Maqsud A. NAJAFOV

MATHEMATICAL STATEMENT OF THE FLUTTER PROBLEM

Abstract

In the paper, mathematical statement of the flutter problem is given, and taking into account momentless state in place of the main state, the obtained system of equations is investigated.

The refined statement of the flutter formula on the base of the new expressions of the parameters of the shell's basic state [2] was cited in [1]. In the present paper we give a mathematical statement of the flutter formula and carry out analysis of the obtained system of equations assuming that a momentoes state was taken in place of the basic state.

Let an annular conic shell in a spherical system of coordinates $r,\ \theta,\ \alpha$ occupy a part $r \leq r \leq r_2$ of the conic surface

$$\{0 \leq r < \infty; \theta = \alpha; 0 \leq \alpha \leq 2\pi\},$$

gas flows interior to the cone in the positive direction of the axis r, and unperturbed flow is assumed to be radial stationary, its parameters-the velocity $u_0(r)$, density $\rho_0(r)$, pressure $p_0(r)$, local velocity $a_0(r)$ are the known functions of the radius.

The flow is supersonic, $M^2 >> 1$, provided small conicity $\alpha << 1$, we'll identify the coordinate r with the coordinate x, counted off from the vertex of the cone.

Further, we consider the shell as elastic, its mechanical characteristics: E is Young's modulus, v is a Poisson ratio, ρ is density, cylindrical rigidity $D = Eh^3/(12(1-v^2))$, where h is the shell's thickness.

We'll describe the stress-strain state of the shell by the equations of technical theory

$$D\Delta^2 w - \frac{1}{xtg\alpha}\frac{\partial^2 F}{\partial x^2} - L(w,F) = p - gh\frac{\partial^2 w}{\partial t^2} \tag{1}$$

$$\Delta^2 F + \frac{Eh}{xtg\alpha}\frac{\partial^2 w}{\partial x^2} - \frac{1}{2}L(w,w) = 0 \tag{2}$$

cited in [3], where

$$L(u,v) = \frac{\partial^2 u}{\partial x^2}\left(\frac{1}{x^2}\frac{\partial^2 v}{\partial \psi^2} + \frac{1}{x}\frac{\partial v}{\partial x}\right) + \frac{\partial^2 v}{\partial x^2}\left(\frac{1}{x^2}\frac{\partial^2 u}{\partial \psi^2} + \frac{1}{x}\frac{\partial v}{\partial x}\right) -$$

$$-2\left(\frac{1}{x}\frac{\partial^2 v}{\partial x \partial \psi} - \frac{1}{x^2}\frac{\partial v}{\partial \psi}\right)\left(\frac{1}{x}\frac{\partial^2 u}{\partial x \partial \psi} - \frac{1}{x^2}\frac{\partial u}{\partial \psi}\right)$$

$\psi = \varphi \sin\alpha$ is Δ is a Laplace operator.

We look for the solution of the nonlinear system in the form

$$w = w_0(x) + w_1(x,\varphi,t),\ \ F = F_0(x) + F_1(x,\varphi,t)$$

where $w_0(x),\ F_0(x)$ is assumed to be the main (quasistatic), $w_1(x,\varphi,t),\ F_1(x,\varphi,t)$ the perturbed (dynamical) state.

Substituting these expansions in (1), (2) and linearizing in small perturbations, after some simplifications we get the following systems.

For the main state

$$D\Delta^2 w_0 - \frac{1}{xtg\alpha}\frac{\partial^2 F_0}{\partial x^2} = p_0(x) \quad (3)$$

$$\Delta^2 F_0 + \frac{Eh}{xtg\alpha}\frac{\partial^2 w_0}{\partial x^2} = 0. \quad (4)$$

For the perturbed state

$$D\Delta^2 w_1 - \frac{1}{xtg\alpha}\frac{\partial^2 F_1}{\partial x^2} - L(w_1, F_0) = \Delta p_1(x,t) - gh\frac{\partial^2 w_1}{\partial t^2} \quad (5)$$

$$\Delta^2 F_1 + \frac{Eh}{xtg\alpha}\frac{\partial^2 w_1}{\partial x^2} = 0. \quad (6)$$

In system (3), (4), $p_0(x)$ is quasistatic pressure, the surplus pressure $\Delta p_1(x,t)$ in system (5), (6) was determined by us in [4,5].

