Malik S. DZHABRAILOV, Sevinc Z. KHALIGOVA

COMMUTATOR OF ANISOTROPIC RIESZ POTENTIAL IN ANISOTROPIC GENERALIZED MORREY SPACES

Abstract

In this paper it is proved that, if $b \in BMO_\sigma$, then commutator of the anisotropic Riesz potential operator $[b, I_{\alpha,\sigma}]$, $0 < \alpha < |\sigma|$ is bounded on anisotropic generalized Morrey spaces $M_{p,\varphi,\sigma}$, where $|\sigma| = \sum\limits_{i=1}^{n} \sigma_i$ is the homogeneous dimension of $\mathbb{R}^n$. We find the conditions on the pair (φ_1, φ_2) which ensure the Spanne-Guliyev type boundedness of $[b, I_{\alpha,\sigma}]$ from the space $M_{p,\varphi_1,\sigma}$ to $M_{q,\varphi_2,\sigma}$, $1 < p < q < \infty$, $1/p - 1/q = \alpha/|\sigma|$. We also find the conditions on the φ which ensure the Adams-Guliyev type boundedness of $I_{\alpha,\sigma}$ from $M_{p,\varphi^{\frac{1}{p}},\sigma}$ to $M_{q,\varphi^{\frac{1}{q}},\sigma}$ for $1 < p < q < \infty$.

1. Introduction

In the present paper we will prove the boundedness of the anisotropic Riesz potential operator in the anisotropic generalized Morrey spaces.

For $x \in \mathbb{R}^n$ and $t > 0$, let $B(x,t)$ denote the open ball centered at x of radius t and ${}^{\complement}B(x,t) = \mathbb{R}^n \setminus B(x,t)$. Let $0 \le b \le 1$, $\sigma = (\sigma_1, \cdots, \sigma_n)$ with $\sigma_i > 0$ for $i = 1, \cdots, n$, $|\sigma| = \sigma_1 + \cdots + \sigma_n$ and $t^\sigma x \equiv (t^{\sigma_1} x_1, \ldots, t^{\sigma_n} x_n)$ for $t > 0$. For $x \in \mathbb{R}^n$ and $t > 0$, let $E_\sigma(x,t) = \prod\limits_{i=1}^{n} (x_i - t^{\sigma_i}, x_i + t^{\sigma_i})$ denote the open parallelepiped centered at x of side length $2t^{\sigma_i}$ for $i = 1, \cdots, n$.

By [3, 11], the function $F(x,\rho) = \sum\limits_{i=1}^{n} x_i^2 \rho^{-2\sigma_i}$, considered for any fixed $x \in \mathbb{R}^n$, is a decreasing one with respect to $\rho > 0$ and the equation $F(x,\rho) = 1$ is uniquely solvable. This unique positive solution will be denoted by $\rho(x)$. Define $\rho(x) = \rho$ and $\rho(0) = 0$. It is a simple matter to check that $\rho(x-y)$ defines a distance between any two points x, $y \in \mathbb{R}^n$. Thus $\mathbb{R}^n$, endowed with the metric ρ, defines a homogeneous metric space ([3, 5, 11]). Note that $\rho(x)$ is equivalent to $|x|_\sigma = \max\limits_{1 \le i \le n} |x_i|^{\frac{1}{\sigma_i}}$ and $|x+y|_\sigma \le c_o (|x|_\sigma + |y|_\sigma)$, where $c_0 = \max\left\{1, 2^{\frac{1}{\sigma_{\min}} - 1}\right\}$.

One of the most important variants of the anisotropic maximal function is the so-called anisotropic fractional maximal function defined by the formula

$$M_{\alpha,\sigma} f(x) = \sup_{t>0} |E_\sigma(x,t)|^{-1+\alpha/|\sigma|} \int_{E_\sigma(x,t)} |f(y)| dy, \quad 0 \le \alpha < |\sigma|,$$

where $|E_\sigma(x,t)| = 2^n t^{|\sigma|}$ is the Lebesgue measure of the parallelepiped $E_\sigma(x,t)$.

It coincides with the anisotropic maximal function $M_\sigma f \equiv M_{0,\sigma} f$ and is intimately related to the anisotropic Riesz potential

$$I_{\alpha,\sigma} f(x) = \int_{\mathbb{R}^n} \frac{f(y)dy}{|x-y|_\sigma^{|\sigma|-\alpha}}, \qquad 0 < \alpha < |\sigma|.$$

If $\sigma = \mathbf{1}$, then $M_\alpha \equiv M_{\alpha,\mathbf{1}}$ and $I_\alpha \equiv I_{\alpha,\mathbf{1}}$ is the fractional maximal operator and Riesz potential, respectively. The operators M_α, $M_{\alpha,\sigma}$, I_α and $I_{\alpha,\sigma}$ play important role in real and harmonic analysis (see, for example [4] and [26]).

Definition 1.1. *Let $0 \le b \le 1$ and $1 \le p < \infty$. We denote by $L_{p,b,\sigma} \equiv L_{p,b,\sigma}(\mathbb{R}^n)$ anisotropic Morrey space, the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n$, with the finite norm*

$$\|f\|_{L_{p,b,\sigma}} = \sup_{x\in\mathbb{R}^n,\, t>0} \left(t^{-b|\sigma|} \int_{E_\sigma(x,t)} |f(y)|^p dy \right)^{1/p}.$$

Remark 1.1. Note that $L_{p,0,\sigma} = L_p(\mathbb{R}^n)$ and $L_{p,1,\sigma} = L_\infty(\mathbb{R}^n)$. If $b < 0$ or $b > 1$, then $L_{p,b,\sigma} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$. In the case $\sigma \equiv \mathbf{1} = (1,\ldots,1)$ and $b = \frac{\lambda}{n}$ we get the classical Morrey space $L_{p,\lambda}(\mathbb{R}^n) = L_{p,\frac{\lambda}{n},\mathbf{1}}(\mathbb{R}^n)$, $0 \le \lambda \le n$.

In the theory of partial differential equations, together with weighted $L_{p,w}(\mathbb{R}^n)$ spaces, Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ play an important role. Morrey spaces were introduced by C. B. Morrey in 1938 in connection with certain problems in elliptic partial differential equations and calculus of variations (see [20]).

