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NECESSARY OPTIMALITY CONDITIONS IN THE PROBLEMS OF CONTROL OF GOURSAT-DARBOUX SYSTEMS WITH INTEGRAL CONDITIONS

Abstract

In the paper, an optimal control problem is considered for the Goursat-Darboux systems with integral boundary conditions. The obvious form of the formulae for the gradient of the functional is found on the base of the functional increment formula.

Introduction. Recently, the differential equations with nonlocal boundary conditions are intensively studied. Usually, the problems wherein instead of classic boundary conditions for partial differential equations, definite relation of the values of the desired function on the boundary of the domain or on its compact subsets is given, are called non-local problems. Often, the conditions containing the integral from the desired solution play as such relations.

In the books [1,2], numerous examples from biology, sociology, agriculture are considered and mathematical models are described by hyperbolic equations with non-local conditions. In the paper [3], a linear hyperbolic system with integral and multipoint boundary conditions is considered, necessary and sufficient solvability conditions of boundary value problems are proved, and concrete processes whose mathematical models are described by these problems, are given. The hyperbolic equations with non-classic conditions in deriving mathematical models of the worn surfaces are obtained in [4]. In [5], one-dimensional nonlinear hyperbolic differential equation with integral equations is considered. The boundary conditions are given or the characteristics of the equation. Existence and uniqueness of the classic solution of the problem under consideration is proved. In the papers [6-10], linear and quasi-linear hyperbolic equations with integral conditions are considered and theorems on existence and uniqueness of classic solutions are proved. An one-dimensional linear hyperbolic equation with integral conditions is considered in the paper [11]. The Fourier method is applied and a theorem on the existence and uniqueness of the classic solution is proved. Different hyperbolic equations with various integral boundary conditions are considered in [12-14]. Thus, if follows from what has been said above that different fields of science, engineering and economy need hyperbolic equations with non-local boundary conditions. Therefore, there arises a necessity on optimal control of such processes.

The optimal control problems with different non-local conditions were not studied quite enough. Different optimal control problems for hyperbolic systems with non-local conditions are considered in the papers [15-19].

In the present paper, a non-linear system of hyperbolic equations on a bounded rectangle is considered. The boundary conditions are given on the characteristics of a hyperbolic system by means of ordinary differential equations. Integral conditions are given in one of the characteristics for a unique solvability of differential equations. The control parameters are contained in the right hand side of the hyperbolic system and boundary conditions. The optimal control problems with such boundary conditions are considered for the first time.

Problem Statement. Let some controlled system be described by the system hyperbolic equations with initial-boundary conditions

$$\frac{\partial^2 y(t,s)}{\partial t \partial s} = f\left(t, s, y(t,s), \frac{\partial y(t,s)}{\partial t}, \frac{\partial y(t,s)}{\partial s}, u(t,s)\right), \quad a.e.\ (t,s) \in Q, \tag{1}$$

$$\frac{\partial y(t,0)}{\partial t} = \varphi(t, y(t,0),\ v(t)), \quad a.e.\ t \in [0,T], \tag{2}$$

$$\frac{\partial y(0,s)}{\partial s} = \psi(s, y(0,s),\ \omega(s0), \quad a.e.\ s \in [0,l]\,, \tag{3}$$

$$y(0,0) + \int_0^T n(t) y(t,0) dt = c, \tag{4}$$

where $Q = \{(t,s)\,; 0 \le t \le T,\ 0 \le s \le l\}$ is the given rectangle, $T, l > 0$ are the given numbers; $y = (y_1, y_2,...,y_n)$ are phase variables; $u = (u_1, u_2,...,u_r)\,,$
$v = (v_1, v_2,...,v_m)\,,\ \omega = (\omega_1, \omega_2,...,\omega_q)$ are the controlling parameters; $\frac{\partial y(t,s)}{\partial t}$, $\frac{\partial y(t,s)}{\partial s}$, $\frac{\partial^2 y(t,s)}{\partial t \partial s}$ are generalized derivatives of the Sobolev function $y = y(t,s)$;

$$f = (f_1, f_2, ..., f_n),\ \varphi = (\varphi_1, \varphi_2, ..., \varphi_n)\,,\ \psi = (\psi_1, \psi_2, ..., \psi_n)$$

are the given vector-functions; is $n(t)$ is $n \times n$-dimensional matrix-function, $c \in R^n$ is the given constant vector.

It is assumed that the controls $w = w(t,s) = (u(t,s),\ v(t),\ \omega(s))$ are chosen from the set

$$W = U \times V \times \Omega, \tag{5}$$

where $U \subseteq L_2^r(Q),\ V \subseteq L_2^m([0,T]),\ \Omega \subseteq L_2^q([0,l])$.

