MATHEMATICS

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DETERMINATION OF THE COEFFICIENT IN THE RIGHT SIDE OF THE SYSTEM OF REACTION –DIFFUSION TYPE IN THE PROBLEM WITH A NONLINEAR BOUNDARY CONDITION

Abstract

The goal of the paper is to investigate the well-posedness of the inverse problem on determination of the coefficient in the right side of the system of reaction-diffusion type in the problem with a nonlinear boundary condition. The additional condition is given in nonlocal-integral form. A theorem on uniqueness and 'conditional' stability of the considered problem is proved.

Let R^n be a real n-dimensional Euclidean space, $x = (x_1, ..., x_n)$ be an arbitrary point of a bounded domain $D \subset R^n$ with rather smooth boundary ∂D, $\Omega = D \times (0;T]$, $S = \partial D \times [0;T]$, $0 < T$ be a fixed number.

The spaces $C^l(\cdot)$, $C^{l+\alpha}(\cdot)$, $C^{l,l/2}(\cdot)$, $C^{l+\alpha,(l+\alpha)/2}(\cdot)$, $l = 0,1,2$, $\alpha \in (0,1)$ and the norms in these spaces were determined for instance in [1, p. 12-20]

$$u = (u_1, ..., u_m), \quad \|u\|_l = \sum_{k=1}^{m} \|u_k\|_{C^l}, \quad u_{kt} = \frac{\partial u_k}{\partial t}, \quad u_{kx_i} = \frac{\partial u_k}{\partial x_i}, \quad i = \overline{1,n},$$

$\Delta u_k = \sum\limits_{i=1}^{n} \dfrac{\partial^2 u_k}{\partial x_i^2}$ is the Laplace operator, $\dfrac{\partial u_k}{\partial v}$ is an internal conormal derivative.

We consider the inverse problem on determination of the pair of functions $\{f_k(t),\ u_k(x,t),\ k = \overline{1,m}\}$ from the conditions:

$$u_{kt} - \Delta u_k = f_k(t) g_k(x,t,u), \qquad (x,t) \in \Omega, \tag{1}$$

$$u_k(x,0) = \varphi_k(x), \quad x \in \overline{D} = D \cup \partial D \tag{2}$$

$$\frac{\partial u_k}{\partial v} = \psi_k(x,t,u), \quad (x,t) \in S \tag{3}$$

$$\int_D u_k(x,t)dx = h_k(t), \ \ t \in [0,T], \tag{4}$$

where $g_k(x,t,p)$, $\varphi_k(x)$, $\psi_k(x,t,p)$, $h_k(t)$ are the given functions.

The coefficient inverse problems were studied in the papers [2-4] (see also references in these papers).

For the input data of problem (1)-(4) we make the following assumptions:

1^0. $g_k(x,t,p) \in C^{\alpha,\alpha/2}_{x,t}(\overline{\Omega} \times R^m)$ there exists $c_1 > 0$, such that for any $(x,t) \in \overline{\Omega}$ and $p_1, p_2 \in R^m : |g_k(x,t,p_1) - g_k(x,t,p_2)| \le c_1 |p_1 - p_2|$;

2^0. $\varphi_k(x) \in C^{2+\alpha}(\overline{D})$;

3^0. $\psi_k(x,t,p) \in C^{\alpha,\alpha/2}_{x,t}(\overline{\Omega} \times R^m)$, there exists $c_2 > 0$ such that for any

$(x,t)\in\overline{\Omega}$ and $p_1,p_2\in R^m : |\psi_k(x,t,p_1)-\psi_k(x,t,p_2)|\le c_2\,|p_1-p_2|$;

4^0. $h_k(t)\in C^{2+\alpha}[0,T]$.

