Hidayat M. HUSEYNOV, Famil Z. DOSTUYEV

ASYMPTOTICS OF EIGEN NUMBERS OF DISCONTINUOUS CONDITON STURM-LIOUVILLE OPERATORS

Abstract

In the paper finded the asimptotics of eigen numbers of discontinuous condition Sturm-Liouville operators, learned the properties of the eigen numbers of Dirichlet's and Dirichlet-Neumann's boundary problems and proved the simplicity of zeros of the characteristic function of Dirichlet-Neumann's boundary problem.

Let's consider on the interval $(0,\pi)$ at the point $a\in(0,\pi)$ the discontinuous condition Sturm-Liouville equation:

$$-y''+q(x)\,y=\lambda^2 y, \tag{1}$$

$$\begin{gathered} y(a+0)=ay(a-0)\,, \\ y'(a+0)=\alpha^{-1}y(a-0)\,, \end{gathered} \tag{2}$$

where (2) are discontinuity conditions, λ is a spectral parameter, $q(x)$ is a real-valued function in the space $L_2(0,\pi)$, $\alpha\in R$ and $\alpha\neq 0,1$.

Note that taking

$$p(x)=\begin{cases} \alpha, & x<a, \\ 1, & x>a, \end{cases}$$

we can write problem (1), (2) in the form of the equation

$$-p(x)\left(\frac{1}{p^2(x)}\,(p(x)\,y)'\right)'+q(x)\,y=\lambda^2 y.$$

Let's consider the following boundary conditions:

$$y(0)=y(\pi)=0 \tag{3}$$

$$y(0)=y'(\pi)=0 \tag{4}$$

$$y(0)-y(\pi)=0,\quad y'(0)-y'(\pi)=0 \tag{5}$$

$$y(0)+y(\pi)=0,\quad y'(0)+y'(\pi)=0. \tag{6}$$

For $\alpha=1$ the asymptotics of eigen numbers for (1), (2), (3) (Dirichlet problem), (1), (2), (4) (Dirichlet-Neumann problem), (1), (2), (5) (periodic problem), (1), (2), (6) (antiperiodic problem) is known (see [3]). For $\alpha\neq 1$, wew'll study the asymptotics of boundary value problems and distribution of eigen numbers of the Dirichlet-Neumann problem on a real axis. For that at first we construct characteristic functions of boundary value problems and their asymptotic expansions. Assume that $s(\lambda,x)$, $c(\lambda,x)$, $s_\pi(\lambda,x)$, $c_\pi(\lambda,x)$ are the solutions satisfying the initial conditions $s(\lambda,0)=c'(\lambda,0)=s_\pi(\lambda,\pi)=c'_\pi(\lambda,\pi)=0,\;\; s'(\lambda,0)=c(\lambda,0)=$

$s'_\pi(\lambda,\pi) = c_\pi(\lambda,\pi) = 1$ of equation (1). Then we can write characteristic functions of boundary value problems (1),(2),(3), (1),(2),(4), (1),(2),(5), (1),(2),(6) as follows:

$$\chi_D(\lambda) = \alpha s(\lambda,a)\, s'_\pi(\lambda,a) - \alpha^{-1} s_\pi(\lambda,a)\, s'(\lambda,a) \tag{7}$$

$$\chi_{DN}(\lambda) = \alpha s(\lambda,a)\, c'_\pi(\lambda,a) - \alpha^{-1} c_\pi(\lambda,a)\, s'(\lambda,a) \tag{8}$$

$$\chi_p(\lambda) = 2 - 2u_+(\lambda) \tag{9}$$

$$\chi_a(\lambda) = 2 + 2u_+(\lambda) \tag{10}$$

where

$$u_+(\lambda) = \frac{1}{2}\left(\alpha\chi_2(\lambda) - \alpha^{-1}\chi_1(\lambda)\right),$$

$$\chi_1(\lambda) = s_\pi(\lambda,a)\, c'(\lambda,a) - c_\pi(\lambda,a)\, s'(\lambda,a)$$

$$\chi_2(\lambda) = s'_\pi(\lambda,a)\, c(\lambda,a) - c'_\pi(\lambda,a)\, s(\lambda,a). \tag{11}$$

When $q(x) \equiv 0$, the characteristic functions

$$\chi_{D,0}(\lambda) = \alpha^+ \frac{\sin\lambda\pi}{\lambda} + \alpha^- \frac{\sin\lambda(2a-\pi)}{\lambda}$$

$$\chi_{DN,0}(\lambda) = -\alpha^+ \cos s\lambda\pi + \alpha^- \cos\lambda(2a-\pi)$$

$$\chi_{p,0}(\lambda) = 2\left(1 - \alpha^+ \cos\lambda\pi\right)$$

$$\chi_{a,0}(\lambda) = 2\left(1 - \alpha^+ \cos\lambda\pi\right),$$

where $\alpha^+ = \frac{1}{2}\left(\alpha + \frac{1}{\alpha}\right)$, $\alpha^- = \frac{1}{2}\left(\alpha - \frac{1}{\alpha}\right)$.

