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SOLVABILITY CONDITIONS OF A BOUNDARY VALUE PROBLEM FOR AN OPERATOR-DIFFERENTIAL EQUATION OF THE THIRD ORDER WITH DISCONTINUOUS COEFFICIENTS

Abstract

In the paper, on a semi-axis we consider a boundary value problem for a class of third order operator-differential equations with discontinuous coefficient and with some linear operator in one of the boundary conditions. Sufficient conditions of well-defined and unique solvability of the given boundary value problem in the Sobolev type space are found. These conditions are expressed by means of the properties of operator coefficients of the boundary value problem.

Let H be a separable Hilbert space, A be a positive-definite self-adjoint operator in H with domain of definition $D(A)$

Under the unit of Hilbert spaces generated by the operator A we'll understande $H_\gamma = D(A^\gamma)$, $(x,y)_{H_\gamma} = (A^\gamma x, A^\gamma y)$, $x, y \in D(A^\gamma)$, $\gamma \geq 0$. Denote by $L(X,Y)$ the space of linear bounded oprators acting from the space X to the space Y, and by $\sigma(\cdot)$ the spectrum of the oprator $(\cdot)$.

Consider in H the boundary value problem

$$-u'''(t) + \rho(t) A^3 u(t) + \sum_{j=1}^{3} A_j \frac{d^{3-j} u(t)}{dt^{3-j}} = f(t), \quad t \in R_+ = [0, +\infty), \tag{1}$$

$$u(0) = K u''(0), \quad u'(0) = 0, \tag{2}$$

where A is a positive-definite self-adjoint operator, $K \in L\left(H_{1/2}, H_{5/2}\right)$, A_j, $j = 1,2,3$ are linear, generally speaking, unbounded operators, $\rho(t) = \alpha$, if $0 \leq t \leq 1$, $\rho(t) = \beta$ if $1 < t < +\infty$ and α, β are positive, generally speaking, unequal to each other numbers, $f(t) \in L_2(R_+; H)$, $u(t) \in W_2^3(R_+; H)$. Here

$$L_2(R_+; H) = \left\{ f(t) : \|f\|_{L_2(R_+;H)} = \left(\int_0^{+\infty} \|f(t)\|_H^2 \, dt \right)^{1/2} < +\infty \right\},$$

$$W_2^3(R_+; H) = \left\{ u(t) : u'''(t),\ A^3 u(t) \in L_2(R_+; H)\right.,$$

$$\left. \|u\|_{W_2^3(R_+;H)} = \left(\left\|u'''\right\|_{L_2(R_+;H)}^2 + \left\|A^3 u\right\|_{L_2(R_+;H)}^2 \right)^{1/2} \right\}$$

(see [1]).

Definition 1. *If the vector-function $u(t) \in W_2^3(R_+; H)$ satisfies equation (1) almost everywhere in R_+, we'll call it a regular solution of equation (1).*

Definition 2. *If for any $f(t)\in L_2(R_+;H)$ there exists a unique regular solution of equation (1) that satisfies boundary conditions (2) in the sense*

$$\lim_{t\to 0}\left\|u(t)-Ku''(t)\right\|_{H_{5/2}}=0,\quad \lim_{t\to 0}\left\|u'(t)\right\|_{H_{3/2}}=0,$$

then boundary value problem (1),(2) is regularly solvable.

In the paper we get regular solvability conditions of boundary value problem (1),(2) expressed by means of the properties of its operator coefficients.

In the paper [2] the regular solvability was studied for a second elliptic operator-differential equation with an operator in the boundary condition. Note that a boundary value problem for equation (1) for $\rho(t)\equiv 1,\ t\in R_+$ when one of the boundary conditions in zero contains some linear oprator, was studied in [3]. In the case $K=0$, problem (1),(2) was considered in [4].

Before passing to consideration of the stated issue, we give additional denotation.