Definition 1. *We call the pair of functions* $\{f_k(t),u_k(x,t)\}$ *the solution of problem (1)-(4) if:*

1) $f_k(t)\in C[0,T]$;

2) $u_k(x,t)\in C^{2,1}(\Omega)\cap C^{1,0}(\overline{\Omega})$;

3) relations (1)-(4) are satisfied for these functions, and conditions (3) is satisfied in the following way:

$$\frac{\partial u_k(x,t)}{\partial v(x,t)}=\lim_{\substack{y\to x\\ y\in\sigma}}\frac{\partial u_k(y,t)}{\partial v(x,t)},$$

where σ *is any closed cone with the vertex* x*, contained in* $D\cup\{x\}$.

The uniqueness theorem and also estimation of stability of the inverse problems solution occupies a central place in investigation of their well-posedness issues. In the paper, under more general assumptions, the uniqueness of the solution of problem (1)-(4) is proved and the estimation characterizing the "conditional" stability of the solution is established.

Let $\left\{f_k^i(t),u_k^i(x,t),\ k=\overline{1,m}\right\}$ be the solution of problem (1)-(4) corresponding to the data $g_k^i(x,t,u^i),\varphi_k^i(x),\psi_k^i(x,t,u^i),h_k^i(t),\ i=1,2$.

Definition 2. *Say that the solution of problem (1)-(4) is stable if for any* $\varepsilon>0$ *there will be found such* $\delta(\varepsilon)>0$ *that for* $\left\|g^1-g^2\right\|<\delta$, $\left\|\varphi^1-\varphi^2\right\|<\delta,\ \left\|\psi^1-\psi^2\right\|<\delta,\ \left\|h^1-h^2\right\|<\delta$ *the inequality* $\left\|u^1-u^2\right\|+\left\|f^1-f^2\right\|\le\varepsilon$ *is fulfilled.*

Theorem. *Let:*

1. $g_k^i(x,t,u^i),\ \varphi_k^i(x),\ \psi_k^i(x,t,u^i),\ h_k^i(t),\ k=\overline{1,m}, i=1,2$ *satisfy conditions* 1^0-4^0*, respectively;*

2. there exist the solutions $\left\{f_k^i(t),u_k^i(x,t),\ k=\overline{1,m},\ i=1,2\right\}$ *of problem (1)-(4) in the sense of definition 1, and they belong to the set*

$$K_\alpha=\left\{(f,u)\,\middle|\,f_k(t)\in C^\alpha[0,T],\ u_k(x,t)\in C^{2+\alpha,1+\alpha/2}\left(\overline{\Omega}\right),\ k=1,m\right\}.$$

Then there exists $T^*>0$ *such that for* $(x,t)\in\overline{D}\times[0,T^*]$ *the solution of problem (1)-(4) is unique and the following stability estimation is true:*

$$\left\|u^1-u^2\right\|_0+\left\|f^1-f^2\right\|_0\le$$
$$\le c_3\left[\left\|g^1-g^2\right\|_0+\left\|\varphi^1-\varphi^2\right\|_2+\left\|\psi^1-\psi^2\right\|_0+\left\|h^1-h^2\right\|_1\right],\qquad(5)$$

where $c_3>0$ *depends on the data of problem (1)-(4) and the set* K_α.

Proof of the theorem. At first prove the validity of estimation (5). In order to get the uniqueness theorem , in the reasonings conducted below the input data should be everywhere supposed to be identically equal to zero.

Allowing for (2), from equation (1) and the theorem's conditions, for the function $f_k(t)$ we get

$$f_k(t)=\left[h_k'(t)-\int\limits_{\partial D}\frac{\partial u_k}{\partial v}d\eta\right]\Big\backslash\int\limits_D g_k(x,t,u)dx,\quad t\in[0;T],\qquad(6)$$

$dx = dx_1...dx_n$, $d\eta = d\eta...d\eta_n$ is the element of the surface ∂D.