Before we pass to asymptotic expansions of characteristic functions, prove the following lemma.

Lemma 1. *For the function $g(x) \in L_2(0,\pi)$ and any sequence $y_n = n + h_n^{(1)} + \frac{h_n^{(2)}}{n}$ the relation*

$$\left\{\beta_1 \int_{b_1}^{b_2} g(x)\sin y_n x dx + \beta_2 \int_{b_1}^{b_2} g(x)\cos y_n x dx\right\} \in l_2$$

is true. Where $\beta_j \in C$, $b_j \in [0,\pi]$, $\sup\left|h_n^{(j)}\right| < \infty$, $j = 1,2$.

Proof. Let's determine the function $G(x)$ satisfying the condition $G(x) \equiv g(x), x \in [b_1,b_2], G(x) \equiv 0, x \in [-\pi,b_1]$ or $x \in [b_2,\pi]$. Then $G(x) \in L_2(-\pi,\pi)$.

$$\beta_1 \int_{b_1}^{b_2} g(x)\sin y_n x dx + \beta_2 \int_{b_1}^{b_2} g(x)\cos y_n x dx =$$

$$= \beta_1 \int_{-\pi}^{\pi} G(x)\sin y_n x dx + \beta_2 \int_{-\pi}^{\pi} G(x)\cos y_n x dx =$$

$$= \frac{\beta_2 - i\beta_1}{2} \int_{-\pi}^{\pi} G(x)\, e^{iy_n x} dx + \frac{\beta_2 + i\beta_1}{2} \int_{-\pi}^{\pi} G(x)\, e^{-iy_n x} dx.$$

Prove that

$$\left\{\int\limits_{-\pi}^{\pi} G(x)\, e^{iy_n x} dx\right\} \in l_2, \quad \left\{\int\limits_{-\pi}^{\pi} G(x)\, e^{-iy_n x} dx\right\} \in l_2$$

$$a_n = \int\limits_{-\pi}^{\pi} G(x)\, e^{iy_n x} dx = \int\limits_{-\pi}^{\pi} G(x)\, e^{inx} e^{ih_n^{(1)} x} e^{i\frac{h_n^{(2)}}{n} x} dx =$$

$$= \int\limits_{-\pi}^{\pi} G(x)\, e^{inx} e^{ih_n^{(1)} x} \left(1 + O\left(\frac{1}{n}\right)\right) dx =$$

$$= \int\limits_{-\pi}^{\pi} G(x)\, e^{inx} e^{ih_n^{(1)} x} dx + O\left(\frac{1}{n}\right), \quad \left\{O\left(\frac{1}{n}\right)\right\} \in l_2$$

$$\int\limits_{-\pi}^{\pi} G(x)\, e^{inx} e^{ih_n^{(1)} x} dx = \int\limits_{-\pi}^{\pi} G(x)\, e^{inx} dx + \sum_{k=1}^{\infty} \frac{\left(ih_n^{(1)}\right)^k}{k!} \int\limits_{-\pi}^{\pi} G(x)\, e^{inx} dx,$$

$$\left\{\int\limits_{-\pi}^{\pi} G(x)\, e^{inx} dx\right\} \in l_2$$

Taking into account $\sup\left|h_n^{(1)}\right| = h < \infty$ in the last sum, apply the Cauchy-Bunyakovski inequality:

$$\left|\sum_{k=1}^{\infty} \frac{\left(ih_n^{(1)}\right)^k}{\sqrt{k!}} \int\limits_{-\pi}^{\pi} G(x)\, e^{inx} \frac{x^k}{\sqrt{k!}} dx\right|^2 \le \sum_{k=1}^{\infty} \frac{h^{2k}}{k!} \sum_{k=1}^{\infty} \frac{1}{k!} \left|\int\limits_{-\pi}^{\pi} G(x)\, e^{inx} x^k dx\right|^2 =$$

$$= \left(e^{h^2} - 1\right) \sum_{k=1}^{\infty} \frac{1}{k!} \left|\int\limits_{-\pi}^{\pi} G(x)\, e^{inx} x^k dx\right|^2.$$