Assume

$$W_{2,K}^3(R_+;H)=\left\{u(t):u(t)\in W_2^3(R_+;H),\quad u(0)=Ku''(0),\quad u'(0)=0\right\}$$

and denote by $P_0,\ P_1$ and P the operators acting from the space $W_{2,K}^3(R_+;H)$ to the space $L_2(R_+;H)$ by the following principles, respectively

$$P_0u(t)=-u'''(t)+\rho(t)A^3u(t),\quad P_1u(t)=A_1u''(t)+A_2u'(t)+A_1u(t),$$

$$Pu(t)=P_0u(t)+P_1u(t),\quad u(t)\in W_{2,K}^3(R_+;H).$$

Prove the following coercive inequality that we'll use later on.

Lemma 1. *Let $C=A^{5/2}KA^{-1/2}$ and $\operatorname{Re}C\geq 0$. Then for any $u(t)\in W_{2,K}^3(R_+;H)$ it holds the inequality*

$$\left\|\rho^{-1/2}(t)u'''\right\|_{L_2(R_+;H)}^2+\left\|\rho^{1/2}(t)A^3u\right\|_{L_2(R_+;H)}^2\leq\frac{1}{\min(\alpha;\beta)}\left\|P_0u\right\|_{L_2(R_+;H)}^2. \quad (3)$$

Proof. Consider the following equalities:

$$\left(P_0u,A^3u\right)_{L_2(R_+;H)}=\left(-u'''+\rho(t)A^3u,A^3u\right)_{L_2(R_+;H)}=\left(-u''',A^3u\right)_{L_2(R_+;H)}+$$

$$+\left(\rho(t)A^3u,A^3u\right)_{L_2(R_+;H)}=\left(-u''',A^3u\right)_{L_2(R_+;H)}+\left\|\rho^{1/2}(t)A^3u\right\|_{L_2(R_+;H)}^2, \quad (4)$$

$$\left(P_0u,-\rho^{-1}(t)u'''\right)_{L_2(R_+;H)}=\left(-u'''+\rho(t)A^3u,-\rho^{-1}(t)u'''\right)_{L_2(R_+;H)}=$$

$$=\left(-u''',-\rho^{-1}(t)u'''\right)_{L_2(R_+;H)}-\left(A^3u,u'''\right)_{L_2(R_+;H)}=$$

$$=\left\|\rho^{-1/2}(t)u'''\right\|_{L_2(R_+;H)}^2-\left(A^3u,u'''\right)_{L_2(R_+;H)}. \quad (5)$$

Note that by integration by parts, for $u(t)\in W_{2,K}^3(R_+;H)$ we have

$$-\operatorname{Re}\left(u''',A^3u\right)_{L_2(R_+;H)}=\operatorname{Re}\left(CA^{1/2}u''(0),A^{1/2}u''(0)\right). \quad (6)$$

Put together equalities (4), (5) and take into account (6):

$$\left(P_0u,A^3u-\rho^{-1}(t)u'''\right)_{L_2(R_+;H)}=\left\|\rho^{1/2}(t)A^3u\right\|_{L_2(R_+;H)}^2+$$

$$+\left\|\rho^{-1/2}(t)\,u'''\right\|^2_{L_2(R_+;H)}+2\operatorname{Re}\left(CA^{1/2}u''(0),A^{1/2}u''(0)\right). \qquad (7)$$