Definition 1.2. *[6] Let $1 \le p < \infty$ and $0 \le b \le 1$. We denote by $WL_{p,b,\sigma} \equiv WL_{p,b,\sigma}(\mathbb{R}^n)$ the weak anisotropic Morrey space as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n$ with finite norm*

$$\|f\|_{WL_{p,b,\sigma}} = \sup_{r>0} r \sup_{x\in\mathbb{R}^n,\, t>0} \left(t^{-b|\sigma|} \; |\{y \in E_\sigma(x,t) : \; |f(y)| > r\}| \right)^{1/p}.$$

Note that

$$WL_p(\mathbb{R}^n) = WL_{p,0,\sigma}(\mathbb{R}^n),$$

$$L_{p,b,\sigma}(\mathbb{R}^n) \subset WL_{p,b,\sigma}(\mathbb{R}^n) \text{ and } \|f\|_{WL_{p,b,\sigma}} \le \|f\|_{L_{p,b,\sigma}},$$

The anisotropic result by Hardy-Littlewood-Sobolev states that if $1 < p < q < \infty$, then $I_{\alpha,\sigma}$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ if and only if $\alpha = |\sigma| \left(\frac{1}{p} - \frac{1}{q}\right)$ and for $p = 1 < q < \infty$, $I_{\alpha,\sigma}$ is bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ if and only if $\alpha = |\sigma| \left(1 - \frac{1}{q}\right)$. Spanne (see [25]) and Adams [1] studied boundedness of the Riesz potential I_α for $0 < \alpha < n$ in Morrey spaces $L_{p,\lambda}$. Later on Chiarenza and Frasca [10] was reproved boundedness of the Riesz potential I_α in these spaces. By more general results of Guliyev [12] (see also [13, 14]) one can obtain the following generalization of the results in [1, 10, 25] to the anisotropic case.

Theorem A. *Let $0 < \alpha < |\sigma|$ and $0 \le b < 1$, $1 \le p < \frac{(1-b)|\sigma|}{\alpha}$.*

1) If $1 < p < \frac{(1-b)|\sigma|}{\alpha}$*, then condition* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{(1-b)|\sigma|}$ *is necessary and sufficient for the boundedness of the operators* $M_{\alpha,\sigma}$ *and* $I_{\alpha,\sigma}$ *from* $L_{p,b,\sigma}(\mathbb{R}^n)$ *to* $L_{q,b,\sigma}(\mathbb{R}^n)$*.*

2) If $p = 1$*, then condition* $1 - \frac{1}{q} = \frac{\alpha}{(1-b)|\sigma|}$ *is necessary and sufficient for the boundedness of the operators* $M_{\alpha,\sigma}$ *and* $I_{\alpha,\sigma}$ *from* $L_{1,b,\sigma}(\mathbb{R}^n)$ *to* $WL_{q,b,\sigma}(\mathbb{R}^n)$*.*

It is known that the anisotropic maximal operator M_σ is also bounded from $L_{p,b,\sigma}$ to $L_{p,b,\sigma}$ for all $1 < p < \infty$ and $0 < b < 1$ (see, for example [12, 13]), which isotropic case proved by F. Chiarenza and M. Frasca [10].

In this work, in the case $b \in BMO$ we prove the boundedness of commutator of the anisotropic Riesz potential operator $[b, I_{\alpha,\sigma}]$, $0 < \alpha < |\sigma|$ from one generalized Morrey space $M_{p,\varphi_1,\sigma}$ to $M_{q,\varphi_2,\sigma}$, $1 < p < q < \infty$, $1/p - 1/q = \alpha/|\sigma|$, and from $M_{1,\varphi_1,\sigma}$ to the weak space $WM_{q,\varphi_2,\sigma}$, $1 < q < \infty$, $1 - 1/q = \alpha/|\sigma|$. We also prove the Adams-Guliyev type boundedness of the operator $I_{\alpha,\sigma}$ from $M_{p,\varphi^{\frac{1}{p}},\sigma}$ to $M_{q,\varphi^{\frac{1}{q}},\sigma}$ for $1 < p < q < \infty$ and from $M_{1,\varphi,\sigma}$ to $WM_{q,\varphi^{\frac{1}{q}},\sigma}$ for $1 < q < \infty$.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Notations

Everywhere in the sequel the functions $\varphi(x,r)$, $\varphi_1(x,r)$ and $\varphi_2(x,r)$ used in the body of the paper, are non-negative measurable function on $\mathbb{R}^n \times (0,\infty)$.

We find it convenient to define the generalized Morrey spaces in the form as follows.

Definition 2.3. *Let* $1 \le p < \infty$*. The anisotropic generalized Morrey space* $M_{p,\varphi,\sigma}$ *is defined of all functions* $f \in L_p^{loc}(\mathbb{R}^n)$ *by the finite norm*

$$\|f\|_{M_{p,\varphi,\sigma}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, |E_\sigma(x,r)|^{-\frac{1}{p}} \, \|f\|_{L_p(E_\sigma(x,r))}.$$

According to this definition, when $\varphi(x,r) = r^{\frac{(b-1)|\sigma|}{p}}$, we can see that

$$M_{p,\varphi,\sigma}(\mathbb{R}^n) = L_{p,b,\sigma}(\mathbb{R}^n).$$

There are many papers discussed the conditions on φ to obtain the boundedness of integral operators on the generalized Morrey spaces, see [12], [13], [14], [21], [22], [23], [24].

In [23] the following statements were proved.

Theorem 2.1. *Let* $1 \le p < \infty, 0 < \alpha < \frac{|\sigma|}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{|\sigma|}$ *and* $\varphi(x,\tau)$ *satisfy conditions*

$$c^{-1}\varphi(x,r) \le \varphi(x,\tau) \le c\,\varphi(x,r), \tag{2.1}$$

whenever $r \le \tau \le 2r$*, where* $c \ge 1$ *does not depend on* r, τ *and* $x \in \mathbb{R}^n$*,*

$$\int_r^\infty \tau^{\alpha p} \varphi(x,\tau)^p \frac{d\tau}{\tau} \le C\, r^{\alpha p} \varphi(x,r)^p. \tag{2.2}$$

Then for $p > 1$ the operator $M_{\alpha,\sigma}$ is bounded from $M_{p,\varphi,\sigma}$ to $M_{q,\varphi,\sigma}$ and for $p = 1$ $M_{\alpha,\sigma}$ is bounded from $M_{1,\varphi,\sigma}$ to $WM_{q,\varphi,\sigma}$.

