1999 VOL. X(XVIII)

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ON THE MULTIDIMENSIONAL WEIGHTED HARDY INEQUALITIES OF FRACTIONAL ORDER

Abstract

In present paper some generalization of two-weighted fractional order of Hardy inequality on multidimensional case over arbitrary domain was obtained.

This paper is devoted to weighted n-dimensional Hardy inequalities of fractional order which found essential application in the problem of compact imbedding of Sobolev space $W_p^1(D)$ into $L_p(D)$, for the domains with unsmooth bound [1,2] (compare with [3, §8.4).

For the one-dimensional case let's note the following results.

The next inequality was proved by Yakovlev G.N. [4] the obtained:

$$\int_0^\infty |u(t)-u(0)|^p t^{-\lambda p} dt \le c \int_0^\infty\int_0^\infty \frac{|u(x)-u(t)|^p}{|x-t|^{1+\lambda p}} dxdt, \tag{1}$$

$1 \le p \le \infty$, $0 \le \lambda \le 1/p$ (see also [5,6]); in [7] the inequality (1) was extended to the case $0 < p < 1, 0 < \lambda < 1$. In the work [2] by V.Burenkov and V.Evans it was proved that for validity of the estimation

$$\int_0^\infty |u(t)|^p \upsilon(t) dt \le c \int_0^\infty\int_0^\infty |u(x)-u(t)|^p \omega(|x-t|) dxdt$$

for $\upsilon(t) = \int_t^\infty \omega(\tau)d\tau$, $1 < p < \infty$, $u \in L_\upsilon^p(0,\infty)$ it is sufficient that

$$\upsilon(t) \le c\upsilon(2t), \quad 0 < t < \infty.$$

The following generalization

$$\int_0^\infty |u(x)|^p \omega_\lambda(x) dx \le c \int_0^\infty\int_0^\infty \omega(x,t) \frac{|u(x)-u(t)|^p}{|x-t|^{1+\lambda p}} dxdt \tag{2}$$

of the previous result for the weights $w(x) = \left(\frac{1}{x}\int_0^x \omega^{-\frac{1}{p-1}}(x,t)dt \right)^{1-p}$, $\omega_\lambda(x) = x^{-\lambda p} w(x)$ and $1 < p < \infty$, $-\frac{1}{p} < \lambda < \infty$ was given by H.Henig, A.Kufner and I.Persson [8], where proposed that if

$$C_p := \sup_{t>0} \left(\int_t^\infty \frac{w(x)}{X^{(1+\lambda)p}} dx \right) \left(\int_0^t w^{-\frac{1}{p-1}}(x) x^{\frac{\lambda p}{p-1}} dx \right)^{\frac{p-1}{p}} < \infty \tag{3}$$

$$k = \frac{pC_p}{(p-1)^{(p-1)/p}} < 1 \tag{4}$$

then the inequality (2) holds for any function $u \in L_{\omega_\lambda}^p(0,\infty)$.

In the same work ([8], p.15) the problem of generalization results of (2)-(4) to the multidimensional case was formulated.

The spaces of the functions with generalized smoothness were considered by some authors [9-12].

In the paper the following denotations have been taken:

Q is an arbitrary ball in $R^n, Q_\tau^x = \{y : |y-x| < \tau\}$. $|A|$ is Lebesque measure of the set A, $v(A) = \int_A v(y)dy$. We also denote by $\psi(A,B) = \int_A \int_B \psi(x,y)dxdy$, for the sets A, B. $L_v^q(D)$ is the space of functions $u(x)$ with the finite norm

$$\|u\|_{L_v^q(D)} = \left(\int_D |u(x)|^q \upsilon(x) dx \right)^{1/q} ;$$

$C(D)$ is the space of continuous functions $u(x)$ in the open domain $D \subset R^n$, $n \geq 1$ with the norm $\|u\|_{C(D)} = \sup_D |u(x)|$; A_∞ is Muckenhoupt class. The note $\psi \in A_\infty$ according to [13] means: there are such $c_1 > 0, c_2 > 0$, $0 < \varepsilon \leq 1, 1 \leq \beta < \infty$ that for any ball Q and compact subset $e \subset Q$ it is true

$$c_1 \left(\frac{|e|}{|Q|} \right)^\beta \leq \frac{\psi(e)}{\psi(Q)} \leq c_2 \left(\frac{|e|}{|Q|} \right)^\delta . \tag{5}$$

By symbols $c, c_1, c_2, c_3, \ldots$, and also $\beta; 1 \leq \beta < \infty, \delta; 0 < \delta \leq 1$ we define the different constants which value is not essential for purpose of the paper.