Applying the Cauchy-Schwarts inequality, then the Young inequality to the left side of (7) and taking into account (6), we get:

$$\left(P_0u, A^3u-\rho^{-1}(t)\,u'''\right)_{L_2(R_+;H)}\le$$

$$\le\|P_0u\|_{L_2(R_+;H)}\left\|\rho^{1/2}(t)\,A^3u-\rho^{-1/2}(t)\,u'''\right\|_{L_2(R_+;H)}\le$$

$$\le\frac{1}{2\min(\alpha;\beta)}\|P_0u\|^2_{L_2(R_+;H)}+\frac{1}{2}\left\|\rho^{1/2}(t)\,A^3u-\rho^{-1/2}(t)\,u'''\right\|^2_{L_2(R_+;H)}=$$

$$=\frac{1}{2\min(\alpha;\beta)}\|P_0u\|^2_{L_2(R_+;H)}+\frac{1}{2}\left\|\rho^{1/2}(t)\,A^3u\right\|^2_{L_2(R_+;H)}+$$

$$+\frac{1}{2}\left\|\rho^{-1/2}(t)\,u'''\right\|^2_{L_2(R_+;H)}+\operatorname{Re}\left(CA^{1/2}u''(0),A^{1/2}u''(0)\right). \qquad (8)$$

Taking into account (8) in (7), we get

$$\left\|\rho^{-1/2}(t)\,u'''\right\|^2_{L_2(R_+;H)}+\left\|\rho^{1/2}(t)\,A^3u\right\|^2_{L_2(R_+;H)}+$$

$$+2\operatorname{Re}\left(CA^{1/2}u''(0),A^{1/2}u''(0)\right)\le\frac{1}{\min(\alpha;\beta)}\|P_0u\|^2_{L_2(R_+;H)}. \qquad (9)$$

Since $\operatorname{Re}C\ge 0$, then from (9) we get the validity of the lemma.

The lemma is proved.

Accept $k(c_1,c_2,c_3)=c_1\sqrt[3]{\beta^2}+c_2\sqrt[3]{\alpha\beta}+c_3\sqrt[3]{\alpha^2}$ and

$$K_{\alpha,\beta}=\left(E+\sqrt[3]{\alpha^2}KA^2\right)\left(k(\omega_1,1,\omega_2)\,e^{\sqrt[3]{\alpha}(\omega_2-1)A}-k(1,1,1)\,E\right)-$$

$$-\left(E+\sqrt[3]{\alpha^2}\omega_2KA^2\right)\left(k(\omega_1,\omega_2,1)\,e^{\sqrt[3]{\alpha}(\omega_1-1)A}-k(1,\omega_2,\omega_1)\,e^{\sqrt[3]{\alpha}(\omega_2-1)A}\right),$$

where $\omega_1=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$ and $\omega_2=-\frac{1}{2}-\frac{\sqrt{3}}{2}i$, α and β are the values accepted by the function $\rho(t)$, E is a unit operator.

In [5] it was astablished that if A is a positive-definite self-adjoint operator, $K\in L\left(H_{1/2},H_{5/2}\right)$, $-\frac{1}{\sqrt[3]{\alpha^2}\omega_2}\notin\sigma(C)$ and the operator $K_{\alpha,\beta}$ is boundedly invertible in the space $H_{5/2}$, then the operator P_0 realizes an isomorphism between the spaces $W^3_{2,K}(R_+;H)$ and $L_2(R_+;H)$. Consequently, the norm $\|P_0u\|_{L_2(R_+;H)}$ is equivalent to the input norm $\|u\|_{W^3_2(R_+;H)}$ in the space $W^3_{2,K}(R_+;H)$. And since the intermediate derivatives operators $A^j\frac{d^{3-j}}{dt^{3-j}}:W^3_{2,K}(R_+;H)\to L_2(R_+;H)$, $j=1,2,3$, are continuous [1], then their norms may be estimated with resoect to the norm $\|P_0u\|_{L_2(R_+;H)}$ as well.

The following theorem is valid.