The following statements, containing results obtained in [23] was proved in [12] (see also [13, 14, 16, 17]).

Theorem 2.2. *Let $1 \le p < \infty$, $0 < \alpha < \frac{|\sigma|}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{|\sigma|}$ and (φ_1, φ_2) satisfy the condition*

$$\int_r^\infty t^{\alpha-1}\varphi_1(x,t)dt \le C\,\varphi_2(x,r), \tag{2.3}$$

where C does not depend on x and r. Then the operator $I_{\alpha,\sigma}$ is bounded from $M_{p,\varphi_1,\sigma}$ to $M_{q,\varphi_2,\sigma}$ for $p > 1$ and from $M_{1,\varphi_1,\sigma}$ to $WM_{q,\varphi_2,\sigma}$ for $p = 1$.

In [14], V.S. Guliyev obtained sufficient conditions on the pair (φ_1, φ_2) for the boundedness of I_α from $M_{p,\varphi,\mathbf{1}}$ to $M_{q,\varphi,\mathbf{1}}$, where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

3. Anisotropic Riesz potential in the spaces $M_{p,\varphi,\sigma}$

3.1. Spanne-Guliyev type result

Sufficient conditions on φ for the boundedness of I_σ and $I_{\alpha,\sigma}$ in generalized Morrey spaces $\mathcal{M}_{p,\varphi,\sigma}$ have been obtained in [2], [7], [14], [16], [17], [23].

The following lemma is true.

Lemma 3.1. *Let $1 \le p < \infty$, $0 < \alpha < \frac{|\sigma|}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{|\sigma|}$. Then for $p > 1$ and any ball $E_\sigma = E_\sigma(x,r)$ the inequality*

$$\|I_{\alpha,\sigma}f\|_{L_q(E_\sigma(x,r))} \lesssim \|f\|_{L_p(E_\sigma(x,2c_0r))} + r^{\frac{|\sigma|}{q}} \int_{2c_0r}^\infty \|f\|_{L_p(E_\sigma(x,t))} t^{-1-\frac{|\sigma|}{q}}dt \tag{3.1}$$

holds for all $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$.

Moreover for $p = 1$ the inequality

$$\|I_{\alpha,\sigma}f\|_{WL_q(E_\sigma(x,r))} \lesssim \|f\|_{L_1(E_\sigma(x,2c_0r))} + r^{\frac{|\sigma|}{q}} \int_{2c_0r}^\infty \|f\|_{L_1(E_\sigma(x,\tau))} t^{-1-\frac{|\sigma|}{q}}dt \tag{3.2}$$

holds for all $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. Let $1 < p < q < \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{|\sigma|}$. For arbitrary ball $E_\sigma = E_\sigma(x,r)$ let $f = f_1 + f_2$, where $f_1 = f\chi_{E_\sigma(x,2c_0r)}$ and $f_2 = f\chi_{\complement\left(E_\sigma(x,2c_0r)\right)}$.

$$\|I_{\alpha,\sigma}f\|_{L_q(E_\sigma)} \le \|I_{\alpha,\sigma}f_1\|_{L_q(E_\sigma)} + \|I_{\alpha,\sigma}f_2\|_{L_q(E_\sigma)}.$$

By the continuity of the operator $I_{\alpha,\sigma} : L_p(\mathbb{R}^n) \to L_q(\mathbb{R}^n)$ we have

$$\|I_{\alpha,\sigma}f_1\|_{L_q(E_\sigma)} \lesssim \|f\|_{L_p(E_\sigma(x,2c_0r))}.$$

Let y be an arbitrary point from E_σ, and z be an arbitrary point from $\complement\big(E_\sigma(x,2c_0r)\big)$, then

$$\frac{1}{2c_0}|x-z|_\sigma \le |y-z|_\sigma \le \frac{1+2c_0}{2}|x-z|_\sigma.$$

We get

$$I_{\alpha,\sigma}f_2(y) \lesssim \int_{\complement\left(E_\sigma(x,2c_0r)\right)} \frac{|f(z)|}{|x-z|_\sigma^{|\sigma|-\alpha}}dz \lesssim$$

$$\lesssim \int_{\complement\left(E_\sigma(x,2c_0r)\right)} |f(z)|dz \int_{|x-z|_\sigma}^{\infty} \frac{dt}{t^{|\sigma|-\alpha+1}} = \int_{2c_0}^{\infty}\int_{2c_0\le|x-z|_\sigma<t} |f(z)|dz \frac{dt}{t^{|\sigma|-\alpha+1}} \lesssim$$

$$\lesssim \int_{2c_0}^{\infty} \|f\|_{L_1(E_\sigma(x,t))}\frac{dt}{t^{|\sigma|-\alpha+1}} \lesssim \int_{2c_0}^{\infty} \|f\|_{L_p(E_\sigma(x,t))} t^{-1-\frac{|\sigma|}{q}}dt. \tag{3.3}$$

Therefore, for all $1 \le p < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{|\sigma|}$ we get

$$\|I_{\alpha,\sigma}f_2\|_{L_q(E_\sigma)} \lesssim r^{\frac{|\sigma|}{q}} \int_{2c_0}^{\infty} \|f\|_{L_p(E_\sigma(x,t))} t^{-1-\frac{|\sigma|}{q}}dt. \tag{3.4}$$

Thus

$$\|I_{\alpha,\sigma}f\|_{L_q(E_\sigma)} \lesssim \|f\|_{L_p(2c_0E_\sigma)} + r^{\frac{|\sigma|}{q}} \int_{2c_0}^{\infty} \|f\|_{L_p(E_\sigma(x,t))} t^{-1-\frac{|\sigma|}{q}}dt \lesssim$$

$$\lesssim r^{\frac{|\sigma|}{q}} \int_{2c_0}^{\infty} \|f\|_{L_p(E_\sigma(x,t))} t^{-1-\frac{|\sigma|}{q}}dt.$$

Let $p = 1$. It is obvious that for any ball $E_\sigma = E_\sigma(x,r)$

$$\|I_{\alpha,\sigma}f\|_{WL_q(E_\sigma)} \le \|I_{\alpha,\sigma}f_1\|_{WL_q(E_\sigma)} + \|I_{\alpha,\sigma}f_2\|_{WL_q(E_\sigma)}.$$

By the continuity of the operator $I_{\alpha,\sigma} : L_1(\mathbb{R}^n) \to WL_q(\mathbb{R}^n)$ we have

$$\|I_{\alpha,\sigma}f_1\|_{WL_q(E_\sigma)} \lesssim \|f\|_{L_1(E_\sigma(x,2c_0r))}.$$

Then by (3.4) we get the inequality (3.2).

