25}(k+s,\gamma)\right)\,\omega_k\left(h_0^{(s)};\pi/n\right)_\gamma,$$

whence the estimation (iii) follows in the case $\gamma \leq 2$.

Consider now the case $2 < \gamma < \infty$. Previously we proof the following estimation ($l \in \mathbb{N}$)

$$n^{1/r-1/\gamma}\lambda_n\varepsilon_n \leq C_{26}(l,\,r,\,\gamma)\;\omega_l\,(h_0;\pi/n)_\gamma\,,\quad n\in\mathbb{N}, \tag{13}$$

under condition that $h_0 \in L_\gamma(\mathbb{T})$.

Since $0 < c_n(h_0) = c_n(Re\,h_0) = c_n(Im\,h_0) = n^{-(1/p'+1/q')}\lambda_n\varepsilon_n \downarrow (n\uparrow)$, then by (ii) of Lemma 3 we have that $n^{1-1/\gamma}c_{2n}(Re\,h_0) \leq C_{27}(\gamma)\,E_n(Re\,h_0)_\gamma$, $n^{1-1/\gamma}c_{2n}(Im\,h_0) \leq C_{27}(\gamma)\,E_n(Im\,h_0)_\gamma$, whence $n^{1-1/\gamma}c_{2n}(h_0) \leq C_{27}(\gamma)\,E_n(h_0)_\gamma$, $n \in \mathbb{N}$.

Taking into account the last estimation and $1/p'+1/q' = 1-1/r$, we obtain that

$$n^{1/r-1/\gamma}\lambda_{2n}\varepsilon_{2n} = 2^{1-1/r}n^{1-1/\gamma}(2n)^{1/r-1}\lambda_{2n}\varepsilon_{2n} =$$

$$= 2^{1-1/r}n^{1-1/\gamma}(2n)^{-(1/p'+1/q')}\lambda_{2n}\varepsilon_{2n} =$$

$$= 2^{1-1/r}n^{1-1/\gamma}c_{2n}(h_0) \leq 2^{1-1/r}C_{27}(\gamma)\,E_n(h_0)_\gamma\,,$$

whence $n^{1/r-1/\gamma}\lambda_{2n}\varepsilon_{2n} \leq 2^{1-1/r}C_{27}(\gamma)\,E_n(h_0)_\gamma$, $n \in \mathbb{N}$.

Further, in virtue of $\lambda_n\downarrow,\ \varepsilon_n\downarrow\ (n\uparrow)$ and by the L_γ - analoque of known D. Jackson-S.B.Stechkin inequality (see [35], Theorem 1, p. 226; [5], Section 5.11, p. 338, inequality (1), and references therein):

$$E_{n-1}(f)_\gamma \leq C_{28}(l)\,\omega_l(f;\pi/n)_\gamma\,,\quad \gamma\in[1,\infty]\,,\quad f\in L_\gamma(\mathbb{T})\,,\quad n\in\mathbb{N}, \tag{14}$$

we have that for $n \geq 2$ ($[t]$ - entire part of $t \in R$)

$$n^{1/r-1/\gamma}\lambda_n\varepsilon_n \leq 3^{1/r-1/\gamma}\,[n/2]^{1/r-1/\gamma}\,\lambda_{2[n/2]}\varepsilon_{2[n/2]} \leq$$

$$\leq 3^{1/r-1/\gamma}2^{1-1/r}C_{27}(\gamma)\,E_{[n/2]}(h_0)_\gamma \leq C_{29}(r,\gamma)\,C_{28}(l)\,\omega_l\left(h_0;\pi/\left([n/2]+1\right)\right)_\gamma \leq$$

$$\leq C_{29}(r,\gamma)\,C_{28}(l)\,\omega_l(h_0;2\pi/n)_\gamma \leq C_{29}(r,\gamma)\,C_{28}(l)\,2^l\omega_l(h_0;\pi/n)_\gamma\,,$$