Here we use the following denotation; R^n stands for $n-$dimensional Euclidean space, $L_2^r(A)$ for a space of Lebesgue measurable and square-summable $r-$ dimensional vector functions on the set A.

Definition. *Under the solution of problem (1)-(4) that corresponds to the control $w \in W$, we understand a vector-function $y = y(t,s;w) \in H_2^{1,m}(Q)$ possessing Sobolev generalized derivative $\frac{\partial^2 y(t,s)}{\partial t \partial s} \in L_2^n(Q)$ and satisfying differential equation (1), conditions (2), (3) almost everywhere, and condition (4) in the classic sense.*

Here $H_2^{1,n}(Q)$ is a Sobolev space of n-dimensional vector-functions (in the sequel we'll omit the indices indicating the dimension of vector-functions) that are Lebesgue square-summable on Q together with their first generalized derivatives, i.e. $y(t,s)$, $\frac{\partial y(t,s)}{\partial t}$, $\frac{\partial y(t,s)}{\partial s} \in L_2(Q)$. Note that from the given definition of the solution of initial boundary value problem (1)-(4) it follows that the given solution also belongs to the space $C^n(Q)$. (By $C^n(Q)$ the space of $n-$dimensional vector-functions continuous in the rectangle Q is denoted).

The optimal control problem is stated as follows: Among the controls $w = w(t,s) \in W$ it is necessary to find a control in order to minimize the functional

$$J(w) = \iint_Q F\left(t,s,y(t,s), \frac{\partial y(t,s)}{\partial t}, \ \frac{\partial y(t,s)}{\partial s}, \ u(t,s)\right) dtds+$$

$$+\sum_{i=1}^{k} \Phi(y(t_i,s_i)), \tag{6}$$

where $F(t,s,y,p,q,u)$ and $\Phi(y)$ are scalar functions; (t_i,s_i), $i=\overline{1,k}$ is an arbitrary collection of points from the rectangle Q; k is a fixed natural number.

Note that the optimization problems of type (1)-(6) are of practical interest. To present day, there exist numerous processes described by means of hyperbolic system of equations: in investigating the sorption, desorption, drying, friction, wearing and etc. processes. Condition (4) is justified by the fact that while modeling the specific process it is impossible to measure some characteristics (states) at the characteristic point, and some mean (integral) value of the characteristic is known [5].

For the given functions we make the following assumptions:

I) Let the functions $f(t,s,y,p,q,u)$ and $F(t,s,y,p,q,u)$ for almost all $(t,s) \in Q$ be continuous with respect to variables $(y,,p,q,u) \in R^{3n} \times R^r$, and for each fixed $(y,p,q,u) \in R^{3n} \times R^r$ be measurable with respect to $(t,s) \in Q$.

II) It is assumed that the function $f(t,s,y,p,q,u)$ for almost all $(t,s) \in Q$ and for any $(y,p,q,u) \in R^{3n} \times R^r$ have continuous derivatives with respect to $(y,p,q) \in R^{3n}$. The followings hold:

$$f\left(\cdot,\cdot,y\left(t,s\right), \ \frac{\partial y(t,s)}{\partial t}, \ \frac{\partial y(t,s)}{\partial s}, \ u(t,s)\right): \ H_2^1(Q) \times U \to L_2(Q);$$

$$\frac{\partial f}{\partial y}\left(\cdot,\cdot,y\left(t,s\right), \ \frac{\partial y(t,s)}{\partial t}, \ \frac{\partial y(t,s)}{\partial s}, \ u(t,s)\right): \ H_2^1(Q) \times U \to L_\infty(Q);$$

$$\frac{\partial f}{\partial p}\left(\cdot,\cdot,y\left(t,s\right), \ \frac{\partial y(t,s)}{\partial t}, \ \frac{\partial y(t,s)}{\partial s}, \ u(t,s)\right): \ H_2^1(Q) \times U \to L_2(Q);$$

$$\frac{\partial f}{\partial q}\left(\cdot,\cdot,y\left(t,s\right), \ \frac{\partial y(t,s)}{\partial t}, \ \frac{\partial y(t,s)}{\partial s}, \ u(t,s)\right): \ H_2^1(Q) \times U \to L_\infty(Q).$$

III) Let for the function $f(t,s,y,p,q,u)$ it hold

$$f(t,s,0,0,0,0) \in L_2(Q);$$

IV) The function $f(t,s,y,p,q,u)$ satisfies the Lipschits condition with respect to variables $(y,p,q,u)\in R^{3n}$, i.e.