Theorem 3.3. *Let $1 \le p < \infty$, $0 < \alpha < \frac{|\sigma|}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{|\sigma|}$, and (φ_1, φ_2) satisfies the condition*

$$\int_r^{\infty} t^{\alpha-1}\,\varphi_1(x,t)dt \le C\,\varphi_2(x,r), \tag{3.5}$$

where C does not depend on x and r. Then for $p > 1$, $I_{\alpha,\sigma}$ is bounded from $M_{p,\varphi_1,\sigma}$ to $M_{q,\varphi_2,\sigma}$ and for $p = 1$, $I_{\alpha,\sigma}$ is bounded from $M_{1,\varphi_1,\sigma}$ to $WM_{q,\varphi_2,\sigma}$.

Proof. By Lemma 4.3 we get

$$\|I_{\alpha,\sigma}f\|_{M_{q,\varphi_2,\sigma}} \lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_r^{\infty} t^{-1-\frac{|\sigma|}{q}} \|f\|_{L_p(E_\sigma(x,t))}dt \lesssim$$

$$\lesssim \|f\|_{M_{p,\varphi_1,\sigma}} \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_r^{\infty} t^{\alpha-1}\,\varphi_1(x,t)dt \lesssim \|f\|_{M_{p,\varphi_1,\sigma}}$$

if $p \in (1,\infty)$ and

$$\|I_{\alpha,\sigma}f\|_{WM_{q,\varphi_2,\sigma}} \lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_r^{\infty} t^{-1-\frac{|\sigma|}{q}} \|f\|_{L_1(E_\sigma(x,t))}dt \lesssim$$

$$\lesssim \|f\|_{M_{1,\varphi_1,\sigma}} \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_r^\infty t^{\alpha-1}\, \varphi_1(x,t)dt \lesssim \|f\|_{M_{1,\varphi_1,\sigma}}$$

if $p = 1$.

Corollary 3.1. *Let $1 \le p < \infty$, $0 < \alpha < \frac{|\sigma|}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{|\sigma|}$, and $\varphi(r)$ satisfies the condition*

$$\int_r^\infty t^{\alpha-1}\, \varphi(t)dt \le C\, r^\alpha \varphi(r),$$

where C does not depend on r. Then for $p > 1$, $I_{\alpha,\sigma}$ is bounded from $M_{p,\varphi,\sigma}$ to $M_{q,r^\alpha\varphi(r),\sigma}$ and for $p = 1$, $I_{\alpha,\sigma}$ is bounded from $M_{1,\varphi,\sigma}$ to $WM_{q,r^\alpha\varphi(r),\sigma}$.

3.2. Adams-Guliyev type result

In [18] the following Lemma was proven.

Lemma 3.2. *Let $1 \le p < \infty$ and (φ_1, φ_2) satisfies the condition*

$$\sup_{r<t<\infty} \varphi_1(x,t) \le C\, \varphi_2(x,r), \tag{3.6}$$

where C does not depend on x and r. Then for $p > 1$, M_σ is bounded from $M_{p,\varphi_1,\sigma}$ to $M_{p,\varphi_2,\sigma}$ and for $p = 1$, M_σ is bounded from $M_{1,\varphi_1,\sigma}$ to $WM_{1,\varphi_2,\sigma}$.

The following is a result of Adams-Guliyev type for the anisotropic Riesz potential.

Theorem 3.4. *Let $1 \le p < q < \infty$, $0 < \alpha < \frac{|\sigma|}{p}$ and let $\varphi(x,t)$ satisfy the condition (4.6) and*

$$\int_r^\infty t^{\alpha-1}\varphi(x,t)^{\frac{1}{p}}dt \le Cr^{-\frac{\alpha p}{q-p}}, \tag{3.7}$$

where C does not depend on $x \in \mathbb{R}^n$ and $r > 0$.

Then the operator $I_{\alpha,\sigma}$ is bounded from $M_{p,\varphi^{\frac{1}{p}},\sigma}$ to $M_{q,\varphi^{\frac{1}{q}},\sigma}$ for $p > 1$ and from $M_{1,\varphi,\sigma}$ to $WM_{q,\varphi^{\frac{1}{q}},\sigma}$.

Proof. Let $1 \le p < q < \infty$, $0 < \alpha < \frac{|\sigma|}{p}$ and $f \in M_{p,\varphi^{\frac{1}{p}},\sigma}$. Write $f = f_1 + f_2$, where $E_\sigma = E_\sigma(x,r)$, $f_1 = f\chi_{E_\sigma(x,2c_0r)}$ and $f_2 = f\chi_{\complement(E_\sigma(x,2c_0r))}$.