Denote

$$z_k(x,t) = u_k^1(x,t) - u_k^2(x,t),\ \lambda_k(t) = f_k^1(t) - f_k^3(t),$$

$$\delta_{1k}(x,t,p) = g_k^1(x,t,p) - g_k^2(x,t,p),\ \delta_{2k}(x) = \varphi_k^1(x) - \varphi_k^2(x),$$

$$\delta_{3k}(x,t,p) = \psi_k^1(x,t,p) - \psi_k^2(x,t,p),\ \delta_{4k}(t) = h_k^1(t) - h_k^2(t).$$

We can verify that the functions $\lambda_k(t)$, $w_k(x,t) = z_k(x,t) - \delta_{2k}(x)$ satisfy the relations of the system:

$$w_{kt} - \Delta w_k = \lambda_k(t) g_k^1(x,t,u^1) + F_k(x,t), \qquad (x,t) \in \Omega, \tag{7}$$

$$w_k(x,0) = 0,\ x \in \overline{D};\ \ \frac{\partial w_k}{\partial v}(x,t) = \Psi_k(x,t), \quad (x,t) \in S \tag{8}$$

$$\lambda_k(t) = \left(-\int\limits_{\partial D} \frac{\partial z_k}{\partial v} d\eta \right) \setminus \int\limits_{D} g_k^1(x,t,u^1)dx + H_k(t), \tag{9}$$

where

$$F_k(x,t) = \Delta\delta_{2k}(x) + f_k^2(t)\delta_{1k}(x,t,u^1) + g_k^2(x,t,u^1) - g_k^2(x,t,u^2),$$

$$\Psi(x,t) = \delta_{3k}(x,t,u^1) - \frac{\partial \delta_2(x)}{\partial v} + \psi_k^2(x,t,u^1) - \psi_k^2(x,t,u^2),$$

$$H_k(t) = \left\{ \delta_{4kt}(t) \int\limits_D g_k^2(x,t,u^2)dx + \left[h_{kt}^2(t) - \int\limits_{\partial D} \frac{\partial u_k^2}{\partial v} d\eta \right] \times \right.$$

$$\left. \times \left[\int\limits_D \left(g_k^1(x,t,u^2) - g_k^1(x,t,u^1) \right) dx - \int\limits_D \delta_{1k}(x,t,u^2)dx \right] \right\} \setminus$$

$$\setminus \left[\int\limits_D g_k^1(x,t,u^1)dx \int\limits_D g_k^2(x,t,u^2)dt \right].$$

From the conditions of the theorem it follows that there exists the classic solution of problem (7),(8) on determination of $w_k(x,t)$ and it may be represented in the form [5, p. 182]:

$$w_k(x,t) = \int\limits_0^t \int\limits_D \Gamma_k(x,t;\xi,\tau) \left[\lambda_k(\tau) g_k^1(\xi,\tau,u^1) + F_k(\xi,\tau) \right] d\xi d\tau +$$

$$+ \int\limits_0^t \int\limits_{\partial D} \Gamma_k(x,t;\xi,\tau) \rho_k(\xi,\tau) d\eta d\tau, \tag{10}$$

where $\Gamma_k(x,t;\xi,\tau)$ is the fundamental solution of the equation, $w_{kt} - \Delta w_k = 0$, $d\xi = d\xi_1 ... d\xi_n$, $d\eta$ is the element of the surface ∂D, $\rho_k(x,t)$ is the continuous bounded solution of the following integral equation [5, p. 183]

$$\rho_k(x,t) = 2\int_0^t \int_{\partial D} \frac{\Gamma_k(x,t;\xi,\tau)}{\partial v(x,t)} \left[\lambda_k(\tau) g_k^1(\xi,\tau,u^1) + F_k(\xi,\tau)\right] d\xi d\tau +$$

$$+2\int_0^t \int_{\partial D} \frac{\Gamma_k(x,t;\xi,\tau)}{\partial v(x,t)} \rho_k(\xi,\tau) d\eta d\tau - 2\Psi_k(x,t). \tag{11}$$