whence it follows the estimation (13) for $n \geq 2$ with constant $C_{26}(l,\,r,\,\gamma) = 2^l C_{28}(l)\,C_{29}(r,\gamma) = 2^l C_{28}(l)\,3^{1/r-1/\gamma}2^{1-1/r}C_{27}(\gamma)$. For $n = 1$ we have that (see f.e. [2], v. 1, p. 129, exercise (6.10)) $\lambda_1\varepsilon_1 = c_1(h_0) \leq E_0(h_0)_1 \leq E_0(h_0)_\gamma \leq C_{28}(l)\,\omega_l(h_0;\pi)_\gamma$.

Now we proof the validity of implication " $\Rightarrow$ " in (ii) for $2 < \gamma < \infty$. In the case $s = 0$ by the known G.Hardy-J.Littlewood's theorem (see f.e. [1], v. 2, p. 193, Lemma (6.6)) we have that ($1 - 1/r = 1/p' + 1/q'$)

$$\left(\sum_{n=1}^{\infty} n^{\gamma(1/r-1/\gamma)-1}\,\lambda_n^\gamma\,\varepsilon_n^\gamma\right)^{1/\gamma} = \left(\sum_{n=1}^{\infty} n^{\gamma(1/r-1)+\gamma-2}\,\lambda_n^\gamma\,\varepsilon_n^\gamma\right)^{1/\gamma} =$$

$$= \left(\sum_{n=1}^{\infty} n^{\gamma-2}n^{-\gamma(1/p'+1/q')}\,\lambda_n^\gamma\,\varepsilon_n^\gamma\right)^{1/\gamma} = \left(\sum_{n=1}^{\infty} n^{\gamma-2}c_n^\gamma(h_0)\right)^{1/\gamma} \leq C_{30}(\gamma)\,\|\,h_0\,\|_\gamma\,.$$

In the case $s > 0$ by (ii) of Lemma 4 (we put $l = s + 1$, $\eta = \max\{2, \gamma\} = \gamma$) and by inequality (13) we have that $(C_{26} = C_{26}(s+1, r, \gamma))$

$$C_{11}(s+1,\, s,\, \gamma) \left\| h_0^{(s)} \right\|_\gamma \ge \left(\sum_{n=1}^{\infty} n^{\gamma s - 1} \omega_{s+1}^\gamma (h_0; \pi/n)_\gamma \right)^{1/\gamma} \ge$$

$$\ge C_{26}^{-1} \left(\sum_{n=1}^{\infty} n^{\gamma s-1} n^{\gamma(1/r-1/\gamma)} \lambda_n^\gamma \varepsilon_n^\gamma \right)^{1/\gamma} = C_{26}^{-1} \left(\sum_{n=1}^{\infty} n^{\gamma \sigma - 1} \lambda_n^\gamma \varepsilon_n^\gamma \right)^{1/\gamma}.$$

Now we proof the estimation (iii). In the case $s = 0$ taking into account $c_n(h_0) = n^{1/r-1}\lambda_n\varepsilon_n$, $n \in \mathbb{N}$, by (iii) of Lemma 3 and inequalities (13) and (14) we have that

$$\left(\sum_{\nu=n+1}^{\infty} \nu^{\gamma(1/r-1/\gamma)-1} \lambda_\nu^\gamma \varepsilon_\nu^\gamma \right)^{1/\gamma} = \left(\sum_{\nu=n+1}^{\infty} \nu^{\gamma-2} c_\nu^\gamma (h_0) \right)^{1/\gamma} \le$$

$$\le \left(\sum_{\nu=n+1}^{2n+1} \nu^{\gamma-2} c_\nu^\gamma (h_0) \right)^{1/\gamma} + \left(\sum_{\nu=2(n+1)}^{\infty} \nu^{\gamma-2} c_\nu^\gamma (h_0) \right)^{1/\gamma} \le$$