Theorem 1. *Let* $\operatorname{Re}C\ge 0$. *Then for any* $u(t)\in W^3_{2,K}(R_+;H)$ *there hold the following inequalities:*

$$\left\|A^j\frac{d^{3-j}u}{dt^{3-j}}\right\|_{L_2(R_+;H)}\le a_j\|P_0u\|_{L_2(R_+;H)},\quad j=1,2,3, \qquad (10)$$

where

$$a_1=\frac{2^{1/3}\max^{1/3}(\alpha;\beta)}{3^{1/2}\min^{2/3}(\alpha;\beta)},\quad a_2=\frac{2^{1/3}\max^{1/6}(\alpha;\beta)}{3^{1/2}\min^{5/6}(\alpha;\beta)},\quad a_3=\frac{1}{\min(\alpha;\beta)}.$$

Proof. Let $u(t)\in W_{2,K}^3(R_+;H)$. Making integration by parts and applying the Cauchy-Schwarts inequality and then the Young inequality, we have

$$\|Au''\|^2_{L_2(R_+;H)}=\int_0^{+\infty}\left(Au'',\ Au''\right)_H dt=\left(Au',\ Au''\right)_H\Big|_0^{+\infty}-\int_0^{+\infty}\left(Au',\ Au'''\right)_H dt=$$

$$=-\int_0^{+\infty}\left(A^2u',\ u'''\right)_H dt\le\|A^2u'\|_{L_2(R_+;H)}\|u'''\|_{L_2(R_+;H)}\le$$

$$\le\max_t\rho^{1/2}(t)\|A^2u'\|_{L_2(R_+;H)}\times\left\|\rho^{-1/2}(t)\,u'''\right\|_{L_2(R_+;H)}\le$$

$$\le\frac{\varepsilon}{2}\max(\alpha;\beta)\|A^2u'\|^2_{L_2(R_+;H)}+\frac{1}{2\varepsilon}\left\|\rho^{-1/2}(t)\,u'''\right\|^2_{L_2(R_+;H)},\quad\varepsilon>0.\tag{11}$$

Behaving in the same way, we get:

$$\|A^2u'\|^2_{L_2(R_+;H)}=\int_0^{+\infty}\left(A^2u',\ A^2u'\right)_H dt=$$

$$=\left(A^2u,\ A^2u'\right)_H\Big|_0^{+\infty}-\int_0^{+\infty}\left(A^2u,\ A^2u''\right)_H dt=$$

$$=-\int_0^{+\infty}\left(A^3u,\ Au''\right)_H dt\le\|A^3u\|_{L_2(R_+;H)}\|Au''\|_{L_2(R_+;H)}\le$$

$$\le\left\|\rho^{1/2}(t)\,A^3u\right\|_{L_2(R_+;H)}\max_t\rho^{-1/2}(t)\|Au''\|_{L_2(R_+;H)}\le$$

$$\le\frac{\eta}{2}\frac{1}{\min(\alpha;\beta)}\|Au''\|^2_{L_2(R_+;H)}+\frac{1}{2\eta}\left\|\rho^{1/2}(t)\,A^3u\right\|^2_{L_2(R_+;H)},\quad\eta>0.\tag{12}$$

Take into attention inequality (12) in inequality (11):

$$\|Au''\|^2_{L_2(R_+;H)}\le\frac{\varepsilon}{2}\max(\alpha;\beta)\left[\frac{\eta}{2}\frac{1}{\min(\alpha;\beta)}\|Au''\|^2_{L_2(R_+;H)}+\right.$$

$$\left.+\frac{1}{2\eta}\left\|\rho^{1/2}(t)\,A^3u\right\|^2_{L_2(R_+;H)}\right]+\frac{1}{2\varepsilon}\left\|\rho^{-1/2}(t)\,u'''\right\|^2_{L_2(R_+;H)}.\tag{13}$$

From inequality (13) we have

$$\left(1-\frac{\varepsilon\eta\max(\alpha;\beta)}{4\min(\alpha;\beta)}\right)\|Au''\|^2_{L_2(R_+;H)}\le$$

$$\leq \frac{\varepsilon \max(\alpha;\beta)}{4\eta}\left\|\rho^{1/2}(t)\,A^3u\right\|^2_{L_2(R_+;H)}+\frac{1}{2\varepsilon}\left\|\rho^{-1/2}(t)\,u'''\right\|^2_{L_2(R_+;H)}. \qquad (14)$$