For $I_{\alpha,\sigma}f_2(x)$ for all $y \in E_\sigma$ from (4.3) we have

$$I_{\alpha,\sigma}(f_2)(y) = \int_{\complement E_\sigma(x,2c_0r)} |y-z|_\sigma^{\alpha-|\sigma|}|f(z)|dz \lesssim$$

$$\lesssim \int_{\complement E_\sigma(x,2c_0r)} |f(z)|dz \int_{|x-z|_\sigma}^\infty t^{\alpha-|\sigma|-1}dt \lesssim$$

$$\lesssim \int_{2c_0r}^\infty \left(\int_{2c_0r<|x-z|_\sigma<t} |f(z)|dz\right) t^{\alpha-|\sigma|-1}dt \lesssim \int_{2c_0r}^\infty t^{-1-\frac{|\sigma|}{q}}\|f\|_{L_p(E_\sigma(x,t))}dt. \tag{3.8}$$

Then from conditions (4.7) and (3.8) for all $y \in E_\sigma$ we get

$$I_{\alpha,\sigma}f(y) \lesssim r^\alpha\, M_\sigma f(y) + \int_{2r}^\infty t^{\alpha-\frac{|\sigma|}{p}-1}\|f\|_{L_p(E_\sigma(x,t))}dt \le$$

$$\leq r^{\alpha}\, M_{\sigma}f(y)+\|f\|_{M_{p,\varphi^{\frac{1}{p}},\sigma}}\int_{2r}^{\infty}t^{\alpha-1}\varphi(x,t)^{\frac{1}{p}}dt\lesssim r^{\alpha}\, M_{\sigma}f(y)+r^{-\frac{\alpha p}{q-p}}\,\|f\|_{M_{p,\varphi^{\frac{1}{p}},\sigma}}.$$

Hence choose $r=\Big(\frac{\|f\|_{M_{p,\varphi^{1/p},\sigma}}}{M_{\sigma}f(y)}\Big)^{\frac{q-p}{\alpha q}}$ for every $y\in E_{\sigma}$, we have

$$|I_{\alpha,\sigma}f(y)|\lesssim (M_{\sigma}f(y))^{\frac{p}{q}}\,\|f\|_{M_{p,\varphi^{\frac{1}{p}},\sigma}}^{1-\frac{p}{q}}.$$

Hence the statement of the theorem follows in view of the boundedness of the anisotropic maximal operator M_{σ} in $M_{p,\varphi^{\frac{1}{p}},\sigma}$ provided by Lemma 4.4 in virtue of condition (4.6).

$$\|I_{\alpha,\sigma}f\|_{M_{q,\varphi^{\frac{1}{q}},\sigma}}=\sup_{x\in\mathbb{R}^n,\, t>0}\varphi(x,t)^{-\frac{1}{q}}t^{-\frac{|\sigma|}{q}}\|I_{\alpha,\sigma}f\|_{L_q(E_{\sigma}(x,t))}\lesssim$$

$$\lesssim\|f\|_{M_{p,\varphi^{\frac{1}{p}},\sigma}}^{1-\frac{p}{q}}\sup_{x\in\mathbb{R}^n,\, t>0}\varphi(x,t)^{-\frac{1}{q}}t^{-\frac{|\sigma|}{q}}\|M_{\sigma}f\|_{L_p(E_{\sigma}(x,t))}^{\frac{p}{q}}=$$

$$=\|f\|_{M_{p,\varphi^{\frac{1}{p}},\sigma}}^{1-\frac{p}{q}}\left(\sup_{x\in\mathbb{R}^n,\, t>0}\varphi(x,t)^{-\frac{1}{p}}t^{-\frac{|\sigma|}{p}}\|M_{\sigma}f\|_{L_p(E_{\sigma}(x,t))}\right)^{\frac{p}{q}}=$$

$$=\|f\|_{M_{p,\varphi^{\frac{1}{p}},\sigma}}^{1-\frac{p}{q}}\,\|M_{\sigma}f\|_{M_{p,\varphi^{\frac{1}{p}},\sigma}}^{\frac{p}{q}}\lesssim\|f\|_{M_{p,\varphi^{\frac{1}{p}},\sigma}}.$$

if $1<p<q<\infty$ and

$$\|I_{\alpha,\sigma}f\|_{WM_{q,\varphi^{\frac{1}{q}},\sigma}}=\sup_{x\in\mathbb{R}^n,\, t>0}\varphi(x,t)^{-\frac{1}{q}}t^{-\frac{|\sigma|}{q}}\|I_{\alpha,\sigma}f\|_{WL_q(E_{\sigma}(x,t))}\lesssim$$

$$\lesssim\|f\|_{M_{1,\varphi,\sigma}}^{1-\frac{1}{q}}\sup_{x\in\mathbb{R}^n,\, t>0}\varphi(x,t)^{-\frac{1}{q}}t^{-\frac{|\sigma|}{q}}\|M_{\sigma}f\|_{WL_1(E_{\sigma}(x,t))}^{\frac{1}{q}}=$$

$$=\|f\|_{M_{1,\varphi,\sigma}}^{1-\frac{1}{q}}\left(\sup_{x\in\mathbb{R}^n,\, t>0}\varphi(x,t)^{-1}t^{-|\sigma|}\|M_{\sigma}f\|_{WL_1(E_{\sigma}(x,t))}\right)^{\frac{1}{q}}=$$

$$=\|f\|_{M_{1,\varphi,\sigma}}^{1-\frac{1}{q}}\,\|M_{\sigma}f\|_{WM_{1,\varphi,\sigma}}^{\frac{1}{q}}\lesssim\|f\|_{M_{1,\varphi,\sigma}},$$

if $1<q<\infty$.

In the case $\varphi(x,t)=t^{(b-1)\frac{|\sigma|}{p}}$, $0<b<1$ from Theorem 4.6 we get the following Adams type result for the anisotropic Riesz potential.

Corollary 3.2. *Let $0<\alpha<|\sigma|$, $1\leq p<\frac{|\sigma|}{\alpha}$, $0<\lambda<|\sigma|-\alpha p$ and $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{|\sigma|-\lambda}$. Then for $p>1$, the operator $I_{\alpha,\sigma}$ is bounded from $L_{p,b,\sigma}$ to $L_{q,b,\sigma}$ and for $p=1$, $I_{\alpha,\sigma}$ is bounded from $L_{1,b,\sigma}$ to $WL_{q,b,\sigma}$.*

4. Commutator of anisotropic Riesz potential in the spaces $M_{p,\varphi,\sigma}$

4.1. Spanne-Guliyev type result

The theory of commutator was originally studied by Coifman, Rochberg and Weiss in [9]. Since then, many authors have been interested in studying this theory. When $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, Canillo [8] proved that the commutator operator $[b, I_\alpha]f = b\, I_\alpha f - I_\alpha(bf)$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ whenever $b \in BMO(\mathbb{R}^n)$.