Since the main state is momentless, we'll not investigate system (3), (4).

For investigating system (5), (6) of the perturbed state, we introduce a new coordinate assuming

$$x = x_1 + y_1,\ 0 \le y_1 \le l,\ l = x_2 - x_1$$

and appropriate dimensionless coordinate

$$x/l = x_1/l + y_1/l \equiv x_0 + y * 0 \le y \le 1.$$

We also introduce dimensionless deflection functions W_1, and forces $\Phi_1 = F_1/\left(Eh^2 l\right)$ assuming

$$W_1 = w_1/h, \quad \Phi_1 = F_1/(Eh^2 l).$$

We'll consider axisymmetric perturbed state. In this case the operator $L(w_1, F_0)$ takes the form

$$L(w_1, F_0) = \frac{\partial^2 w_1}{\partial x^2}\cdot\frac{1}{x}\cdot\frac{\partial F_0}{\partial x} + \frac{1}{x}\cdot\frac{\partial w_1}{\partial x}\cdot\frac{\partial^2 F_0}{\partial x^2}. \quad (7)$$

By definition, the force functions

$$\frac{1}{x}\cdot\frac{\partial F_0}{\partial x} = T_1; \quad \frac{\partial^2 F_0}{\partial x^2} = T_2, \quad (8)$$

where T_1, T_2 are forces in median surface determined as follows (see [6]):

$$T_2 = p_0(x)xtg\alpha$$

$$T_1 = -\frac{tg\alpha}{x}\int_x^{x_z} p_0(x)x dx. \quad (9)$$

Then from (6) we get the equation

$$\Delta^2 \Phi_1 + \frac{1}{tg\alpha(x_0 + y)}\frac{\partial^2 W_1}{\partial y^2} = 0. \quad (10)$$

But we can write these expressions of forces x in dimensionless coordinates in the form

$$T_1=-\frac{A_0}{2(\gamma-1)}\frac{1}{(x_0+1)^{2\gamma-1}}\left(\left(\frac{x_0+1}{x_0+y}\right)^{2\gamma-1}-\frac{x_0+1}{x_0+y}\right) \quad (11)$$

$$T_2=\frac{A_0 l}{(x_0+y)^{2\gamma-1}},$$

where

$$A_0=\left(\frac{\gamma-1}{\gamma+1}\right)^{\gamma/2}p_{\widehat{e\partial}}tg\alpha\cdot x_{\widehat{e\partial}}^{2\gamma}. \quad (12)$$

Substitute (7), (8) and (11) in (5), then in dimensionless parameters we get:

$$\frac{Dh}{t^4}\Delta^2W_1-\frac{Eh^2}{l^2tg\alpha(x_0+y)}\Phi_1'+\frac{A_0}{2(\gamma-1)}\frac{h}{(x_0+1)^{2\gamma-1}}\times$$

$$\times\left(\left(\frac{x_0+1}{x_0+y}\right)^{2\gamma-1}\left(\frac{x_0+1}{x_0+y}\right)\right)W_1''-\frac{A_0}{l(x_0+y)^{2\gamma-1}}W_1'=\Delta p_1-\rho h^2\frac{\partial^2W_1}{\partial t^2}. \quad (13)$$