Assume

$$\chi = \left\|u^1 - u^2\right\|_0 + \left\|f^1 - f^2\right\|_0 .$$

Estimate the function $|z_k(x,t)|$. Taking into account $z_k(x,t) = w_k(x,t) + +\delta_{2k}(x)$, from (10) we get:

$$|z_k(x;t)| \le |w_k(x,t)| + |\delta_{2k}(x)| \le |\delta_{2k}(x)| +$$

$$+\int_0^t \int_D |\Gamma_k(x,t,\xi,\tau)| \left[\left|\lambda_k(\tau) g_k^1(\xi,\tau,u^1)\right| + |F_k(\xi,\tau)|\right] d\xi d\tau +$$

$$+\int_0^t \int_{\partial D} |\Gamma_k(x,t,\xi,\tau)| \cdot |\rho_k(\xi,\tau)|\, d\eta d\tau. \tag{12}$$

For the expression $\int_D |\Gamma_k(x,t,\xi,\tau)|\, d\xi$ in the second addend of the right side of (12), the following estimation is true [5, p. 20]

$$\int_D |\Gamma_k(x,t,\xi,\tau)|\, d\xi \le c_4. \tag{13}$$

The integrand function $\left|\lambda_k(t) g_k^1(\xi,t,u^1)\right| + |F_k(x,t)|$ in the second addend of the right side of (12), by the requirements imposed on the input data and on the set K_α satisfies the estimation

$$\left|\lambda_k(t) g_k^1(x,t,u^1)\right| + |F_k(x,t)| \le \left|\lambda_k(t) g_k^1(x,t,u^1)\right| +$$

$$+ |\Delta\delta_{2k}(x)| + \left|f_k^2(t)\right| \cdot \left|\delta_{1k}(x,t,u^1)\right| + \left|g_k^2(x,t,u^1) - g_k^2(x,t,u^2)\right| \le$$

$$\le c_5 \left[\left\|g^1 - g^2\right\|_0 + \left\|\varphi^1 - \varphi^2\right\|_2\right] + c_6 \cdot \chi, \quad (x,t) \in \overline{\Omega}, \tag{14}$$

where $c_5, c_6 > 0$ depend on the data of problem (1)-(4) and the set K_α.

The expression $\int_{\partial D} |\Gamma_k(x,t;\xi,\tau)|\, d\eta$ in the third addend of the right side of (12) satisfies the estimation [5, p. 20]

$$\int_{\partial D} |\Gamma_k(x,t;\xi,\tau)|\, d\eta \le c_7. \tag{15}$$

Taking into account expressions (11), the theorem's conditions, definition of the set K_α and the following estimation [5, p. 20]:

$$\int\limits_D \left| \frac{\partial \Gamma_k(x,t;\xi,\tau)}{\partial v(x,t)} \right| d\xi \le c_8 (t-\tau)^{-\mu}, \qquad \frac{1}{2} < \mu < 1$$

for the function $\rho_k(x,t)$ we get:

$$|\rho_k(x,t)| \le c_9 \left[\|\delta_1\|_0 + \|\delta_2\|_2 + \|\delta_3\|_0 + \chi \right] + c_{10} \|\rho\|_0 \cdot t^{1-\mu}, \ (x,t) \in S,$$

where $c_9, c_{10} > 0$ depend on the data of problem (1)-(4) and the set K_α.

The last inequality is fulfilled for all $(x,t) \in \overline{D} \times [0,T]$, therefore the following estimation is true:

$$\|\rho\|_0 \le c_9 \left[\|\delta_1\|_0 + \|\delta_2\|_2 + \|\delta_3\|_0 + \chi \right] + c_{10} t^{1-\mu} \|\rho\|_0 .$$

Let $0 < T_1 \le T$ be such a number that $c_{10} T_1^{1-\mu} < 1$. Then for all $(x,t) \in \overline{D} \times [0,T_1]$ we have

$$\|\rho\|_0 \le c_{11} \left[\|\delta_1\|_0 + \|\delta_2\|_2 + \|\delta_3\|_0 + \chi \right], \tag{16}$$

where $c_{11} > 0$ depends on the data of problem (1)-(4) and the set K_α.