$$|f(t,s,y,p,q,u)-f(t,s,\overline{y},\overline{p},\overline{q},\overline{u})|\leq k\left(|y-y|+|p-\overline{p}|+|q-\overline{q}|+|u-\overline{u}|\right),$$

for all $(y,p,q,u)\,,\ (\overline{y},\overline{p},\overline{q},\overline{u})\in R^{3n}\times R^n$;

V) The function $\varphi(t,y,v)$ for almost all $t\in[0,T]$ is continuous with respect to $(y,v)\in R^n\times R^m$ and measurable with respect to $t\in[0,T]$ for each fixed $(y,v)\in R^n\times R^r$;

VI) The function $\varphi(t,y,v)$ for almost all $t\in[0,T]$ and for any $(y,v)\in R^n\times R^r$ has continuous derivatives with respect to y and

$$\varphi\left(\cdot,y(t,0),\ v(t)\right)\colon C\left([0,T]\right)\times V\rightarrow L_2\left([0,T]\right),$$

$$\frac{\partial\varphi}{\partial y}\left(\cdot,y(t,0),\ v(t)\right)\colon C\left([0,T]\right)\times V\rightarrow L_\infty\left([0,T]\right);$$

VII) Let for the function $\varphi(t,y,v)$ it hold

$$\varphi\left(t,0,0\right)\in L_2\left([0,T]\right);$$

VIII) The function $\varphi(t,y,v)$ satisfies the Lipschits condition with respect to the variables $(y,v)\in R^n\times R^m$, i.e.

$$|\varphi(t,y,v)-\varphi(t,\overline{y},\overline{v})|\leq L\left(|y-\overline{y}|+|v-\overline{v}|\right),$$

for all (t,y,v) and $(t,\overline{y},\overline{v})\in[0,T]\times R^n\times R^m$;

IX) Let the function $\psi(s,y,\omega)$ for almost all $s\in[0,l]$ be continuous with respect to $(y,\omega)\in R^n\times R^q$;

X) The function $\psi(s,y,\omega)$ for almost all $s\in[0,l]$ and for any $(y,\omega)\in R^n\times R^q$ has continuous derivatives with respect to y and

$$\psi\left(\cdot,y(0,s),\omega(s)\right)\colon C\left([0,l]\right)\times\Omega\rightarrow L_2\left([0,l]\right),$$

$$\frac{\partial\psi}{\partial y}\left(\cdot,y(0,s),\omega(s)\right)\colon C\left([0,l]\right)\times\Omega\rightarrow L_\infty\left([0,l]\right);$$

XI) Let $\psi(s,0,0)\in L_2\left([0,l]\right)$;

XII) The function $\psi(s,y,\omega)$ satisfies the Lipchits condition with respect to the variables $(y,\omega)\in R^n\times R^q$, i.e.

$$|\psi(s,y,\omega)-\psi(s,\overline{y},\overline{\omega})|\leq N\left(|y-\overline{y}|+|\omega-\overline{\omega}|\right),$$

for all (s,y,ω) and $(s,\overline{y},\overline{\omega})\in[0,l]\times R^n\times R^q$;

XIII) $n(t)$ is a matrix of the function of order $n\times n$ and $n_{ij}(t)\in L_\infty\left([0,T\right)$, $i,j=\overline{1,n}$, moreover, $\left\|\int\limits_0^T n(t)dt\right\|<1$. Obviously, $\widetilde{n}(T)=E+\int\limits_0^t n(t)dt$ is a non-degenerate matrix and

$$LT\left(1+\left\|\widetilde{n}^{-1}(T)\right\|\cdot|n(t)|_{L_\infty}\frac{T}{2}\right)<1;$$

XIV) The function $\Phi(y)$ for any $y \in R^n$ has continuous derivatives;

XV) Let the scalar function $F(t,s,y,p,q,u)$ for almost all $(t,s) \in Q$ and for any $(y,p,q,u) \in R^{3n} \times R^r$ be continuous with respect to $y,p,q,u \in R^{3n} \times R^r$ and for fixed $y,p,q,u \in R^{3n} \times R^r$ have continuous derivatives with respect to (y,p,q,u) and

$$F\left(\cdot,\cdot,\ y(t,s,\frac{\partial y(t,s)}{\partial t},\ \frac{\partial y(t,s)}{\partial s},u(t,s)\right): H_2^1(Q)\times U \to L_1(Q);$$

$$\frac{\partial F}{\partial y}\left(\cdot,\cdot,\ y(t,s,\frac{\partial y(t,s)}{\partial t},\ \frac{\partial y(t,s)}{\partial s},u(t,s)\right): H_2^1(Q)\times U \to L_2(Q);$$