Choosing $\eta=\dfrac{\varepsilon^2\max(\alpha;\beta)}{2}$, from inequality (14) we get

$$\left\|Au''\right\|^2_{L_2(R_+;H)}\leq$$

$$\leq \frac{4\min(\alpha;\beta)}{8\varepsilon\min(\alpha;\beta)-\varepsilon^4\max^2(\alpha;\beta)}\left[\left\|\rho^{1/2}(t)\,A^3u\right\|^2_{L_2(R_+;H)}+\left\|\rho^{-1/2}(t)\,u'''\right\|^2_{L_2(R_+;H)}\right].$$

Then, minimizing with respect to ε, we find $\varepsilon=\sqrt[3]{\dfrac{2\min(\alpha;\beta)}{\max^2(\alpha;\beta)}}$. Consequently,

$$\left\|Au''\right\|^2_{L_2(R_+;H)}\leq$$

$$\leq \frac{2^{2/3}\max^{2/3}(\alpha;\beta)}{3\min^{1/3}(\alpha;\beta)}\left[\left\|\rho^{1/2}A^3u\right\|^2_{L_2(R_+;H)}+\left\|\rho^{-1/2}(t)\,u'''\right\|^2_{L_2(R_+;H)}\right]. \qquad (15)$$

Now, taking into account inequality (3), form inequality (15) we get

$$\left\|Au''\right\|^2_{L_2(R_+;H)}\leq \frac{2^{2/3}\max^{2/3}(\alpha;\beta)}{3\min^{4/3}(\alpha;\beta)}\left\|P_0u\right\|^2_{L_2(R_+;H)}.$$

Thus,

$$\left\|Au''\right\|_{L_2(R_+;H)}\leq \frac{2^{1/3}\max^{1/3}(\alpha;\beta)}{3^{1/2}\min^{2/3}(\alpha;\beta)}\left\|P_0u\right\|_{L_2(R_+;H)}.$$

To estimate the norm $\left\|A^2u'\right\|_{L_2(R_+;H)}$, we'll take into account inequality (11) in inequality (12):

$$\left(1-\frac{\varepsilon\eta\max(\alpha;\beta)}{4\min(\alpha;\beta)}\right)\left\|A^2u'\right\|^2_{L_2(R_+;H)}\leq \frac{\eta}{4\varepsilon\min(\alpha;\beta)}\left\|\rho^{-1/2}(t)\,u'''\right\|^2_{L_2(R_+;H)}+$$

$$+\frac{1}{2\eta}\left\|\rho^{1/2}(t)\,A^3u\right\|^2_{L_2(R_+;H)}. \qquad (16)$$

Choosing $\varepsilon=\dfrac{\eta^2}{2\min(\alpha;\beta)}$, form inequality (16) we have

$$\left\|A^2u'\right\|^2_{L_2(R_+;H)}\leq$$

$$\leq \frac{4\min^2(\alpha;\beta)}{8\eta\min^2(\alpha;\beta)-\eta^4\max(\alpha;\beta)}\left[\left\|\rho^{-1/2}(t)\,u'''\right\|^2_{L_2(R_+;H)}+\left\|\rho^{1/2}(t)\,A^3u\right\|^2_{L_2(R_+;H)}\right].$$

In tis case, minimizing with respect to η, we find $\eta=\sqrt[3]{\dfrac{2\min^2(\alpha;\beta)}{\max(\alpha;\beta)}}$.