Theorem 1. (Sobolev type). *Let* $-\frac{n}{p} \leq \lambda < \infty, 1 \leq p \leq q \leq r < \infty$ *and* $v : R^n \to [0,\infty), \omega : R^n \times R^n \to [0,\infty)$, *moreover* v *belongs to* A_∞*-class.*

Assume, that there exists such function $\psi(x,y)$, *which belongs to Muckenhoupt* A_∞*-class relative to each variables and exists such constant* $A > 0$ *that, for any ball* $Q \subset R^n$ *the condition* $A_{(p,q),r}^{\lambda,\psi}$ *is fulfilled, i.e.*

$$\frac{|Q|^{1/p+\lambda/n} v(Q)^{1/r}}{\psi(Q,Q)} \left\{ \int_Q \left[\int_Q \left(\frac{\psi^p(x,y)}{\omega(x,y)} \right)^{\frac{1}{p-1}} dy \right]^{\frac{q(p-1)}{p(q-1)}} dx \right\}^{\frac{q-1}{q}} < A . \tag{6}$$

Then for any function $u \in C(R^n)$, $\lim_{x\to\infty} u(x) = 0$ *the following estimate is valid*

$$\left(\int_{R^n} |u(x)|^r v(x) dx \right)^{1/r} \leq cA \left\{ \int_{R^n} dx \left[\int_{R^n} \frac{\left||u(x)| - |u(y)|\right|^p}{|x-y|^{n+p\lambda}} \omega(x,y) dy \right]^{\frac{q}{p}} \right\}^{\frac{1}{q}} , \tag{7}$$

where the constant $c > 0$ *depends only on* n, q, λ *and on* A_∞*-constants of functions* v, ψ.

Proof. Fix the real number α, where $\alpha > 0$. Let $e_\alpha = \{x \in R^n : |u(x)| > \alpha\}$ and $\gamma,\ 0 < \gamma \le \frac{1}{2}$ be sufficiently small number. The precise value of γ will be chosen later. Show that for any point $x \in e_{2\alpha}$ there is such ball $Q^x_{\rho(x,\alpha)}$ that

$$\left|Q^x_{\rho(x,\alpha)} \setminus e_\alpha\right| = \gamma \left|Q^x_{\rho(x,\alpha)}\right|. \tag{8}$$

Consider the auxiliary function

$$F(t) = \frac{\left|Q^x_t \setminus e_\alpha\right|}{\left|Q^x_t\right|} - \gamma, \tag{9}$$

which is continuous on $[0, \infty)$. So as $u(x)$ is continuous function, then the set e_α is open. Therefore such $\varepsilon > 0$ will be found that $F(\varepsilon) < 0$. The function $u(x)$ tends to zero at infinity, and for this reason the set e_α is also bounded and $d(\alpha) = diam\, e_\alpha < +\infty$, then

$$F[2d(\alpha)] = \frac{\left|Q^x_{2d(\alpha)} \setminus e_\alpha\right|}{\left|Q^x_{2d(\alpha)}\right|} - \gamma \ge \frac{\left|Q^x_{d(\alpha)}\right|}{\left|Q^x_{2d(\alpha)}\right|} - \gamma = \frac{1}{2^n} - \gamma \ge 0. \tag{10}$$

With the help of Cauchy theorem on continuous functions such $\rho(x,\alpha),\ 0 < \rho(x,\alpha) \le 2d(\alpha)$ exists for which $F[\rho(x,\alpha)] = 0$, therefore (8) is true.

There exists the family of balls $\{Q^x_{\rho(x,\alpha)}\}, x \in e_{2\alpha}$, whose diameters are bounded by number $d(\alpha)$, and the centers belong to $e_{2\alpha}$. By Bezikovich lemma [14] on covering from the system of balls $\{Q^x_{\rho(x,\alpha)}\}, x \in e_{2\alpha}$ the subcovering $\{Q^i(\alpha)\}$, can be chosen, $i \in N$ with finite multiplicity - $\aleph_n$. Fix some ball $Q^i(\alpha)$, by construction $e_{2\alpha} \cap Q^i(\alpha)$ is not empty and

$$\left|Q^i(\alpha) \setminus e_\alpha\right| = \gamma \left|Q^i(\alpha)\right|, \tag{11}$$

moreover either 1)

$$\left|e_{2\alpha} \cap Q^i(\alpha)\right| \le \gamma \left|Q^i(\alpha)\right| \tag{12}$$

or 2) $\left|e_{2\alpha} \cap Q^i(\alpha)\right| > \gamma \left|Q^i(\alpha)\right|$. In the first case by virtue of $v \in A_\infty$

$$v\left(e_{2\alpha} \cap Q^i(\alpha)\right) \le c \left(\frac{\left|e_{2\alpha} \cap Q^i(\alpha)\right|}{\left|Q^i(\alpha)\right|}\right)^\delta v\left(Q^i(\alpha)\right) \le c\gamma^\delta v\left(Q^i(\alpha)\right). \tag{13}$$

From A_∞-condition on v and (11)

$$v\left(Q^i(\alpha) \setminus e_\alpha\right) \le c\gamma^\delta v\left(Q^i(\alpha)\right)$$

follows.