$$\le c_{n+1}(h_0) (\gamma-1)^{-1/\gamma} \left(2^{\gamma-1} - 1\right)^{1/\gamma} (n+1)^{1-1/\gamma} + C_{31}(\gamma) E_{n+1}(h_0)_\gamma \le$$

$$\le (\gamma-1)^{-1/\gamma} \left(2^{\gamma-1}-1\right)^{1/\gamma} 2^{1-1/\gamma} n^{1-1/\gamma} c_n(h_0) + C_{31}(\gamma) E_n(h_0)_\gamma =$$

$$= (\gamma-1)^{-1/\gamma} \left(2^{\gamma-1}-1\right)^{1/\gamma} 2^{1-1/\gamma} n^{1/r-1/\gamma} \lambda_n \varepsilon_n + C_{31}(\gamma) E_n(h_0)_\gamma \le$$

$$\le C_{32}(\gamma) C_{26}(k,\, r,\, \gamma) \omega_k (h_0; \pi/n)_\gamma + C_{31}(\gamma) C_{28}(k) \omega_k (h_0; \pi/(n+1))_\gamma \le$$

$$\le \{ C_{32}(\gamma) C_{26}(k,\, r,\, \gamma) + C_{31}(\gamma) C_{28}(k) \}\; \omega_k (h_0; \pi/n)_\gamma .$$

In the case $s > 0$ by (iii) of Lemma 4 (we put $l = k + s$) and inequality (13) we obtain that $(C_{33} = C_{26}(k+s,\, r,\, \gamma))$

$$C_{12}(k+s,\, k,\, s,\, \gamma) \omega_k \left(h_0^{(s)}; \pi/n \right)_\gamma \ge \left(\sum_{\nu=n+1}^{\infty} \nu^{\gamma s - 1} \omega_{k+s}^\gamma (h_0; \pi/\nu)_\gamma \right)^{1/\gamma} \ge$$

$$\ge C_{33}^{-1} \left(\sum_{\nu=n+1}^{\infty} \nu^{\gamma s - 1} \nu^{\gamma(1/r-1/\gamma)} \lambda_\nu^\gamma \varepsilon_\nu^\gamma \right)^{1/\gamma} = C_{33}^{-1} \left(\sum_{\nu=n+1}^{\infty} \nu^{\gamma\sigma-1} \lambda_\nu^\gamma \varepsilon_\nu^\gamma \right)^{1/\gamma}.$$

From obtained estimations follows the estimation of the first summand in (iii) for $s \in \mathbb{Z}_+$ and $2 < \gamma < \infty$.

Further, by (i) of Lemma 4 (we put $l = k + s + 1$, $\;s \in \mathbb{Z}_+$) and inequality (13) we have that $(C_{10} = C_{10}(k+s+1,\, k+s,\, \gamma), C_{34} = C_{26}(k+s+1,\, r,\, \gamma))$

$$C_{10}\, \pi^s \omega_k \left(h_0^{(s)}; \pi/n \right)_\gamma \ge n^{-k} \left(\sum_{\nu=1}^{n} \nu^{\gamma(k+s)-1} \omega_{k+s+1}^\gamma (h_0; \pi/\nu)_\gamma \right)^{1/\gamma} \ge$$

$$\ge C_{34}^{-1} n^{-k} \left(\sum_{\nu=1}^{n} \nu^{\gamma(k+s)-1} \nu^{\gamma(1/r-1/\gamma)} \lambda_\nu^\gamma \varepsilon_\nu^\gamma \right)^{1/\gamma} = C_{34}^{-1} n^{-k} \left(\sum_{\nu=1}^{n} \nu^{\gamma(k+\sigma)-1} \lambda_\nu^\gamma \varepsilon_\nu^\gamma \right)^{1/\gamma},$$

whence it follows the estimation of the second summand in (iii) for $2 < \gamma < \infty$.