According to Parseval's equality,

$$\sum_{n=-\infty}^{\infty} \left|\sum_{k=1}^{\infty} \frac{\left(ih_n^{(1)}\right)^k}{\sqrt{k!}} \int\limits_{-\pi}^{\pi} G(x)\, e^{inx} x^k dx\right|^2 \le \left(e^{h^2} - 1\right) \sum_{k=1}^{\infty} \frac{1}{k!} \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \left|G(x)\, x^k\right|^2 dx \le$$

$$\le \frac{1}{2\pi} \left(e^{h^2} - 1\right) \sum_{k=1}^{\infty} \frac{\pi^{2k}}{k!} \int\limits_{-\pi}^{\pi} |G(x)|^2 dx = \frac{1}{2\pi} \left(e^{h^2} - 1\right) \left(e^{\pi^2} - 1\right) \int\limits_{-\pi}^{\pi} |G(x)|^2 dx < \infty.$$

So, $\{a_n\} \in l_2$. The lemma is proved

Using Lemma 1 and the inequalities $|\sin \lambda x| \le e^{|\mathrm{Im}\, \lambda \pi|}$, $|\cos \lambda x| \le e^{|\mathrm{Im}\, \lambda \pi|}$, $x \in [-\pi, \pi]$, take into account asymptotic expansions of the functions $s(\lambda, x)$, $c(\lambda, x)$,

$s_\pi(\lambda,x)$, $c_\pi(\lambda,x)$ (see: [3], p. 18) in expressions (7)-(11). Then for the characteristic functions of boundary value problems we can write the following asymptotic expansions:

$$\chi_D(\lambda)=\chi_{D,0}(\lambda)-\frac{A\alpha^+\cos\lambda\pi-B\alpha^-\cos\lambda(2a-\pi)}{\lambda^2}+e^{|\mathrm{Im}\,\lambda\pi|}\frac{f_1(\lambda)}{\lambda^2}, \tag{12}$$

$$\chi_{DN}(\lambda)=\chi_{DN,0}(\lambda)-\frac{A\alpha^+\sin\lambda\pi+B\alpha^-\sin\lambda(2a-\pi)}{\lambda}+e^{|\mathrm{Im}\,\lambda\pi|}\frac{f_2(\lambda)}{\lambda}, \tag{13}$$

$$\chi_p(\lambda)=\chi_{p,0}(\lambda)-\frac{2A\alpha^+\sin\lambda\pi}{\lambda}+e^{|\mathrm{Im}\,\lambda\pi|}\frac{f_3(\lambda)}{\lambda}, \tag{14}$$

$$\chi_a(\lambda)=\chi_{a,0}(\lambda)+\frac{2A\alpha^+\sin\lambda\pi}{\lambda}+e^{|\mathrm{Im}\,\lambda\pi|}\frac{f_4(\lambda)}{\lambda}, \tag{15}$$

where

$$A=\frac{1}{2}\int_0^\pi q(t)\,dt,\quad B=\frac{1}{2}\left(\int_a^\pi q(t)\,dt-\int_0^a q(t)\,dt\right),$$

$$\sup|f_k(\lambda)|<\infty,\ \{f_k(y_n)\}\in l_2,\quad k=1,2,3,4,\ y_n=n+h_n^{(1)}+\frac{h_n^{(2)}}{n}$$

is a sequence satisfying the condition $\sup\left|h_n^{(j)}\right|<\infty,\ j=1,2$. In [2], the distribution of eigen numbers of periodic and anti-periodic boundary value problems on a real axis was shown. Simplicity of all the zeros of the function $\chi_D\left(\sqrt{\lambda}\right)$ and the zeros of the functions $\chi_p\left(\sqrt{\lambda}\right)$ and $\chi_a\left(\sqrt{\lambda}\right)$ begining from some term, was proved. Now prove the simplicity of all the zeros of the function $\chi_{DN}\left(\sqrt{\lambda}\right)$ and show its distribution an a real axis.