$$\frac{\partial F}{\partial p}\left(\cdot,\cdot,\ y(t,s,\frac{\partial y(t,s)}{\partial t},\ \frac{\partial y(t,s)}{\partial s},u(t,s)\right): H_2^1(Q)\times U \to L_2(Q);$$

$$\frac{\partial F}{\partial q}\left(\cdot,\cdot,\ y(t,s,\frac{\partial y(t,s)}{\partial t},\ \frac{\partial y(t,s)}{\partial s},u(t,s)\right): H_2^1(Q)\times U \to L_2(Q);$$

$$\frac{\partial F}{\partial u}\left(\cdot,\cdot,\ y(t,s,\frac{\partial y(t,s)}{\partial t},\ \frac{\partial y(t,s)}{\partial s},u(t,s)\right): H_2^1(Q)\times U \to L_2(Q).$$

Functional increment formula. Let $w=(v,\omega,u)$ and

$$w+\overline{w}=(v+\overline{v},\omega+\overline{\omega},u+\overline{u})$$

be two admissible controls, i.e. w and $w+\overline{w}\in W=V\times\Omega\times U$. The solution of problem (1)-(4) that corresponds to these controls is denoted by $y(t,s)=y(t,s;w)$ and $y(t,s)+\overline{y}(t,s)=y(t,s;w+\overline{w})$. Then, according to (6), for the increment of the functional we get the following formula:

$$\begin{aligned} J\left(w+w\right)-J(w) &= \sum_{i=1}^{k}\left[\Phi(y(t_i,s_i)+y(t_i,s_i))-\Phi(y(t_i,s_i))\right]+ \\ &+\iint_Q F\left(t,s,y(t,s)+\overline{y}(t,s),\frac{\partial y(t,s)}{\partial t}+\frac{\partial \overline{y}(t,s)}{\partial t},\right. \\ &\left.\frac{\partial y(t,s)}{\partial s}+\frac{\partial \overline{y}(t,s)}{\partial s},u(t,s)+\overline{u}(t,s)\right)- \\ &-F\left(t,s,y(t,s),\frac{\partial y(t,s)}{\partial t},\frac{\partial y(t,s)}{\partial t},u(t,s)\right)dtds, \end{aligned} \tag{7}$$

where $\overline{y}(t,s)$ is a solution of the following system:

$$\begin{aligned} \frac{\partial^2\overline{y}(t,s)}{\partial t\partial s} &= f\left(t,s,y(t,s)+\overline{y}(t,s),\frac{\partial y(t,s)}{\partial t}+\frac{\partial \overline{y}(t,s)}{\partial t},\right. \\ &\left.\frac{\partial y(t,s)}{\partial s}+\frac{\partial \overline{y}(t,s)}{\partial s},u(t,s)+\overline{u}(t,s)\right)- \\ &-f\left(t,s,y(t,s),\frac{\partial y(t,s)}{\partial t},\frac{\partial y(t,s)}{\partial s},u(t,s)\right)\Bigg],\quad a.e.\ (t,s)\in Q \end{aligned} \tag{8}$$

with initial boundary conditions

$$\frac{\partial y(t,s)}{\partial t}=\varphi(t,y(t,0)+\overline{y}(t,0),\ \overline{v}(t)+v(t))- \\ -\varphi(t,y(t,0),\ v(t)),\quad a.e.\ t\in[0,T], \tag{9}$$

$$\frac{\partial \overline{y}(0,s)}{\partial s}=\psi(s,y(0,s)+\overline{y}(0,s),\ \omega(s)-\overline{\omega}(t))- \\ -\psi(s,y(0,s),\ \omega(t)),\quad a.e.\ \ s\in[0,l], \tag{10}$$

$$y(0,0)+\int\limits_0^T n(t)\overline{y}(t,0)dt=0. \tag{11}$$

For further simplification of mathematical formulae, we introduce the denotation:

$$\widetilde{H}(t,s)=H(t,s,y(t,s),\frac{\partial y}{\partial t}(t,s),\frac{\partial y}{\partial s}(t,s),\psi(t,s),\ u(t,s)),$$

$$\frac{\partial \widetilde{f}}{\partial y}(t,s)=\frac{\partial f}{\partial y}(t,s,y(t,s),\frac{\partial y}{\partial t}(t,s),\frac{\partial y}{\partial s}(t,s),u(t,s))$$

and etc.