We look for the solution of system (10), (13) in the form

$$W_1=W(y)\exp(\omega t),\quad \Phi_1=\Phi(y)\exp(\omega t).$$

Therewith, on the base of the results from [7,8], the surplus pressure $\Delta\rho_1$ takes the form

$$\Delta p_1=\Delta q\exp(\omega t),$$

where

$$\Delta q=-\frac{\gamma p^*}{l}\left[\frac{l\omega}{a^*}\left(\frac{2}{(\gamma+1)}\right)^{k_1}\cdot\frac{x_{\widehat{e\partial}}^2}{Mx^2}W+\left(\frac{2}{(\gamma+1)}\right)^{k_2}\cdot\left(\frac{x_{\widehat{e\partial}}^2}{x^2}\right)^{k_3}\frac{1}{M^{\varepsilon}}\frac{dW}{dx}+\right.$$

$$\left.+\left(\frac{2}{(\gamma+1)}\right)^{k_2}\frac{1}{2xtg\alpha}\left(\frac{x_{\widehat{e\partial}}^2}{Mx^2}\right)^{k_3}W-\frac{2^{k_2}}{(\gamma+1)^{k_4}}\frac{1}{(M^2-1)x_{\widehat{e\partial}}}\left(\frac{x_{\widehat{e\partial}}}{x}\right)^3W\right]. \quad (14)$$

Since in formula (14) the flexure W is dimensional quantity, we should make the substitution $W\Longrightarrow hW$.

Dividing the both parts of equation (13) by Dh/l^4, we get

$$\Delta^2W-\frac{B}{x_0+y}\Phi''+\frac{B_1}{2(\gamma-1)(x_0+y)}\left(\left(\frac{x_0+1}{x_0+y}\right)^{2\gamma-1}-\frac{x_0+1}{x_0+y}\right)W''-$$

$$-\frac{B_1}{(x_0+y)^{2\gamma-1}}W'=\frac{12(1-v^2)l^4\Delta q}{Eh^4}-B_2\Omega^2W. \quad (15)$$

Here we introduced the denotation:

$$B_0=\frac{12(1-v^2)l^2}{h^2};\ \Omega=\frac{l\omega}{a_{\widehat{e\partial}}} \quad (16)$$

$$B_1=\frac{12(1-v^2)l^3A_0}{Eh^3} \quad (17)$$

$$B_2=\frac{12(1-v^2)l^2a^2_{\widehat{e\partial}}}{h^2\widetilde{n}^2},\ \widetilde{n}^2=E/\rho, \tag{18}$$

the primes denote the derivatives with respect to y. Using (14), assign each of the addends of the expression

$$\frac{12(1-v^2)l^4\Delta q}{Eh^4} \tag{19}$$

taking into account that x is dimensionless, so $x=x_0+y$;

(I) $\dfrac{12(1-v^2)\gamma p_{\widehat{e\partial}}l^3}{Eh^3}\frac{x_{\widehat{e\partial}}}{M(x_0+y)^2}\Omega W=\dfrac{A_1x_{\widehat{e\partial}}}{M(x_0+y)^2}\Omega W;$

(II) $\dfrac{A_1x^{2k_3}_{\widehat{e\partial}}}{M^{\varepsilon}(x_0+y)^{2k_3}}W';$

(III) $\dfrac{A_1x^{2k_3}_{\widehat{e\partial}}}{2tg\alpha M^{k_3}(x_0+y)^{k_5}}W;\ \ k_5=\dfrac{3\gamma+1}{\gamma+1};$

(IV) $\dfrac{A_1(\gamma+1)^{k_6}x^2_{\widehat{e\partial}}}{M^2(x_0+y)^3}W;\ \ k_6=\dfrac{2\gamma-1}{2(\gamma 11)},$

where

$$A_1=\frac{12(1-v^2)\gamma p_{\widehat{e\partial}}l^3}{Eh^3}. \tag{20}$$

Note that two sequential addends are the same by mechanical sense-of Winkler base type, bus they are opposite in sign. It makes sense to compare (in modulus) these addends. We have

$$\frac{(III)}{(IV)}=\frac{x^{2\varepsilon}_{\widehat{e\partial}}}{2tg\alpha}\left[\frac{M(x_0+y)}{\gamma+1}\right]^{k_6}. \tag{21}$$

This result is expected: for small conisity and large M this ratio will greater than a unit.