Lemma 2. *The zeros of the function $\chi_{DN}(\sqrt{z})$ are simple.*

Proof. Assume that the numbers ν_k are the zeros of the function $\chi_{DN}(\sqrt{z})$. Let $\varphi_1(z,x)\overset{def}{=}s(\sqrt{z},x)\,c'_\pi(\sqrt{z},a)$, $\varphi_2(z,x)\overset{def}{=}c_\pi(\sqrt{z},x)\,s'(\sqrt{z},a)$. It is easy to prove that (see [2], p. 26) the inequality

$$\alpha\int_0^a\varphi_1^2(\nu_k,x)\,dx+\alpha^{-1}\int_a^\pi\varphi_2^2(\nu_k,x)\,dx=\dot{\chi}_{DN}(\sqrt{\nu_k})\,s'(\sqrt{\nu_k},a)\,c'_\pi(\sqrt{\nu_k},a)$$

is true. It is clear that ν_k, $k=1,2,...$ are real numbers. From the last equality we get that $\dot{\chi}_{DN}\left(\sqrt{\nu_k}\right)\neq0$, $k=1,2,...$ is satisfied. Thus, the lemma is proved,

Let the numbers λ_k be the zeros of the function $\chi_{DN}(\sqrt{z})$.

Lemma 3. *For the numbers λ_k and ν_k the relation*

$$-\infty<\nu_1<\lambda_1<\nu_2<\lambda_2<\nu_3<\lambda_3<...<\nu_k<\lambda_k<...$$

is true.

Proof. It is clear that $\alpha s\left(\sqrt{\lambda_k},a\right)=-\frac{1}{\alpha}\frac{s_\pi\left(\sqrt{\lambda_k},a\right)}{s'_\pi\left(\sqrt{\lambda_k},a\right)}s'\left(\sqrt{\lambda_k},a\right)$. If we take into account this expression in

$$\chi_{DN}\left(\sqrt{\lambda_k}\right)=\alpha s\left(\sqrt{\lambda_k},a\right)c'_\pi\left(\sqrt{\lambda_k},a\right)-\alpha^{-1}c_\pi\left(\sqrt{\lambda_k},a\right)s'\left(\sqrt{\lambda_k},a\right),$$

we get the equality

$$\chi_{DN}\left(\sqrt{\lambda_k}\right)=-\frac{1}{\alpha}\frac{s_\pi\left(\sqrt{\lambda_k},a\right)}{s'_\pi\left(\sqrt{\lambda_k},a\right)}.$$

It is clear that $sign\dfrac{s'\left(\sqrt{\lambda_k},a\right)}{s'_\pi\left(\sqrt{\lambda_k},a\right)}=sign\dot{\chi}_D\left(\sqrt{\lambda_k}\right)=(-1)^k$ (see [2], p. 28). So,

$$sign\chi_{DN}\left(\sqrt{\lambda_k}\right)=(-1)^{k+1}. \tag{17}$$

Allowing for (13),

$$\lim_{z\to-\infty}\chi_{DN}\left(\sqrt{z}\right)=-\infty\,(\alpha>0).$$

This equality and (17) shows the validity of the lemma.

Assume that the numbers $z_k^{(1)}$ and $z_k^{(2)}$ are the zeros of the functions $F_1(z)=\sin\pi z+\nu\sin bz$, $F_2(z)=\cos\pi z+\nu\cos bz$, $(|\nu|<1, 0\le b<\pi)$, respectively

$$G_k=\left\{z:\left|\operatorname{Re}z-\frac{k}{2}\right|<\frac{1}{2}\right\},\quad \Gamma_k=\left\{z:\operatorname{Re}z=k\pm\frac{1}{2}\right\}.$$

Lemma 4. *On each domain G_{2n+j-1} the function $F_j(z)$ has a unique zero, all zeros are real, and they satisfy the relation*

$$\inf_{s\neq m}\left|z_s^{(j)}-z_m^{(j)}\right|\ge 1-2\theta>0, \tag{18}$$

where $\theta=\frac{1}{\pi}\arcsin|\nu|$, $j=1,2$.