Consequently,

$$\left\|A^2u'\right\|^2_{L_2(R_+;H)}\leq \frac{2^{2/3}\max^{1/3}(\alpha;\beta)}{3\min^{2/3}(\alpha;\beta)}\times$$

$$\times\left[\left\|\rho^{-1/2}(t)\,u'''\right\|^2_{L_2(R_+;H)}+\left\|\rho^{1/2}(t)\,A^3u\right\|^2_{L_2(R_+;H)}\right]. \tag{17}$$

From inequality (17), taking into account inequality (3), we get

$$\left\|A^2u'\right\|^2_{L_2(R_+;H)}\le\frac{2^{2/3}\max^{1/3}(\alpha;\beta)}{3\min^{5/3}(\alpha;\beta)}\left\|P_0u\right\|^2_{L_2(R_+;H)}.$$

As a result,

$$\left\|A^2u'\right\|_{L_2(R_+;H)}\le\frac{2^{1/3}\max^{1/6}(\alpha;\beta)}{3^{1/2}\min^{5/6}(\alpha;\beta)}\left\|P_0u\right\|_{L_2(R_+;H)}.$$

Now estimate the norm $\left\|A^3u\right\|_{L_2(R_+;H)}$. From inequality (3) we have

$$\frac{1}{\min(\alpha;\beta)}\left\|P_0u\right\|^2_{L_2(R_+;H)}\ge\left\|\rho^{1/2}(t)\,A^3u\right\|^2_{L_2(R_+;H)}\ge\min(\alpha;\beta)\left\|A^3u\right\|^2_{L_2(R_+;H)}.$$

Hence we get

$$\left\|A^3u\right\|^2_{L_2(R_+;H)}\le\frac{1}{\min^2(\alpha;\beta)}\left\|P_0u\right\|^2_{L_2(R_+;H)}$$

or

$$\left\|A^3u\right\|_{L_2(R_+;H)}\le\frac{1}{\min(\alpha;\beta)}\left\|P_0u\right\|_{L_2(R_+;H)}.$$

Then theorem is proved.

Now study the operator P_1.

Lemma 2. *Let $A_jA^{-j}\in L(H,H)$, $j=1,2,3$. Then P_1 is a bounded operator form the space $W^3_{2,K}(R_+;H)$ to the space $L_2(R_+;H)$.*

Proof. For any $u(t)\in W^3_{2,K}(R_+;H)$ we have:

$$\left\|P_1u\right\|_{L_2(R_+;H)}=\left\|A_1u''+A_2u'+A_3u\right\|_{L_2(R_+;H)}\le\left\|A_1A^{-1}\right\|_{H\to H}\left\|Au''\right\|_{L_2(R_+;H)}+$$

$$+\left\|A_2A^{-2}\right\|_{H\to H}\left\|A^2u'\right\|_{L_2(R_+;H)}+\left\|A_3A^{-3}\right\|_{H\to H}\left\|A^3u\right\|_{L_2(R_+;H)}.$$

Taking into attention the theorem on intermeate derivatives, [1, chapter 1], form the last inequality we get

$$\left\|P_1u\right\|_{L_2(R_+;H)}\le const\left\|u\right\|_{W_2^3(R_+;H)}.$$

The lemma is proved.

Now formulate the conditions of regular solvability of boundary value problem (1),(2).

Theorem 2. *Let A be a positive-definite self-adjoint operator, $K\in L\left(H_{1/2},H_{5/2}\right)$, $\operatorname{Re}C\ge0$, $-\frac{1}{\sqrt[3]{\alpha^2\omega_2}}\notin\sigma(C)$, the operator $K_{\alpha,\beta}$ be boundedly invertible in the space $H_{5/2}$, $A_jA^{-j}\in L(H,H)$, $j=1,2,3$, and the following inequality be fulfilled:*

$$a_1\left\|A_1A^{-1}\right\|_{H\to H}+a_2\left\|A_2A^{-2}\right\|_{H\to H}+a_3\left\|A_3A^{-3}\right\|_{H\to H}<1,$$

where the numbers a_j, $j=1,2,3$, were determined in theorem 1. Then boundary value problem (1),(2) is regularly solvable.