Locally integrable function b is said to be in $BMO_\sigma(\mathbb{R}^n)$ if

$$\|f\|_{BMO_\sigma} = \sup_{x\in\mathbb{R}^n,\, r>0} \frac{1}{|E_\sigma(x,r)|} \int_{E_\sigma(x,r)} |f(y) - f_{E_\sigma(x,r)}| dy < \infty,$$

where

$$f_{E_\sigma(x,r)} = \frac{1}{|E_\sigma(x,r)|} \int_{E_\sigma(x,r)} f(y) dy.$$

The following lemma is true.

Lemma 4.3. *Let $1 < p < \infty$, $0 < \alpha < \frac{|\sigma|}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{|\sigma|}$ and $b \in BMO_\sigma$. Then for any ball $E_\sigma = E_\sigma(x,r)$ the inequality*

$$\|[b, I_{\alpha,\sigma}]f\|_{L_q(E_\sigma(x,r))} \lesssim \|f\|_{BMO_\sigma}\, r^{\frac{|\sigma|}{q}} \int_{2c_0 r}^{\infty} \left(1 + \ln\frac{t}{r}\right) \|f\|_{L_p(E_\sigma(x,t))} t^{-1-\frac{|\sigma|}{q}} dt \quad (4.1)$$

holds for all $f \in L_p^{\rm loc}(\mathbb{R}^n)$.

Proof. Let $1 < p < q < \infty$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{|\sigma|}$ and $b \in BMO_\sigma$. For arbitrary ball $E_\sigma = E_\sigma(x,r)$ let $f = f_1 + f_2$, where $f_1 = f\chi_{E_\sigma(x,2c_0r)}$ and $f_2 = f\chi_{\complement(E_\sigma(x,2c_0r))}$.

$$\|[b, I_{\alpha,\sigma}]f\|_{L_q(E_\sigma)} \leq \|[b, I_{\alpha,\sigma}]f_1\|_{L_q(E_\sigma)} + \|[b, I_{\alpha,\sigma}]f_2\|_{L_q(E_\sigma)}.$$

By the continuity of the operator $[b, I_{\alpha,\sigma}] : L_p(\mathbb{R}^n) \to L_q(\mathbb{R}^n)$ we have

$$\|[b, I_{\alpha,\sigma}]f_1\|_{L_q(E_\sigma)} \lesssim \|f\|_{BMO_\sigma} \|f\|_{L_p(E_\sigma(x,2c_0r))}.$$

Let $b \in BMO_\sigma(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that

$$\left|b_{E_\sigma(x,r)} - b_{E_\sigma(x,t)}\right| \leq C\|b\|_{BMO_\sigma} \ln\frac{t}{r} \quad \text{for} \quad 0 < 2r < t, \quad (4.2)$$

where C is independent of b, x, r and t (see, for example, [23]). The John-Nirenberg inequality implies that

$$\|b\|_{BMO_\sigma} \approx \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{|E_\sigma(x,r)|} \int_{E_\sigma(x,r)} |b(y) - b_{E_\sigma(x,r)}|^p dy\right)^{\frac{1}{p}} \quad (4.3)$$

for $1 < p < \infty$.

For $y \in E_\sigma$ we get

$$[b, I_{\alpha,\sigma}]f_2(y) \leq (2c_0)^{|\sigma|-\alpha} \int_{\complement\left(E_\sigma(x,2c_0r)\right)} \frac{|b(y) - b(z)|\, f(z)|}{|x - z|_\sigma^{|\sigma|-\alpha}} dz.$$

Then

$$\|[b,I_{\alpha,\sigma}]f_2\|_{L_q(E_\sigma)}\lesssim\left(\int_{E_\sigma}\left(\int_{\complement\left(E_\sigma(x,2c_0r)\right)}\frac{|b(y)-b(z)|}{|x-z|_\sigma^{|\sigma|-\alpha}}|f(z)|dz\right)^q dy\right)^{\frac{1}{q}}\lesssim$$

$$\lesssim\left(\int_{E_\sigma}\left(\int_{\complement\left(E_\sigma(x,2c_0r)\right)}\frac{|b(y)-b_{E_\sigma}|}{|x-z|_\sigma^{|\sigma|-\alpha}}|f(z)|dz\right)^q dy\right)^{\frac{1}{q}}+$$

$$+\left(\int_{E_\sigma}\left(\int_{\complement\left(E_\sigma(x,2c_0r)\right)}\frac{|b(z)-b_{E_\sigma}|}{|x-z|_\sigma^{|\sigma|-\alpha}}|f(z)|dz\right)^q dy\right)^{\frac{1}{q}}=D_1+D_2.$$

At first estmiate D_1.

$$D_1=\left(\int_{E_\sigma}|b(y)-b_{E_\sigma}|^q dy\right)^{\frac{1}{q}}\int_{\complement\left(E_\sigma(x,2c_0r)\right)}\frac{|f(z)|}{|x-z|_\sigma^{|\sigma|-\alpha}}dz.$$

By (4.3) and (3.3), we get

$$D_1\lesssim\|b\|_{BMO_\sigma}\,r^{\frac{|\sigma|}{q}}\int_{\complement\left(E_\sigma(x,2c_0r)\right)}\frac{|f(z)|}{|x-z|_\sigma^{|\sigma|-\alpha}}dz\le$$

$$\le(|\sigma|-\alpha)2^n(2c_0)^{|\sigma|-\alpha}\,\|b\|_{BMO_\sigma}\int_{2c_0}^{\infty}\|f\|_{L_p(E_\sigma(x,t))}t^{-1-\frac{|\sigma|}{q}}dt.$$

Let us estimate D_2.