Introduce the system of equations in variations:

$$\frac{\partial^2 z(t,s)}{\partial t\partial s}=\frac{\partial \overline{f}}{\partial y}(t,s),z(t,s)+\frac{\partial \overline{f}}{\partial p}(t,s)\frac{\partial z}{\partial t}(t,s)+ \\ +\frac{\partial \overline{f}}{\partial q}(t,s)\frac{\partial z}{\partial s}(t,s)+\frac{\partial \overline{f}}{\partial q}(t,s)\overline{u}(t,s),\ a.e.\ \ (t,s)\in Q, \tag{12}$$

$$\frac{\partial z(t,0)}{\partial t}=\frac{\partial \overline{\varphi}}{\partial y}(t)z(t)+\frac{\partial \overline{\varphi}}{\partial v}(t)\overline{v}(t),\ a.e.\ \ t\in[0,T], \tag{13}$$

$$\frac{\partial z(0,s)}{\partial s}=\frac{\partial \overline{\psi}}{\partial y}(s)z(s)+\frac{\partial \overline{\psi}}{\partial \omega}(s)\overline{\omega}(s),\ a.e.\ \ s\in[0,l], \tag{14}$$

$$z(0,0)+\int\limits_0^T n(t)z(t,0)dt=0. \tag{15}$$

By means of the solutions of the system of variational equations (12)-(15), we can rewrite the increment formula of functional (7) in the following equivalent form:

$$J(w+\overline{w})-J(w)=\sum_{i=1}^k\left\langle\frac{\partial \Phi}{\partial y}(y(t_i,s_i),z(t_i,s_i))\right\rangle+\iint\limits_Q\left\langle\frac{\partial \widetilde{F}}{\partial y}(t,s),z(t,s)\right\rangle dtds+$$

$$+\iint\limits_Q\left\langle\frac{\partial \widetilde{F}}{\partial p}(t,s),\frac{\partial z}{\partial t}(t,s)\right\rangle dtds+\iint\limits_Q\left\langle\frac{\partial \widetilde{F}}{\partial q}(t,s),\frac{\partial z}{\partial s}(t,s)\right\rangle dtds+$$

$$+\iint\limits_Q\left\langle\frac{\partial \widetilde{F}}{\partial u}(t,s),\overline{u}(t,s)\right\rangle dtds+$$

$$+\sum_{i=1}^{k}\left[\Phi(y(t_i,s_i)+y(t_i,s_i))-\Phi(y(t_i,s_i))-\left\langle\frac{\partial\Phi}{\partial y}(y(t_i,s_i),\overline{y}(t_i,s_i))\right\rangle\right]+$$

$$+\int\limits_{Q}\int\left[F\left(t,s,y(t,s+\overline{y}(t,s),\frac{\partial y}{\partial t}(t,s)+\frac{\partial\overline{y}}{\partial t}(t,s),\right.\right.$$

$$\left.\frac{\partial y}{\partial s}(t,s)+\frac{\partial\overline{y}}{\partial s}(t,s),u(t,s)+\overline{u}(t,s)\right)-$$

$$-\widetilde{F}(t,s)-\left\langle\frac{\partial\widetilde{F}}{\partial y}(t,s),\overline{y}(t,s)\right\rangle-\left\langle\frac{\partial\widetilde{F}}{\partial p}(t,s),\frac{\partial\overline{y}}{\partial t}(t,s)\right\rangle-$$

$$\left.-\left\langle\frac{\partial\widetilde{F}}{\partial q}(t,s),\frac{\partial\overline{y}}{\partial s}(t,s)\right\rangle-\left\langle\frac{\partial\widetilde{F}}{\partial u}(t,s),\overline{u}(t,s)\right\rangle\right]dtds+$$

$$+\sum_{i=1}^{k}\left\langle\frac{\partial\Phi}{\partial y}(y(t_i,s_i)),\overline{y}(t_i,s_i)-z(t_i,s_i)\right\rangle+\int\limits_{Q}\int\left[\left\langle\frac{\partial\widetilde{F}}{\partial y}(t,s),\overline{y}(t,s)-z(t,s)\right\rangle+\right.$$

$$\left.+\left\langle\frac{\partial\widetilde{F}}{\partial p}(t,s),\frac{\partial\overline{y}}{\partial t}(t,s)-\frac{\partial z}{\partial t}(t,s)\right\rangle+\left\langle\frac{\partial\widetilde{F}}{\partial q}(t,s),\frac{\partial\overline{y}}{\partial s}(t,s)-\frac{\partial z}{\partial s}(t,s)\right\rangle\right]dtds. \quad (16)$$