Proof. Represent boundary value problem (1),(2) in the form of the operator equation

$$P_0 u(t) + P_1 u(t) = f(t),$$

where $f(t) \in L_2(R_+; H)$, $u(t) \in W_{2,K}^3(R_+; H)$.

In [5] it was proved that if A is a positive-definite self-adjoint operator, $K \in L\left(H_{1/2}, H_{5/2}\right)$, $-\frac{1}{\sqrt[3]{\alpha^2 \omega_2}} \notin \sigma(C)$, and the operator $K_{\alpha,\beta}$ is boundedly invertible in the space $H_{5/2}$, then the operator P_0 has a bounded inverse P_0^{-1} acting form the space $L_2(R_+; H)$ to the space $W_{2,K}^3(R_+; H)$. Then after replacement $u(t) = P_0^{-1}\nu(t)$, where $\nu(t) \in L_2(R_+; H)$, in $L_2(R_+; H)$ we have the equation

$$\left(E + P_1 P_0^{-1}\right)\nu(t) = f(t),$$

where E is a unit operaotr. Show that subject to the theorem's conditions, the norm of the operator $P_1 P_0^{-1}$ is less than a unit. Taking into-account inequality (10), we have:

$$\left\|P_1 P_0^{-1}\nu\right\|_{L_2(R_+;H)} = \|P_1 u\|_{L_2(R_+;H)} \leq \left\|A_1 u''\right\|_{L_2(R_+;H)} +$$

$$+\left\|A_2 u'\right\|_{L_2(R_+;H)} + \|A_3 u\|_{L_2(R_+;H)} \leq \left\|A_1 A^{-1}\right\|_{H\to H} \left\|A u''\right\|_{L_2(R_+;H)} +$$

$$+\left\|A_2 A^{-2}\right\|_{H\to H} \left\|A^2 u'\right\|_{L_2(R_+;H)} + \left\|A_3 A^{-3}\right\|_{H\to H} \left\|A^3 u\right\|_{L_2(R_+;H)} \leq$$

$$\leq a_1 \left\|A_1 A^{-1}\right\|_{H\to H} \|P_0 u\|_{L_2(R_+;H)} + a_2 \left\|A_2 A^{-2}\right\|_{H\to H} \|P_0 u\|_{L_2(R_+;H)} +$$

$$+a_3 \left\|A_3 A^{-3}\right\|_{H\to H} \|P_0 u\|_{L_2(R_+;H)} = \sum_{j=1}^{3} a_j \left\|A_j A^{-j}\right\|_{H\to H} \|\nu\|_{L_2(R_+;H)}.$$

Thus,

$$\left\|P_1 P_0^{-1}\right\|_{L_2(R_+;H)\to L_2(R_+;H)} \leq \sum_{j=1}^{3} a_j \left\|A_j A^{-j}\right\|_{H\to H} < 1.$$

Consequently, subject to this inequality, the operator $E + P_1 P_0^{-1}$ is invertible in the space $L_2(R_+; H)$ and $u(t)$ is determined by the formula

$$u(t) = P_0^{-1}\left(E + P_1 P_0^{-1}\right)^{-1} f(t),$$

moreover

$$\|u\|_{W_2^3(R_+;H)} \leq \left\|P_0^{-1}\right\|_{L_2(R_+;H)\to W_2^3(R_+;H)} \left\|\left(E + P_1 P_0^{-1}\right)^{-1}\right\|_{L_2(R_+;H)\to L_2(R_+;H)} \times$$

$$\times \|f\|_{L_2(R_+;H)} \leq const \|f\|_{L_2(R_+;H)}.$$

The theorem is proved.

Corollary. *At conditions of theorem 2, the operator P is an isomorphism from $W_{2,K}^3(R_+; H)$ onto $L_2(R_+; H)$.*

Note that in the case $K = 0$, the obtained results coincide with the results of [4], in the case $K = 0$ and $A_3 = 0$ with the results of [6], in the case $K = 0$ and $\alpha = \beta = 1$ with the results of [7].

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