Proof. It is easy to show that the inequality $\left|\nu\cos\left(\frac{\pi}{2}j+bz\right)\right|<\left|\cos\left(\frac{\pi}{2}j+\pi z\right)\right|$, $z\in\Gamma_{2n+j-1}$, $j=1,2$ is true. It is clear that at each domain G_{2n+j-1} the function $\cos\left(\frac{\pi}{2}j+\pi z\right)$, $j=1,2$ has a unique zero. According to the Rouche theorem, at each domain G_{2n+j-1}, $j=1,2$ the function $F_j(z)$, $j=1,2$ has a unique zero

$$\sin\pi z=|\nu|,\quad z=\omega_n^{(1)}=(-1)^n\theta+n;$$

$$\sin\pi z=-|\nu|,\quad z=\sigma_n^{(1)}=(-1)^{n+1}\theta+n;$$

$$\cos\pi z=|\nu|,\quad z=\omega_{2n}^{(2)}=2n+\theta-\frac{1}{2},\quad z=\sigma_{2n+1}^{(2)}=2n-\theta+\frac{1}{2};$$

$$\cos\pi z=-|\nu|,\; z=\omega_{2n+1}^{(2)}=2n+\theta+\frac{1}{2},\quad z=\sigma_{2n}^{(2)}=2n-\theta-\frac{1}{2},$$

$$\theta=\frac{1}{\pi}\arcsin|\nu|,\quad 0<\theta<\frac{1}{2}.$$

It is clear that $F_j\left(\sigma_n^{(j)}\right)F_j\left(\omega_n^{(j)}\right)<0, j=1,2$. So, for each number n there exists a real number $z_n^{(j)}$ such that $z_n^{(j)}\in\left[\sigma_n^{(j)},\omega_n^{(j)}\right]$ and $F_j\left(z_n^{(j)}\right)=0$, $j=1,2$ i.e.

$$z_n^{(j)}\in G_{2n+j-1}, j=1,2.$$

We proved the first part of the lemma. Now prove the validity of relation (18)

$$\inf_{s\neq m}\left|z_s^{(1)}-z_m^{(1)}\right|=\inf_k\left|z_{k+1}^{(1)}-z_k^{(1)}\right|\ge$$

$$\geq \min\left\{\inf_k \left|\sigma_{k+1}^{(1)}-\sigma_k^{(1)}\right|, \inf_k \left|\omega_{k+1}^{(1)}-\omega_k^{(1)}\right|\right\} = 1-2\theta>0,$$

$$\inf_{s\neq m}\left|z_s^{(2)}-z_m^{(2)}\right| = \inf_k \left|z_{k+1}^{(2)}-z_k^{(2)}\right| \geq \inf_k \left|\sigma_{k+1}^{(2)}-\omega_k^{(2)}\right| = 1-2\theta>0.$$

Thus, the lemma is proved completely.

Lemma 5. *There exists a number $c>0$ independent of the numbers $\delta>0$ and n such that for $z:\left|z-z_n^{(j)}\right|<\delta$ the inequality*

$$|F_j(z)| \geq c\left|z-z_n^{(j)}\right| e^{|\mathrm{Im}\, z\pi|}, \quad j=1,2 \tag{19}$$

is satisfied, where $\delta << 1-2\theta$.

Proof. At first prove that the relation

$$\inf_n \left|F_j\left(z_n^{(j)}\right)\right| \geq c'>0, \quad j=1,2 \tag{20}$$

is true. Assume that $j=1$

$$\frac{\left|F_1'\left(z_n^{(1)}\right)\right|}{\pi} = \left|\cos\pi z_n^{(1)} + \frac{b}{\pi}\nu\cos b z_n^{(1)}\right| \geq \left|\left|\cos\pi z_n^{(1)}\right| - \left|\frac{b}{\pi}\right|\left|\nu\cos b z_n^{(1)}\right|\right| =$$

$$= \frac{\left|\cos^2\pi z_n^{(1)} - \left(\frac{b}{\pi}\right)^2\nu^2\cos^2 b z_n^{(1)}\right|}{\left|\cos\pi z_n^{(1)}\right| + \left|\frac{b\nu}{\pi}\right|\left|\nu\cos b z_n^{(1)}\right|} \geq \frac{1-\sin^2\pi z_n^{(1)} - \left(\frac{b}{n}\right)^2\nu^2 + \left(\frac{b}{n}\right)^2\sin^2\pi z_n^{(1)}}{1+\left|\frac{b\nu}{\pi}\right|} \geq$$

$$\geq \frac{1-\left(\frac{b}{\pi}\right)^2\nu^2 - 1 + \left(\frac{b}{\pi}\right)^2}{1+\left|\frac{b\nu}{\pi}\right|},$$

$$\left|F_1'\left(z_n^{(1)}\right)\right| \geq \frac{b^2\left(1-\nu^2\right)}{\pi+|b\nu|}>0, \quad b>0$$

for $b=0$, $\left|F_1'\left(z_n^{(1)}\right)\right| = |F_1'(n)| = \pi>0$. For $j=2$ (20) is proved in the same way. Thus,

$$\inf_n \left|F_j\left(z_n^{(j)}\right)\right| = c'>0, \quad j=1,2.$$

It is known that

$$F_j(z) = F_j'\left(z_n^{(j)}\right)\left(z-z_n^{(j)}\right) + o\left(z-z_n^{(j)}\right), \quad |z|\to\left|z_n^{(j)}\right|, \quad j=1,2.$$