At last we consider the case $\gamma = \infty$ ($\Rightarrow$ $\tau = 1$). We proof the validity of implication " $\Rightarrow$ " in (ii). If $h_0 = f_0 * g_0 \in W^s_\infty(\mathbb{T}) \equiv C^s(\mathbb{T})$, then taking into account equality $c_n(h_0) = c_n(f_0)\, c_n(g_0) = n^{-(1/p'+1/q')}\lambda_n\varepsilon_n = n^{1/r-1}\lambda_n\varepsilon_n$, $n \in \mathbb{N}$, and by (iii) of Lemma 2 we have that ($s \in \mathbb{Z}_+$, $\sigma = s + 1/r$)

$$\sum_{n=1}^{\infty} n^{\sigma-1}\lambda_n\varepsilon_n = \sum_{n=1}^{\infty} n^{s+1/r-1}\lambda_n\varepsilon_n = \sum_{n=1}^{\infty} n^s c_n(h_0) \le \left\| \psi^{(s)} \right\|_\infty .$$

Further, by (iv) of Lemma 2 we obtain that ($s \in \mathbb{Z}_+$)

$$\sum_{\nu=n+1}^{\infty} \nu^{\sigma-1}\lambda_\nu\varepsilon_\nu = \sum_{\nu=n+1}^{\infty} \nu^{s+1/r-1}\lambda_\nu\varepsilon_\nu = \sum_{\nu=n+1}^{\infty} \nu^s c_\nu(h_0) \le C_{35}(k)\, \omega_k\left(h_0^{(s)}; \pi/n\right)_\infty ,$$

whence it follows the estimation of the first summand in (iii). Now we estimate the second summand in (iii). For $s \in \mathbb{Z}_+$ and $k \in \mathbb{N}$ by (i) of Lemma 2 for even $k+s$ and by (ii) of Lemma 2 for odd $k+s$ we have that

$$n^{-k}\sum_{\nu=1}^{n} \nu^{k+\sigma-1}\lambda_\nu\varepsilon_\nu = n^{-k}\sum_{\nu=1}^{n} \nu^{k+s+1/r-1}\lambda_\nu\varepsilon_\nu = n^{-k}\sum_{\nu=1}^{n} \nu^{k+s} c_n(h_0) \le$$

$$\le C_{36}(k+s)\, n^s\, \omega_{k+s}(h_0; \pi/n)_\infty \le C_{36}(k+s)\, \pi^s\, \omega_k\left(h_0^{(s)}; \pi/n\right)_\infty ,$$

whence it follows the estimation of the second summand in (iii).

Lemma 5 is proved.

Given p, $q \in [1,\infty]$ and λ, $\varepsilon \in M_0$, put

$$E_p[\lambda] * E_q[\varepsilon] = \{h = f * g : \ f \in E_p[\lambda], \ g \in E_q[\varepsilon]\}.$$

The following theorem shows that estimation (5) of Theorem 1 is exact in the sence of order on classes $E_p[\lambda] * E_q[\varepsilon]$ in the case p, $q \in (1,\infty)$ under conditions that $\lambda \in M_0(\alpha)$ and $\varepsilon \in M_0(\beta)$ for some α, $\beta \in (0,\infty)$.

Theorem 2. *Let p, $q \in (1,\infty)$, $r = pq/(p+q-pq) \in (1,\infty)$, $\gamma \in (r,\infty]$, $k \in \mathbb{N}$, $s \in \mathbb{Z}_+$, $\sigma = s + 1/r - 1/\gamma$, $\tau = \tau(\gamma) = \gamma$ for $\gamma < \infty$ and $\tau(\infty) = 1$, $\lambda = \{\lambda_n\}_{n=1}^\infty \in M_0(\alpha)$ and $\varepsilon = \{\varepsilon_n\}_{n=1}^\infty \in M_0(\beta)$ for some α, $\beta \in (0,\infty)$, and*

$$\sum_{n=1}^{\infty} n^{\tau\sigma-1}\lambda_n^\tau\varepsilon_n^\tau < \infty. \tag{15}$$