$$D_2=r^{\frac{|\sigma|}{q}}\int_{\complement\left(E_\sigma(x,2c_0r)\right)}\frac{|b(z)-b_{E_\sigma}|}{|x-z|_\sigma^{|\sigma|-\alpha}}|f(z)|dz\approx$$

$$\approx r^{\frac{|\sigma|}{q}}\int_{\complement\left(E_\sigma(x,2c_0r)\right)}|b(z)-b_{E_\sigma}||f(z)|\int_{|x-z|_\sigma}^{\infty}\frac{dt}{t^{|\sigma|+1-\alpha}}dz\approx$$

$$\approx r^{\frac{|\sigma|}{q}}\int_{2r}^{\infty}\int_{2c_0r\le|x-z|_\sigma\le t}|b(z)-b_{E_\sigma}||f(z)|dz\frac{dt}{t^{|\sigma|+1-\alpha}}\lesssim$$

$$\lesssim r^{\frac{|\sigma|}{q}}\int_{2c_0r}^{\infty}\int_{E_\sigma(x,t)}|b(z)-b_{E_\sigma}||f(z)|dz\frac{dt}{t^{|\sigma|+1-\alpha}}.$$

Applying Hölder's inequality and by (4.2), (4.3), we get

$$D_2\lesssim r^{\frac{|\sigma|}{q}}\int_{2c_0r}^{\infty}\int_{E_\sigma(x,t)}|b(z)-b_{E_\sigma(x,t)}||f(z)|dz\frac{dt}{t^{|\sigma|+1-\alpha}}+$$

$$+r^{\frac{|\sigma|}{q}}\int_{2c_0r}^{\infty}|b_{E_\sigma(x,r)}-b_{E_\sigma(x,t)}|\frac{dt}{t^{|\sigma|+1-\alpha}}\int_{E_\sigma(x,t)}|f(z)|dz\lesssim$$

$$\lesssim r^{\frac{|\sigma|}{q}}\int_{2c_0r}^{\infty}\left(\int_{E_\sigma(x,t)}|b(z)-b_{E_\sigma(x,t)}|^{p'}dy\right)^{\frac{1}{p'}}\|f\|_{L_p(E_\sigma(x,t))}\frac{dt}{t^{|\sigma|+1-\alpha}}+$$

$$+r^{\frac{|\sigma|}{q}}\int_{2c_0r}^{\infty}|b_{E_\sigma(x,r)}-b_{E_\sigma(x,t)}|\|f\|_{L_p(E_\sigma(x,t))}\frac{dt}{t^{\frac{|\sigma|}{p}+1-\alpha}}\lesssim$$

$$\lesssim \|b\|_{BMO_\sigma} r^{\frac{|\sigma|}{q}} \int_{2c_0r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L_p(E_\sigma(x,t))} t^{-1-\frac{|\sigma|}{q}} dt.$$

Summing up D_1 and D_2, for all $p \in (1,\infty)$ we get

$$\|[b, I_{\alpha,\sigma}]f_2\|_{L_q(E_\sigma)} \lesssim \|b\|_{BMO_\sigma} r^{\frac{|\sigma|}{q}} \int_{2c_0r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L_p(E_\sigma(x,t))} t^{-1-\frac{|\sigma|}{q}} dt. \qquad (4.4)$$

Finally,

$$\|[b, I_{\alpha,\sigma}]f\|_{L_q(E_\sigma)} \lesssim \|b\|_{BMO_\sigma} \|f\|_{L_p(E_\sigma(x,2c_0r))} +$$
$$+\|b\|_{BMO_\sigma} r^{\frac{|\sigma|}{q}} \int_{2c_0r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L_p(E_\sigma(x,t))} t^{-1-\frac{|\sigma|}{q}} dt \lesssim$$
$$\lesssim \|b\|_{BMO_\sigma} r^{\frac{|\sigma|}{q}} \int_{2c_0r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L_p(E_\sigma(x,t))} t^{-1-\frac{|\sigma|}{q}} dt.$$

Theorem 4.5. *Let $1 < p < \infty$, $0 < \alpha < \frac{|\sigma|}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{|\sigma|}$, $b \in BMO_\sigma$, and (φ_1, φ_2) satisfies the condition*

$$\int_r^{\infty} t^{\alpha-1} \Big(1+\ln\frac{t}{r}\Big) \varphi_1(x,t) dt \le C\, \varphi_2(x,r), \qquad (4.5)$$

where C does not depend on x and r. Then $[b, I_{\alpha,\sigma}]$ is bounded from $M_{p,\varphi_1,\sigma}$ to $M_{q,\varphi_2,\sigma}$.

Proof. By Lemma 4.3 we get

$$\|[b, I_{\alpha,\sigma}]f\|_{M_{q,\varphi_2,\sigma}} \lesssim$$
$$\lesssim \|b\|_{BMO_\sigma} \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_r^{\infty} \Big(1+\ln\frac{t}{r}\Big) t^{-1-\frac{|\sigma|}{q}} \|f\|_{L_p(E_\sigma(x,t))} dt \lesssim$$
$$\lesssim \|b\|_{BMO_\sigma} \|f\|_{M_{p,\varphi_1,\sigma}} \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_r^{\infty} \Big(1+\ln\frac{t}{r}\Big) t^{\alpha-1} \varphi_1(x,t) dt \lesssim$$
$$\lesssim \|b\|_{BMO_\sigma} \|f\|_{M_{p,\varphi_1,\sigma}}.$$

Corollary 4.3. *Let $1 < p < \infty$, $0 < \alpha < \frac{|\sigma|}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{|\sigma|}$, $b \in BMO_\sigma$, and $\varphi(r)$ satisfies the condition*

$$\int_r^{\infty} t^{\alpha-1} \Big(1+\ln\frac{t}{r}\Big) \varphi(t) dt \le C\, r^{\alpha} \varphi(r),$$

where C does not depend on r. Then $[b, I_{\alpha,\sigma}]$ is bounded from $M_{p,\varphi,\sigma}$ to $M_{q,r^{\alpha}\varphi(r),\sigma}$.

4.2. Adams-Guliyev type result

In [18] the following Lemma was proven.

Lemma 4.4. *Let $1 < p < \infty$, $b \in BMO_\sigma$ and (φ_1, φ_2) satisfies the condition*

$$\sup_{r<t<\infty} \Big(1+\ln\frac{t}{r}\Big) \varphi_1(x,t) \le C\, \varphi_2(x,r), \qquad (4.6)$$

where C does not depend on x and r. Then $M_{b,\sigma}$ is bounded from $M_{p,\varphi_1,\sigma}$ to $M_{p,\varphi_2,\sigma}$.

The following is a result of Adams-Guliyev type for the anisotropic Riesz potential.