We scalarly multiply equation (12) by the function $\psi\,(t,s)$, multiply (13) scalarly by $\mu\,(t)$, (14) multiply scalarly by $\eta(s)$, (15) multiply by a constant vector λ, and consequently, integrate the obtained equalities on the rectangle Q and on the segments $[0,T]$ and $[0,l]$, put the obtained one together with (16) and introduce the system of conjugated equations:

$$\begin{gathered}\sum_{i=1}^{k}\frac{\partial\Phi}{\partial y}\left(y\left(t_i,s_i\right)\right)-\int\limits_{Q}\int\frac{\partial\widetilde{H}}{\partial y}(t,s)dtds-\\ -\int\limits_{0}^{T}\frac{\partial}{\partial y}\widetilde{H}^1(t)dt-\int\limits_{0}^{l}\frac{\partial}{\partial y}\widetilde{H}^2(s)ds+(\widetilde{n}(T))'\lambda=0,\end{gathered} \quad (17)$$

$$\begin{gathered}\sum_{i=1}^{k}E\chi(t-t_i)\frac{\partial\Phi}{\partial y}\left(y\left(t_i,s_i\right)\right)-\int\limits_{0}^{l}\int\limits_{t}^{T}\frac{\partial\widetilde{H}}{\partial y}(\tau,s)d\tau ds-\\ -\int\limits_{0}^{l}\frac{\partial}{\partial p}\widetilde{H}(t,s)ds-\int\limits_{t}^{T}\frac{\partial}{\partial y}\widetilde{H}^1(\tau)d\tau+\left(\int\limits_{t}^{T}n(\tau)d\tau\right)'\lambda+\mu(t)=0,\end{gathered} \quad (18)$$

$$\begin{gathered}\sum_{i=1}^{k}E\chi(s-s_i)\frac{\partial\Phi}{\partial y}\left(y\left(t_i,s_i\right)\right)-\int\limits_{0}^{T}\int\limits_{s}^{l}\frac{\partial\widetilde{H}}{\partial y}(t,r)dr-\\ -\int\limits_{0}^{T}\frac{\partial}{\partial q}\widetilde{H}^1(t,s)dt-\int\limits_{s}^{l}\frac{\partial}{\partial y}\widetilde{H}^2(r)dr+\eta(s)=0,\end{gathered} \quad (19)$$

$$\sum_{i=1}^{k} E\chi(t-t_i)\chi(s-s_i)\frac{\partial \Phi}{\partial y}\left(y\left(t_i,s_i\right)\right)-\int_{l}^{T}\int_{s}^{l}\frac{\partial \widetilde{H}}{\partial y}(\tau,r)d\tau dr-$$
$$-\int_{s}^{l}\frac{\partial \widetilde{H}}{\partial p}(t,r)dr-\int_{t}^{T}\frac{\partial}{\partial q}\widetilde{H}(\tau,s)d\tau+\psi(t,s)=0, \tag{20}$$

where

$$\widetilde{H}(t,s)=\left\langle \psi(t,s),\widetilde{f}(t,s)\right\rangle-\widetilde{F}\left(t,s\right),$$

$$\widetilde{H}^{1}(t)=\left\langle \mu(t),\widetilde{\varphi}(t)\right\rangle, \quad \widetilde{H}^{2}(s)=\left\langle \eta(s),\widetilde{\psi}(s)\right\rangle.$$

$\chi(t)$ is Heaviside's function. E is a unit matrix.

As a result, we have the following equality:

$$J(w+\overline{w})-J(w)=-\int_{Q}\int\left\langle \frac{\partial}{\partial u}\widetilde{H}(t,s),\overline{u}(t,s)\right\rangle dtds-$$
$$-\int_{0}^{T}\left\langle \frac{\partial}{\partial v}\widetilde{H}^{1}(t),\overline{v}(t)\right\rangle dt-\int_{0}^{l}\left\langle \frac{\partial}{\partial \omega}\widetilde{H}^{2}(s),\overline{\omega}(s)\right\rangle ds+R, \tag{21}$$

where R is a residual formula of functional increment and is determined by the equality