Then

$$\sup\left\{ \omega_k\left(h^{(s)}; \pi/n\right)_\gamma : \ h \in E_p[\lambda] * E_q[\varepsilon] \right\} \asymp$$

$$\asymp \left(\sum_{\nu=n+1}^{\infty} \nu^{\tau\sigma-1}\lambda_\nu^\tau\varepsilon_\nu^\tau\right)^{1/\tau} + n^{-k}\left(\sum_{\nu=1}^{n} \nu^{\tau(k+\sigma)-1}\lambda_\nu^\tau\varepsilon_\nu^\tau\right)^{1/\tau}, \quad n \in \mathbb{N}.$$

Proof. Indeed, the upper estimation for every p, $q \in [1,\infty)$ and for arbitrary λ, $\varepsilon \in M_0$ immediately follows by inequality (5) of Theorem 1. The lower estimation is realized by function

$$h_0(\,\cdot\,; p, q; \lambda, \varepsilon) = (C_{21}(p,\alpha))^{-1} f_0(\,\cdot\,; p; \alpha) * (C_{21}(q,\beta))^{-1} g_0(\,\cdot\,; q; \varepsilon) \in E_p[\lambda] * E_q[\varepsilon]$$

in virtue of (iii) of Lemma 5.

Remark. The condition convergence of the series (15) it is necessary and sufficiently for imbedding $E_p[\lambda] * E_q[\varepsilon] \subset W^s_\gamma(\mathbb{T})$. The sufficiency for arbitrary $\lambda,\ \varepsilon \in M_0$ immediately follows from the first part of the statement of Theorem 1. The necessity under conditions $\lambda \in M_0(\alpha)$ and $\varepsilon \in M_0(\beta)$ follows from the statement (ii) of Lemma 5.

Given $p,\ q \in [1,\infty]$ and $\alpha,\ \beta \in (0,\infty)$ we denote $E_{p,\alpha} = E_p\left[\{n^{-\alpha}\}_{n=1}^{\infty}\right]$, $E_{q,\beta} = E_q\left[\{n^{-\beta}\}_{n=1}^{\infty}\right]$. The following statement follows from Theorem 2.

Corollary. *Let* $p,\ q \in (1,\infty)$, $1/r = 1/p + 1/q - 1 > 0$ $(\Rightarrow r \in (1,\infty))$, $\gamma \in (r,\infty]$, $k \in \mathbb{N}$, $s \in \mathbb{Z}_+$, $\sigma = s + 1/r - 1/\gamma$, $\tau = \tau(\gamma) = \gamma$ *for* $\gamma < \infty$ *and* $\tau(\infty) = 1$, $\alpha,\ \beta \in (0,\infty)$, $\rho = \alpha + \beta - \sigma > 0$. *Then for* $\delta \in (0,\pi]$

(i) $\sup\left\{\omega_k\left(h^{(s)};\delta\right)_\gamma :\ h \in E_{p,\alpha} * E_{q,\beta}\right\} \asymp$

$$\asymp \left\{\delta^\rho \text{ for } \rho < k;\ \ \delta^k\left(\ln(\pi e/\delta)\right)^{1/\tau} \text{ for } \rho = k;\ \ \delta^k \text{ for } \rho > k\right\}.$$

(ii) $\sup\left\{\omega_{k+1}\left(h^{(s)};\delta\right)_\gamma :\ h \in E_{p,\alpha} * E_{q,\beta}\right\} \asymp \delta^k$ *for* $\rho = k$.