From this equality we get that the relation

$$|F_j(z)| \geq \left(\left|F_j'\left(z_n^{(j)}\right)\right| - \frac{\left|o\left(z-z_n^{(j)}\right)\right|}{\left|z-z_n^{(j)}\right|}\right)\left|z-z_n^{(j)}\right| \geq$$

$$\geq \left(c' - \frac{\left|o\left(z-z_n^{(j)}\right)\right|}{\left|z-z_n^{(j)}\right|}\right)\left|z-z_n^{(j)}\right|, \quad |z|\to\left|z_n^{(j)}\right|, \quad j=1,2$$

is true. It is clear that one can find a number $\delta_0 > 0$ such that for each integer n, when $z : \left|z - z_n^{(j)}\right| < \delta_0$ the inequality $\frac{o\ z - z_n^{(j)}}{z - z_n^{(j)}} < \frac{c'}{2}$ be true. Denote $\delta = \min\left\{\delta_0, \frac{1-2\theta}{2}\right\}$. According to the last inequality, the relation

$$|F_j(z)| \geq \frac{c'}{2}\left|z - z_n^{(j)}\right|, \quad j = 1, 2$$

is true. Thus,

$$|F_j(z)| \geq \frac{c'}{2}\left|z - z_n^{(j)}\right| = \frac{c'}{2e^{\pi}}\left|z - z_n^{(j)}\right| e^{\pi} \geq$$

$$\geq \frac{c'}{2e^{\pi}}\left|z - z_n^{(j)}\right| e^{|\mathrm{Im}\, z\pi|} = c\left|z - z_n^{(j)}\right| e^{|\mathrm{Im}\, z\pi|}, \ c = \frac{c'}{2e^{\pi}} > 0, \quad j = 1, 2.$$

The lemma is proved.

Note that the estimates (18) and (20) may be obtained by other ways as well (see: [1]).

Assume that the function $\chi_0(z)$ is one of the functions $F_1(z)$ or $F_2(z)$, the numbers $z_{k,0}$ are its zeros, $\widetilde{\chi}(z)$ is a function consisting of any linear combination of the functions $e^{-i\pi z}$, $e^{i\pi z}$, e^{-ibz}, e^{ibz}, the function $f(z)$ is a function satisfying the conditions sup sup $|f(z)| < \infty$ and $\left\{f\left(z_{k,0} + \frac{h'_k}{z_{k,0}}\right)\right\} \in l_2$, $\sup|h'_k| < \infty$

$$\chi(z) \stackrel{def}{=} \chi_0(z) + \frac{\widetilde{\chi}(z)}{z} + e^{|\mathrm{Im}\, z\pi|}\frac{f(z)}{z}. \tag{21}$$

Let the numbers z_k be the zeros of the function $\chi(z)$.

Theorem 1. *For the numbers z_k the asymptotic expansion formula*

$$z_k = z_{k,0} + \frac{\alpha_k^{(1)}}{z_{k,0}} + \frac{\alpha_k^{(2)}}{z_{k,0}}, \ \alpha_k^{(1)} = -\frac{\widetilde{\chi}(z_{k,0})}{\chi'_0(z_{k,0})}, \quad \left\{\alpha_k^{(2)}\right\} \in l_2 \tag{22}$$

is true.

Proof. First we prove that

$$z_k = z_{k,0} + \frac{h_k}{z_{k,0}}, \quad \sup|h_k| < \infty \tag{23}$$

is true. It is easy to prove that $\widetilde{\chi}(z) = e^{|\mathrm{Im}\, z\pi|}O(1)$, $|z| \to \infty$ is true. Then we can find a natural number k_0 and a number $c_0 > 0$ such that for $z : |z| > k_0$ the inequality $\left|e^{-|\mathrm{Im}\, z\pi|}\widetilde{\chi}(z) + f(z)\right| < c_0$ is satisfied, and furthermore $\delta_1 = \frac{2c_0}{c}\frac{1}{k} < \frac{\delta}{2}$. Thus, for $z : |z - z_{k,0}| < \delta_1$, $|z| > \frac{\kappa}{2}$ is true. According to (19), the inequality