Substitute expressions (I)-(IV) in (19), insert all these in (15). As a result we get:

$$\Delta^2W-\frac{B_0}{x_0+y}\Phi''+\frac{B}{2(\gamma-1)(x_0+1)}\left(\left(\frac{x_0+1}{x_0+y}\right)^{2\gamma-1}-\frac{x_0+1}{x_0+y}\right)W''-$$

$$-\frac{B_1}{(x_0+y)^{2\gamma-1}}W'+\frac{A_1x_{\widehat{e\partial}}}{M(x_0+y)^2}\Omega W=\frac{A_1x^{2k_3}_{\widehat{e\partial}}}{M^{\varepsilon}(x_0+y)^{2k_3}}W'+$$

$$+\frac{A_1x^{2k_3}_{\widehat{e\partial}}}{2tg\alpha M^{k_3}(x_0+y)^{k_5}}W-\frac{A_1(\gamma+1)^{k_6}x^2_{\widehat{e\partial}}}{M^2(x_0+y)^3}W+B_2\Omega^2W=0. \tag{22}$$

Equation (10) is written in the form

$$\Delta^2\Phi+\frac{1}{tg\alpha(x_0+y)}W''=0. \tag{23}$$

Together with boundary conditions, system (22), (23) composes an eigen value problem. A problem on critical parameters of the flutter, as it was noted in [4], will be determined by the "surface" $\operatorname{Re}\Omega_n(M_{\min})=0$ that defines the area of stable and unstable vibrations. By definition, $M_{\min}=M_{\widehat{e\partial}}$.

In nozzle flow aerodynamics, the Mach number in critical cross section is assumed to be critical; by definition it equals unit. Here, the value of the Mach number on the left end of the shell will be considered as critical, i.e. $M_{\widehat{e\partial}} = M_{\widehat{e\partial}}(x_0) = M_0^{\widehat{e\partial}}$.

Based around such assumptions make an analysis of system (22), (23).

Pay a special attention to equation (22). Rewrite it in the form

$$\Delta^2 W - \frac{B_0}{x_0+y}\Phi'' + \frac{B_1}{2(\gamma-1)(x_0+1)}\left(\left(\frac{x_0+1}{x_0+y}\right)^{2\gamma-1} - \frac{x_0+1}{x_0+y}\right)W'' -$$

$$-\left(\frac{B_1}{(x_0+y)^{2\gamma-1}} - \frac{A_1 x_{\widehat{e\partial}}^{2k_3}}{M^{\varepsilon}(x_0+y)^{2k_3}}\right)W' + \frac{A_1 x_{\widehat{e\partial}}^{2k_3}}{2tg\alpha M^{\varepsilon}(x_0+y)^{k_5}}\times$$

$$\times(1-\psi(y))W + \left(\frac{A_1 x_{\widehat{e\partial}}}{M(x_0+y)^2}\Omega + B_2\Omega^2\right)W = 0, \tag{24}$$

where

$$\psi(y) = \frac{2tg\alpha}{x_{\widehat{e\partial}}^{2\varepsilon}}\frac{\gamma+1}{M(x_0+y)^{k_6}}. \tag{25}$$

As we noted above, the critical velocity of the flutter is defined by the number $M_0^{\widehat{e\partial}}$, where M_0 is the value of the Mach number on the left end of the shell $x = x_0$.

Since for great supersonic velocities $(\gamma-1)\,M^2 >> 1$, then for this case we can write the main relations $\rho^*/\rho = (T^*/T)^{\frac{1}{\gamma-1}}$ as follows

$$\frac{x}{x_{\widehat{e\partial}}} = \varepsilon^{k_1/2}M^{k_7}, \quad k_7 = k_1 - \frac{1}{2} = \frac{1}{\gamma-1}. \tag{26}$$

For $x = x_0$ hence we have

$$\frac{x_0}{x_{\widehat{e\partial}}} = \varepsilon^{k_1/2}M_0^{k_7}. \tag{27}$$

Then for x_{cr} from (26) we have

$$M = M_0\left(\frac{x}{x_0}\right)^{1/k_7} = M_0\left(\frac{x_0+y}{x_0}\right)^{1/k_7} \equiv M_0 g(y). \tag{28}$$

From (27) define x_{cr}

$$x_{\widehat{e\partial}} = \frac{x_0}{\varepsilon^{k_1/2}M_0^{k_7}}. \tag{29}$$

Obviously, $x_{cr} < x_0$. Therefore in (29) there should be

$$\varepsilon^{k_1/2}M_0^{k_7} > 1. \tag{30}$$

Note that for concrete calculations it is necessary to follow that this is fulfilled.