$$R=\sum_{i=1}^{k}\left[\Phi\left(y(t_i,s_i)+\overline{y}(t_i,s_i)\right)-\Phi(y(t_i,s_i))-\right.$$
$$\left.-\left\langle \frac{\partial \Phi}{\partial y}\left(y(t_i,s_i)\right),\overline{y}(t_i,s_i)\right\rangle\right]+\sum_{i=1}^{k}\left\langle \frac{\partial \Phi}{\partial y}\left(y(t_i,s_i)\right),\overline{y}(t_i,s_i)-z(t_i,s_i)\right\rangle+$$
$$+\int_{Q}\int\left[F\left(t,s,y(t,s)+\overline{y}(t,s),\frac{\partial y}{\partial t}(t,s)+\right.\right.$$
$$\left.+\frac{\partial \overline{y}}{\partial t}(t,s),\frac{\partial y}{\partial s}(t,s)+\frac{\partial \overline{y}.}{\partial s}(t,s),u(t,s)+\overline{u}(t,s)\right)-$$
$$\left.-\left\langle \frac{\partial \widetilde{F}}{\partial q}(t,s),\frac{\partial \overline{y}.}{\partial s}(t,s)\right\rangle-\left\langle \frac{\partial \widetilde{F}}{\partial u}(t,s),\overline{u}(t,s)\right\rangle\right]dtds+$$
$$+\int_{Q}\int\left\langle \frac{\partial \widetilde{F}}{\partial y}(t,s),\overline{y}(t,s)-z(t,s)\right\rangle+\left\langle \frac{\partial \widetilde{F}}{\partial p}(t,s),\frac{\partial \overline{y}.}{\partial t}(t,s)-\frac{\partial z.}{\partial t}(t,s)\right\rangle+$$
$$\left.+\left\langle \frac{\partial \widetilde{F}}{\partial q}(t,s),\frac{\partial \overline{y}.}{\partial s}(t,s)-\frac{\partial z.}{\partial s}(t,s)\right\rangle\right]dtds. \tag{22}$$

Gradient in an optimal control problem. Introduce the following conditions:

XVI) Let the function $f(t,s,y,p,q,u)$ satisfy the condition

$$\Big| f\left(t,s,y+\overline{y},p+\overline{p},q+\overline{q},u+\overline{u}\right)-\widetilde{f}(t,s)-$$

$$-\frac{\partial f}{\partial y}(t,s)\overline{y}-\frac{\partial f}{\partial p}(t,s)\overline{p}-\frac{\partial f}{\partial q}(t,s)\overline{q}-\frac{\partial f}{\partial u}(t,s)\overline{u}\Big|\leq$$

$$\leq k\left(\left|\overline{y}\right|^2+\left|\overline{p}\right|^2+\left|\overline{q}\right|^2+\left|\overline{u}\right|^2\right);$$

XVII) Let the function $\varphi\left(t,y,v\right)$ satisfy the condition

$$\left|\varphi\left(t,y+\overline{y},v+\overline{v}\right)-\widetilde{\varphi}(t)-\frac{\partial}{\partial y}\widetilde{\varphi}(t)\overline{y}-\frac{\partial}{\partial v}\widetilde{\varphi}(t)v\right|\leq\overline{L}\left(\left|\overline{y}\right|^2+\left|\overline{v}\right|^2\right);$$

XVIII) Let the function $\psi\left(s,y,\omega\right)$ satisfy the condition

$$\left|\psi\left(s,y+\overline{y},\omega+\overline{\omega}\right)-\psi(s,y,\omega)-\frac{\partial}{\partial y}\psi(s,y,\omega)\overline{y}-\frac{\partial}{\partial \omega}\psi(s,y,\omega)\overline{\omega}\right|\leq\overline{N}\left(\left|\overline{y}\right|^2+\left|\overline{\omega}\right|^2\right);$$

XIX) Let the function $\Phi\left(y\right)$ satisfy the condition

$$\left|\Phi\left(y+\overline{y}\right)-\Phi\left(y\right)-\left\langle\frac{\partial\Phi}{\partial y}(y),\overline{y}\right\rangle\right|\leq M\left|y-\overline{y}\right|^2;$$

XX) Let the function $F\left(t,s,y,p,q,u\right)$ satisfy the condition

$$\Big|F(t,s,y+\overline{y},p+\overline{p},q+\overline{q},u+\overline{u})-\widetilde{F}(t,s)-$$

$$-\left\langle\frac{\partial\widetilde{F}(t,s)}{\partial y}\right\rangle-\left\langle\frac{\partial\widetilde{F}(t,s)}{\partial p}\right\rangle-\left\langle\frac{\partial\widetilde{F}(t,s)}{\partial q},q\right\rangle-\left\langle\frac{\partial\widetilde{F}(t,s)}{\partial u},u\right\rangle\Big|\leq$$

$$\leq\overline{M}\left(\left|\overline{y}\right|^2+\left|\overline{p}\right|^2+\left|\overline{q}\right|^2+\left|\overline{u}\right|^2\right)$$

for all

$$\left(y,v\right),\ \ \left(y+\overline{y},v+\overline{v}\right)\in R^n\times R^m,\quad \left(y,\omega\right),\ \ \left(y+\overline{y},\omega+\overline{\omega}\right)\in R^n\times R^q,$$

$$\left(y,p,q,u\right),\quad \left(y+\overline{y},p+\overline{p},q+\overline{q},u+\overline{u}\right)\in R^{3n}\times R^r.$$