$$|\chi(z) - \chi_0(z)| = \frac{\left|e^{-|\mathrm{Im}\, z\pi|}\widetilde{\chi}(z) + f(z)\right|}{|z|}e^{|\mathrm{Im}\, z\pi|} < \frac{c_0}{|z|}e^{|\mathrm{Im}\, z\pi|} <$$

$$< \frac{2c_0}{k}e^{|\mathrm{Im}\, z\pi|} < c\delta_1 e^{|\mathrm{Im}\, z\pi|} < |\chi_0(z)|$$

is true. According to the Rouche theorem, for each natural number $k > 2k_0$ the function $\chi(z)$ has a unique zero at each domain $|z - z_{k,0}| < \delta_1$. Denote this zero by

z_k. According to Lemma 4, we can write the relation $z_{k,0} = O(k)$, $k \to \infty$. So, the relation

$$z_k - z_{k,0} = \frac{h_k}{z_{k,0}}, \quad \sup |h_k| < \infty$$

or

$$z_k = z_{k,0} + \frac{h_k}{z_{k,0}}, \quad \sup |h_k| < \infty$$

is true.

Write the following equalities:

$$e^{itz_k} = e^{itz_{k,0}} e^{it\frac{h_k}{z_{k,0}}} = e^{itz_{k,0}} \left(1 + \frac{h_k}{z_{k,0}} it\right) + O\left(\left(\frac{1}{z_{k,0}}\right)^2\right), \quad k \to \infty$$

or

$$e^{itz_k} = \left(e^{itz} + \frac{h_k}{z}\left(e^{itz}\right)'\right)|_{z=z_{k,0}} + O\left(\left(\frac{1}{z_{k,0}}\right)^2\right), \quad k \to \infty.$$

According to this equality, we can state the validity of relations

$$\chi_0(z_k) = \chi_0(z_{k,0}) + \frac{h_k}{z_{k,0}} \chi_0'(z_{k,0}) + O\left(\left(\frac{1}{z_{k,0}}\right)^2\right) =$$

$$= \frac{h_k}{z_{k,0}} \chi_0'(z_{k,0}) + O\left(\left(\frac{1}{z_{k,0}}\right)^2\right), \quad k \to \infty,$$

$$\chi_0(z_k) = \frac{h_k}{z_{k,0}} \chi_0'(z_{k,0}) + O\left(\left(\frac{1}{z_{k,0}}\right)^2\right), \quad k \to \infty \tag{24}$$

$$\widetilde{\chi}(z_k) = \widetilde{\chi}(z_{k,0}) + \frac{h_k}{z_{k,0}} \widetilde{\chi}'(z_{k,0}) + O\left(\left(\frac{1}{z_{k,0}}\right)^2\right), \quad k \to \infty$$

From the last equality, we can write the validity of

$$\frac{\widetilde{\chi}(z_k)}{z_k} = \frac{\widetilde{\chi}(z_{k,0})}{z_{k,0}} + O\left(\left(\frac{1}{z_{k,0}}\right)^2\right), \quad k \to \infty \tag{25}$$

According to (24) and (25),

$$0 = \chi(z_k) = \frac{h_k}{z_{k,0}} \chi_0'(z_{k,0}) + \frac{\widetilde{\chi}(z_{k,0})}{z_{k,0}} + \frac{f(z_k)}{z_{k,0}} + O\left(\left(\frac{1}{z_{k,0}}\right)^2\right),$$

$$h_k = \alpha_k^{(1)} + \alpha_k^{(2)},$$

$$\alpha_k^{(1)} = -\frac{\widetilde{\chi}(z_{k,0})}{\chi_0'(z_{k,0})}, \quad \alpha_k^{(2)} = \frac{f(z_k) + O\left(\frac{1}{z_{k,0}}\right)}{\chi_0'(z_{k,0})}.$$

According to (20), $\sup \left| \frac{1}{\chi_0'(z_{k,0})} \right| < \infty$. Hence $\left\{\alpha_k^{(2)}\right\} \in l_2$.

The theorem is proved.