Allowing for (28), (29), the addends in (24) may be written as:

$$(I)\frac{A_1 x_{\widehat{e\partial}}^{2k_3}}{M^{\varepsilon}(x_0+y)^{2k_3}} = \frac{A_1}{\varepsilon^{k_2}}\left(\frac{x_0}{x_0+y}\right)^{k_8}\frac{1}{M_0^{1/\varepsilon}} \equiv \frac{\alpha_1}{(x_0+y)^{k_8}M_0^{1/\varepsilon}};\ k_8 = \frac{\gamma^2+4\gamma-1}{\gamma+1}$$

$$(II)\frac{A_1 x_{\widehat{e\partial}}^{2k_3}}{2tg\alpha M^{k_3}(x_0+y)^{k_5}} = \frac{A_1\alpha_0^{3k_3}}{2\varepsilon^{k_2}tg\alpha}\frac{1}{(x_0+y)^{k_9}M_0^{k_8/2}} \equiv \frac{\alpha_2}{(x_0+y)^{k_9}M_0^{k_8/2}}; k_9 = \frac{5\gamma+1}{\gamma+1}$$

$$(III)\frac{A_1 x_{\widehat{e\partial}}}{M(x_0+y)^2}=\frac{A_1 x_0^{\gamma}}{\varepsilon^{k_1/2}}\frac{1}{(x_0+y)^{\gamma+1}M_0^{k_2}}\equiv\frac{\alpha_3}{(x_0+y)^{\gamma+1}M_0^{k_2}};$$

$$(IV)\left|\begin{array}{l}\psi=\dfrac{(\gamma+1)tg\alpha\sqrt{\varepsilon}}{x_0^{k_{11}}}\dfrac{M_0^{k_{10}}}{(x_0+y)^{k_{12}}}\equiv\dfrac{\alpha_4 M_0^{k_{10}}}{(x_0+y)^{k_{12}}};\\ k_{10}=\dfrac{2-\gamma}{\gamma-1};\quad k_{11}=\dfrac{(\gamma-1)^2}{\gamma+1};\quad k_{12}=\dfrac{1+(\gamma-1)^2}{2(\gamma-1)}.\end{array}\right.$$

Substituting these expressions in (24), we finally get:

$$\Delta^2 W-\frac{B_0}{x_0+y}\Phi''+\frac{B}{2(\gamma-1)(x_0+1)}\left(\left(\frac{x_0+1}{x_0+y}\right)^{2\gamma-1}-\frac{x_0+1}{x_0+y}\right)-$$

$$-\left(\frac{B_0}{(x_0+y)^{2\gamma-1}}-\frac{\alpha_1}{(x_0+y)^{k_8}M^{1/\varepsilon}}\right)W'+$$

$$+\frac{\alpha_1}{(x_0+y)^{k_8}M^{k_8/2}}\left(1-\frac{\alpha_4 M^{k_{10}}}{(x_0+y)^{k_{12}}}\right)W+\left(\frac{\alpha_3}{(x_0+y)^{\gamma+1}M^{k_2}}\Omega+B_2\Omega^2\right)W=0.$$

Thus, we get the system of equations (23), (31) that together with homogeneous boundary conditions makes up an eigen value problem.

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Maqsud A. Najafov
Azerbaijan State Pedogogical University
34, U. Hajibeyov str., AZ1000, Baku, Azerbaijan.
Tel.: (99412) 493 33 69 (off.).

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