Taking into account inequalities (13), (14), (15) and (16) for $|z_k(x,t)|$ from (12) we get:

$$|z_k(x,t)| \le c_{12} \left[\|\delta_1\|_0 + \|\delta_2\|_2 + \|\delta_3\|_0 \right] + c_{13} \chi t, \quad (x,t) \in \overline{\Omega}, \tag{17}$$

where $c_{12}, c_{13} > 0$ depend on the data of problem (1)-(4) and the set K_α.

Now estimate the function $|\lambda_k(t)|$. From (9) it follows

$$|\lambda_k(t)| \le \int\limits_{\partial D} \left| \frac{\partial z_k}{\partial v} \right| d\eta \Big\backslash \int\limits_D \left| g_k^1(x,t,u^1) \right| dx +$$

$$+ \left\{ |\delta_{4kt}(t)| \int\limits_D \left| g_k^2(x,t,u^2) \right| dx + \left(\left| h_{kt}^2 \right| + \int\limits_{\partial D} \left| \frac{\partial u_k^2}{\partial v} \right| d\eta \right) \times \right.$$

$$\left. \times \int\limits_D \left[\left| \delta_{1k}(x,t,u^2) \right| + \left| g_k^1(x,t,u^1) - g_k^2(x,t,u^2) \right| \right] dx \right\} \Big\backslash$$

$$\Big\backslash \left[\int\limits_D \left| g_k^1(x,t,u^1) \right| dx \int\limits_D \left| g_k^2(x,t,u^2) \right| dx, \right.$$

or

$$|\lambda_k(t)| \le c_{14} \left[\left\| g^1 - g^2 \right\|_0 + \left\| h^1 - h^2 \right\|_1 \right] + c_{15} \left[|z_k| + \left| \frac{\partial z_k}{\partial v} \right| \right], \quad t \in [0,T_1]$$

$c_{14}, c_{15} > 0$ depend on the data of problem (1)-(4) and the set K_α.

Taking into account (17) and the following estimation:

$$\left|\frac{\partial z_k}{\partial v}\right| \leq c_{16}\left[\|\delta_1\|_0 + \|\delta_2\|_2 + \|\delta_3\|_0\right] + c_{17}\chi t^{1-\mu}, \quad t \in [0, T_1]$$

for $|\lambda_k(t)|$ we get:

$$|\lambda_k(t)| \leq c_{18}\left[\|\delta_1\|_0 + \|\delta_2\|_2 + \|\delta_3\|_0 + \|\delta_4\|_1\right] + c_{19}\chi t^{1-\mu}, \quad t \in [0, T_1] \tag{18}$$

care independent $c_{18}, c_{19} > 0$ depend on the data of problem (1)-(4) and the set K_α.

Inequalities (17) and (18) are satisfied for any values of $(x,t) \in \overline{D} \times [0, T_1]$.

Consequently, combining these inequalities, we get

$$\chi \leq c_{20}\left[\|\delta_1\|_0 + \|\delta_2\|_2 + \|\delta_3\|_0 + \|\delta_4\|_1\right] + c_2\chi t^{1-\mu}, \tag{19}$$

where $c_{20}, c_{21} > 0$ depend on the data of problem (1)-(4) and the set K_α.

Now let $T_2 \quad (0 < T_2 \leq T)$ be such a number that $c_{21}T_2 < 1$. Then from (19) we get that for $(x,t) \in \overline{D} \times [0, T]^*$, $\quad T^* = \min(T_1, T_2)$, stability estimation (5) for the solution of problem (1)-(4) is true.

The uniqueness of solution of problem (1)-(4) follows from estimation (5) for

$$g_k^1(x,t,u) = g_k^2(x,t,u),\ \varphi_k^1(x) = \varphi_k^2(x),\ \psi_k^1(x,t,u) = \psi_k^2(x,t,u),\ h_k^1(t) = h_k^2(t).$$

The theorem is completely proved.

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