If in expression (21) of the function $\chi(z)$ we accept

1) $\chi(z)=\frac{z}{\alpha^+}\chi_D(z)$, $\chi_0(z)=\frac{z}{\alpha^+}\chi_{D,0}(z)$,
$\widetilde{\chi}(z)=\frac{1}{\alpha^+}\left(A\alpha^+\cos z\pi-B\alpha^-\cos z(2a-\pi)\right)$, $f(z)=\frac{1}{\alpha^+}f_1(z)$;
2) $\chi(z)=-\frac{1}{\alpha^+}\chi_{DN}(z)$, $\chi_0(z)=-\frac{1}{\alpha^+}\chi_{DN,0}(z)$,
$\widetilde{\chi}(z)=\frac{1}{\alpha^+}\left(A\alpha^+\sin z\pi+B\alpha^-\sin z(2a-\pi)\right)$, $f(z)=-\frac{1}{\alpha^+}f_2(z)$;
3) $\chi(z)=-\frac{1}{2\alpha^+}\chi_p(z)$, $\chi_0(z)=\cos z\pi-\frac{1}{\alpha^+}$, $\widetilde{\chi}(z)=A\sin z\pi$, $f(z)=-\frac{1}{2\alpha^+}f_3(z)$;
4) $\chi(z)=\frac{1}{2\alpha^+}\chi_a(z)$, $\chi_0(z)=\cos z\pi+\frac{1}{\alpha^+}$, $\widetilde{\chi}(z)=A\sin z\pi$, $f(z)=-\frac{1}{2\alpha^+}f_4(z)$,

we prove the following theorem for asymptotics of eigen numbers of Dirichlet, Dirichlet – Neumann periodic and antiperiodic problems.

Theorem 2. *For the eigen values λ_n, ν_n, $\mu_n^{\pm}$ of problems* (1),(2),(3), (1),(2),(4), (1),(2),(5),(6)

$$\sqrt{\lambda_n}=\sqrt{\lambda_{n,0}}+\frac{a_n}{\sqrt{\lambda_{n,0}}}+\frac{\alpha_n}{\sqrt{\lambda_{n,0}}},$$

$$a_n=\frac{A\alpha^+\cos\sqrt{\lambda_{n,0}}\pi-B\alpha^-\cos\sqrt{\lambda_{n,0}}(2a-\pi)}{\sqrt{\lambda_{n,0}}\chi_D'\left(\sqrt{\lambda_{n,0}}\right)},\{\alpha_n\}\in l_2$$

$$\sqrt{\nu_n}=\sqrt{\nu_{n,0}}+\frac{b_n}{\sqrt{\nu_{n,0}}}+\frac{\beta_n}{\sqrt{\nu_{n,0}}},$$

$$b_n=\frac{A\alpha^+\sin\sqrt{\nu_{n,0}}\pi+B\alpha^-\sin\sqrt{\nu_{n,0}}(2a-\pi)}{\chi_{DN}'\left(\sqrt{\nu_{n,0}}\right)},\{\beta_n\}\in l_2$$

$$\sqrt{\mu_n^{\pm}}=\sqrt{\mu_{n,0}^{\pm}}+\frac{A}{\pi\sqrt{\mu_{n,0}^{\pm}}}+\frac{\gamma_n}{\sqrt{\mu_{n,0}^{\pm}}},$$

$$\mu_{n,0}^{\pm}=n\pm\theta_1,\ \theta_1=\frac{1}{\pi}\arccos\frac{1}{\alpha^+},\{\gamma_n\}\in l_2$$

$$A=\frac{1}{2}\int\limits_0^{\pi}q(t)\,dt,\quad B=\frac{1}{2}\left(\int\limits_a^{\pi}q(t)\,dt-\int\limits_0^{a}q(t)\,dt\right),$$

$$\alpha^{\pm}=\frac{1}{2}\left(\alpha\pm\frac{1}{\alpha}\right)$$

where the numbers $\lambda_{n,0}$, $\nu_{n,0}$, $\mu_{2n,0}^{\pm}$, $\mu_{2n+1,0}^{\pm}$ are the zeros of the functions $\chi_{D,0}(\sqrt{z})$, $\chi_{DN,0}(\sqrt{z})$, $\chi_{p,0}(\sqrt{z})$ and $\chi_{a,0}(\sqrt{z})$, respectively.

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Hidayat M. Huseynov
Institute of Mathematics and Mechanics of NAS of Azerbaijan
9, B.Vahabzade str., AZ 1141, Baku, Azerbaijan

Baku State University,
23, Z. Khalilov str., AZ 1148, Baku, Azerbaijan

Famil Z. Dostuyev
Baku State University,
23, Z. Khalilov str., AZ 1148, Baku, Azerbaijan
Tel.: (99412) 539 47 20 (off.).

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