Theorem 4.6. *Let $1 < p < q < \infty$, $0 < \alpha < \frac{|\sigma|}{p}$, $b \in BMO_\sigma$ and let $\varphi(x,t)$ satisfy the condition (4.6) and*

$$\int_r^\infty \left(1+\ln\frac{t}{r}\right) t^{\alpha-1}\varphi(x,t)^{\frac{1}{p}}dt \le Cr^{-\frac{\alpha p}{q-p}}, \tag{4.7}$$

where C does not depend on $x \in \mathbb{R}^n$ and $r > 0$.

Then the operator $[b, I_{\alpha,\sigma}]$ is bounded from $M_{p,\varphi^{\frac{1}{p}},\sigma}$ to $M_{q,\varphi^{\frac{1}{q}},\sigma}$.

Proof. Let $1 \le p < q < \infty$, $0 < \alpha < \frac{|\sigma|}{p}$, $b \in BMO_\sigma$ and $f \in M_{p,\varphi^{\frac{1}{p}},\sigma}$. Write $f = f_1 + f_2$, where $E_\sigma = E_\sigma(x,r)$, $f_1 = f\chi_{E_\sigma(x,2c_0r)}$ and $f_2 = f\chi_{\complement(E_\sigma(x,2c_0r))}$.

For $[b, I_{\alpha,\sigma}]f_2(x)$ for all $y \in E_\sigma$ we have

$$\left|[b, I_{\alpha,\sigma}](f_2)(y)\right| \lesssim \int_{\complement E_\sigma(x,2c_0r)} \frac{|b(y)-b(z)|}{|x-z|_\sigma^{|\sigma|-\alpha}}|f(z)|dz.$$

Analogously section 4.1, for all $p \in (1,\infty)$ and $y \in E_\sigma$ we get

$$|[b, I_\alpha^P]f_2(y)| \lesssim \|b\|_{BMO_\sigma} \int_{2c_0r}^\infty \left(1+\ln\frac{t}{r}\right) t^{\alpha-\frac{|\sigma|}{p}-1}\|f\|_{L_p(E_\sigma(x,t))}dt. \tag{4.8}$$

Then from conditions (4.7) and (3.8) for all $y \in E_\sigma$ we get

$$[b, I_{\alpha,\sigma}]f(y) \lesssim \|b\|_{BMO_\sigma}\left(r^\alpha M_\sigma f(y) + \int_{2c_0r}^\infty \left(1+\ln\frac{t}{r}\right) t^{\alpha-\frac{|\sigma|}{p}-1}\|f\|_{L_p(E_\sigma(x,t))}dt\right) \le$$

$$\le \|b\|_{BMO_\sigma}\left(r^\alpha M_\sigma f(y) + \|f\|_{M_{p,\varphi^{\frac{1}{p}},\sigma}} \int_{2c_0r}^\infty \left(1+\ln\frac{t}{r}\right) t^{\alpha-1}\varphi(x,t)^{\frac{1}{p}}dt\right) \lesssim$$

$$\lesssim \|b\|_{BMO_\sigma}\left(r^\alpha M_\sigma f(y) + r^{-\frac{\alpha p}{q-p}}\ \|f\|_{M_{p,\varphi^{\frac{1}{p}},\sigma}}\right).$$

Hence choose $r = \left(\frac{\|f\|_{M_{p,\varphi^{1/p},\sigma}}}{M_\sigma f(y)}\right)^{\frac{q-p}{\alpha q}}$ for every $y \in E_\sigma$, we have

$$|[b, I_{\alpha,\sigma}]f(y)| \lesssim \|b\|_{BMO_\sigma}\,(M_\sigma f(y))^{\frac{p}{q}}\,\|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}},\sigma}}.$$

Hence the statement of the theorem follows in view of the boundedness of the anisotropic maximal operator M_σ in $M_{p,\varphi^{\frac{1}{p}},\sigma}$ provided by Lemma 4.4 in virtue of condition (4.6).

$$\|[b, I_{\alpha,\sigma}]f\|_{M_{q,\varphi^{\frac{1}{q}},\sigma}} = \sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{q}}t^{-\frac{|\sigma|}{q}}\|[b, I_{\alpha,\sigma}]f\|_{L_q(E_\sigma(x,t))} \lesssim$$

$$\lesssim \|b\|_{BMO_\sigma}\|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}},\sigma}} \sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{q}}t^{-\frac{|\sigma|}{q}}\|M_\sigma f\|^{\frac{p}{q}}_{L_p(E_\sigma(x,t))} =$$

$$= \|b\|_{BMO_\sigma} \|f\|_{M_{p,\varphi^{\frac{1}{p}},\sigma}}^{1-\frac{p}{q}} \left(\sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{p}} t^{-\frac{|\sigma|}{p}} \|M_\sigma f\|_{L_p(E_\sigma(x,t))} \right)^{\frac{p}{q}} =$$

$$= \|b\|_{BMO_\sigma} \|f\|_{M_{p,\varphi^{\frac{1}{p}},\sigma}}^{1-\frac{p}{q}} \|M_\sigma f\|_{M_{p,\varphi^{\frac{1}{p}},\sigma}}^{\frac{p}{q}} \lesssim \|b\|_{BMO_\sigma} \|f\|_{M_{p,\varphi^{\frac{1}{p}},\sigma}}.$$

In the case $\varphi(x,t) = t^{(b-1)\frac{|\sigma|}{p}}$, $0 < b < 1$ from Theorem 4.6 we get the following Adams type result for the commutator of anisotropic Riesz potential.

Corollary 4.4. *Let* $0 < \alpha < |\sigma|$, $1 < p < \frac{|\sigma|}{\alpha}$, $0 < \lambda < |\sigma| - \alpha p$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{|\sigma|-\lambda}$ *and* $b \in BMO_\sigma$. *Then the operator* $[b, I_{\alpha,\sigma}]$ *is bounded from* $L_{p,b,\sigma}$ *to* $L_{q,b,\sigma}$.

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Malik S. Dzhabrailov, Sevinc Z. Khaligova
Azerbaijan State Pedagogical University
34, U.Hajibekov str., Az 1000, Baku, Azerbaijan
E-mail adresses: seva.xaligova@hotmail.com

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