For calculating the gradient of functional (6) under constraints (1)-(5) it is enough to show that the residual formula of functional (21) has order $O\left(\left\|\overline{w}\right\|^2\right)$. Using conditions I)- XIII), one can prove that there exist the non-negative numbers $\sigma_1,\ \sigma_2,\ \sigma_3$ such that

$$\max_{[0,T]}\left|\overline{y}(t,0)-z(t,0)\right|\leq\sigma_1\left\|\overline{v}\right\|^2, \tag{23}$$

$$\max_{[0,l]}\left|\overline{y}(0,s)-z(0,s)\right|\leq\sigma_2\left(\left\|\overline{v}\right\|^2+\left\|\overline{\omega}\right\|^2\right), \tag{24}$$

$$\max_{Q}|\overline{y}(t,s)-z(t,s)|+\underset{[0,l]}{vrai\max}\int\limits_0^T\left|\frac{\partial}{\partial t}\left(\overline{y}(t,s)-z(t,s)\right)\right|dt+$$
$$+\underset{[0,T]}{vrai\max}\int\limits_0^l\left|\frac{\partial}{\partial s}\left(\overline{y}(t,s)-z(t,s)\right)\right|ds\le\sigma_3\left[\|\overline{u}\|^2+\|\overline{v}\|^2+\|\overline{\omega}\|^2\right] \tag{25}$$

Taking into account estimates (23)-(25)and conditions XIV)-XX), from (22) we can obtain the estimation

$$|R|\le C\left|\overline{w}\right|^2,$$

where C is a constant independent of control parameters. Thus, we proved

Theorem 1. *Let conditions 1)-XX) be fulfilled. Then functional (6) under constraints (1)-(5) is differentiable and its gradient is of the form:*

$$J'(w)=-\left(\frac{\partial\widetilde{H}}{\partial u}(t,s),\frac{\partial\widetilde{H}^1}{\partial v}(t),\frac{\partial\widetilde{H}^2}{\partial\omega}(s)\right)\in L_2^r(Q)\times L_2^m\left([0,T]\right)\times L_2^q\left([0,l]\right).$$

Necessary optimality conditions. Having the gradient formula for functional (6) under constraints (1)-(5) we can obtain necessary optimality conditions for optimal control problem (1)-(6).

Theorem 2. *Let $w_*=(u_*(t,s),v_*(t),\omega_*(s))\in W$ be an optimal control in problem (1)-(6). Then it holds the inequality*

$$\iint\limits_Q\left\langle\frac{\partial H}{\partial u}\left(t,s,y\left(t,s;w_*\right),\frac{\partial y}{\partial t}(t,s;w_*),\right.\right.$$
$$\left.\left.\frac{\partial y}{\partial s}(t,s;w_*),\psi(t,s;w_*),u_*(t,s)\right),\overline{u}(t,s)\right\rangle\, dtds+$$
$$+\int\limits_0^T\left\langle\frac{\partial H^1}{\partial v}\left(t,y(t,0;w_*),\mu(t;w_*),v_*(t)\right),\overline{v}(t)\right\rangle dt+$$
$$+\int\limits_0^l\left\langle\frac{\partial H^2}{\partial\omega}\left(s,y(0,s;w_*),\eta(s;w_*),\omega_*(s)\right),\overline{\omega}(s)\right\rangle ds\le 0,$$

where $y(t,s;w_*)$ is a solution of boundary value problem (1)-(6) for $w_*=(u_*,v_*,\omega_*)\in W=U\times V\times\Omega$, and the triple $\left(\psi\left(t,s;w_*\right),\mu(t;w_*),\eta\left(s;w_*\right)\right)$ be a solution of conjugated system (17)-(20) that corresponds to the control $w_*=(u_*,v_*,\omega_*)\in W$. If $w_*\in int\ W$, the last inequality is equivalent to

$$\frac{\partial H}{\partial u}\left(t,s,y\left(t,s;w_*\right),\ \frac{\partial y}{\partial t}(t,s;w_*),\frac{\partial y}{\partial s}(t,s;w_*),\psi(t,s;w_*),u_*(t,s)\right)=0,$$
$$\frac{\partial H^1}{\partial v}\left(t,y(t,0;w_*),\mu(t;w_*),v_*(t)\right)=0,$$
$$\frac{\partial H^2}{\partial\omega}\left(s,y(0,s;w_*),\eta(s;w_*),\omega_*(s)\right)=0.$$

The proof is carried out by means of the scheme from [20, p. 524].

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