Proof. For the proof it is sufficiently to note the following (see f.e. [22], the proof of Theorem 3). For every $\delta \in (0,\pi]$ there exists an $n \in \mathbb{N}$ such that $\pi/(n+1) < \delta \le \pi/n$, whence we have the following estimations:

$$2^{-k}\omega_k\left(h^{(s)};\pi/n\right)_\gamma \le \omega_k\left(h^{(s)};\delta\right)_\gamma \le \omega_k\left(h^{(s)};\pi/n\right)_\gamma;$$

$$2^{-\rho}(\pi/n)^\rho < \delta^\rho \le (\pi/n)^\rho \quad \text{for every} \quad \rho \in (0,\infty);$$

$$\delta^k\left(\ln(\pi e/\delta)\right)^{1/\tau} \le (\pi/n)^k\left(\ln(e(n+1))\right)^{1/\tau} =$$

$$= \pi^k n^{-k}\left(1 + \ln(n+1)\right)^{1/\tau} \le 3^{1/\tau}\pi^k n^{-k}\left(\ln(n+1)\right)^{1/\tau};$$

$$n^{-k}\left(\ln(en)\right)^{1/\tau} \le (2/\pi)^k(\pi/(n+1))^k\left(\ln(\pi e/\delta)\right)^{1/\tau} < (2/\pi)^k\delta^k\left(\ln(\pi e/\delta)\right)^{1/\tau}.$$

Furthermore the following estimations hold:

$$(\tau\rho)^{-1/\tau}2^{-\rho}n^{-\rho} \le (\tau\rho)^{-1/\tau}(n+1)^{-\rho} \le \left(\sum_{\nu=n+1}^{\infty}\nu^{-\tau\rho-1}\right)^{1/\tau} \le (\tau\rho)^{-1/\tau}n^{-\rho},\ n \in \mathbb{N};$$

$\varphi_n(k-\rho;\tau) \le n^{-k}\left(\sum\limits_{\nu=1}^{n}\nu^{\tau(k-\rho)-1}\right)^{1/\tau} \le \psi_n(k-\rho;\tau)$, $n \in \mathbb{N}$, where $\varphi_n(k-\rho;\tau) = (\tau\ (k-\rho))^{-1/\tau}n^{-\rho}$, $\psi_n(k-\rho;\tau) = (\tau\ (k-\rho))^{-1/\tau}n^{-k}\left((n+1)^{\tau(k-\rho)}-1\right)^{1/\tau} \le$ $\le (\tau(k-\rho))^{-1/\tau}2^{k-\rho}n^{-\rho}$ either $\psi_n(k-\rho;\tau) \le n^{-\rho}$ for $\rho < k$ and $\tau(k-\rho) \ge 1$;

$$\varphi_n(k-\rho;\tau) = (\tau\ (k-\rho))^{-1/\tau}n^{-k}\left((n+1)^{\tau(k-\rho)}-1\right)^{1/\tau} \ge$$

$$\ge (\tau(k-\rho))^{-1/\tau}n^{-k}\left(\tau(k-\rho)2^{\tau(k-\rho)-1}n^{\tau(k-\rho)}\right)^{1/\tau} = 2^{k-\rho-1/\tau}n^{-\rho},$$

$$\psi_n(k-\rho;\tau) = (\tau\ (k-\rho))^{-1/\tau}n^{-\rho} \text{ for } \rho < k \text{ and } \tau(k-\rho) \le 1;$$

$$\varphi_n(k-\rho;\tau) = n^{-k}\left(\ln(n+1)\right)^{1/\tau},\quad \psi_n(k-\rho;\tau) = n^{-k}\left(\ln(en)\right)^{1/\tau} \text{ for } \rho = k;$$

$$\varphi_n (k-\rho;\tau) = n^{-k}, \psi_n (k-\rho;\tau) = \left(1+(\tau(\rho-k))^{-1}\right)^{1/\tau} n^{-k} \text{ for } \rho > k;$$

$$\tau^{-1/\tau} n^{-k} \leq n^{-(k+1)} \left(\sum_{\nu=1}^{n} \nu^{\tau(k+1-\rho)-1}\right)^{1/\tau} \leq n^{-k} \text{ for } \rho = k.$$

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Received February 17, 2010; Revised May 12, 2010.