Using the latter in identity

$$v\left(Q^i(\alpha)\right) = v\left(Q^i(\alpha) \cap e_\alpha\right) + v\left(Q^i(\alpha) \setminus e_\alpha\right)$$

we obtain

$$v\left(Q^i(\alpha)\right) \le \frac{1}{1 - c\gamma^\delta} v\left(e_\alpha \cap Q^i(\alpha)\right).$$

Then from (13) we find

$$v\left(e_{2\alpha} \cap Q^i(\alpha)\right) \le \frac{c\gamma^\delta}{1 - c\gamma^\delta} v\left(e_\alpha \cap Q^i(\alpha)\right) \tag{14}$$

If $\left|Q^i(\alpha)\cap e_\alpha\right|>\gamma\left|Q^i(\alpha)\right|$, then by virtue of (11) and $\psi\in A_\infty$

$$\int\limits_{Q^i(\alpha)\setminus e_\alpha}\left(\int\limits_{Q^i(\alpha)\cap e_{2\alpha}}\psi(x,y)dy\right)dx\geq\left(c_1\gamma^\beta\right)^2\psi\left(Q^i(\alpha),Q^i(\alpha)\right).$$

Using the Hölder's inequality we conclude that

$$1\leq\left(\frac{1}{c_1^2\gamma^{2\beta}\psi\left(Q^i(\alpha),Q^i(\alpha)\right)}\right)\left[\int\limits_{Q^i(\alpha)\setminus e_\alpha}\left(\int\limits_{Q^i(\alpha)\cap e_{2\alpha}}\omega\,(x,y)dy\right)^{\frac{q}{p}}dx\right]^{\frac{1}{q}}\times$$

$$\times\left\{\int\limits_{Q^i(\alpha)\setminus e_\alpha}\left[\int\limits_{Q^i(\alpha)\cap e_{2\alpha}}\left(\frac{\psi^p(x,y)}{\omega(x,y)}\right)^{\frac{1}{p-1}}dy\right]^{\frac{q(p-1)}{p(q-1)}}dx\right\}^{\frac{q-1}{q}}$$

Since for $x\in Q^i(\alpha)\setminus e_\alpha$, $y\in Q^i(\alpha)\cap e_{2\alpha}$ we have $|x-y|\leq c_0\left|Q^i(\alpha)\right|^{1/n}$, then from the latter estimation we obtain

$$v\left(Q^i(\alpha)\cap e_{2\alpha}\right)\leq v\left(Q^i(\alpha)\right)\left(\frac{\left|Q^i(\alpha)\right|^{\frac{n+p\lambda}{pn}}}{c_1^2\gamma^{2\beta}\psi\left(Q^i(\alpha),Q^i(\alpha)\right)}\right)^r\times$$

$$\left[\int\limits_{Q^i(\alpha)\setminus e_\alpha}\left(\int\limits_{Q^i(\alpha)\cap e_{2\alpha}}\frac{\omega(x,y)}{|x-y|^{n+p\lambda}}dy\right)^{\frac{q}{p}}dx\right]^{\frac{r}{q}}\times\left\{\int\limits_{Q^i(\alpha)}\left[\int\limits_{Q^i(\alpha)}\left(\frac{\psi^p}{\omega}\right)^{\frac{1}{p-1}}dy\right]^{\frac{q(p-1)}{p(q-1)}}dx\right\}^{\frac{q-1}{q}r}$$

By virtue of condition (6) we have

$$v\left(Q^i(\alpha)\cap e_{2\alpha}\right)\leq\frac{cA^r}{\gamma^{2\beta r}}\left[\int\limits_{Q^i(\alpha)\setminus e_\alpha}\left(\int\limits_{Q^i(\alpha)\cap e_{2\alpha}}\frac{\psi(x,y)}{|x-y|^{n+p\lambda}}dy\right)^{\frac{q}{p}}dx\right]^{\frac{r}{q}}\tag{15}$$

From (14) and (15) it follows

$$v\left(e_{2\alpha}\cap Q^i(\alpha)\right)\leq\frac{c\gamma^\delta}{1-c\gamma^\delta}v\left(e_\alpha\cap Q^i(\alpha)\right)+\frac{cA^r}{\gamma^{2\beta r}}\left[\int\limits_{Q^i(\alpha)\setminus e_\alpha}\left(\int\limits_{Q^i(\alpha)\cap e_{2\alpha}}\frac{\omega(x,y)}{|x-y|^{n+p\lambda}}dy\right)^{\frac{q}{p}}dx\right]^{\frac{r}{q}}\tag{16}$$

Summing all the inequalities (16) by i taking into account the finite multiplicity of the covering $\left\{Q^i(\alpha)\right\}$, $i\in N$ and condition $1\leq p\leq q\leq r<\infty$ we have

$$\nu(e_{2\alpha}) \le \frac{\aleph_n c\gamma^\delta}{1-c\gamma^\delta}\nu(e_\alpha) + \frac{c\aleph_n A^r}{\gamma^{2\beta r}}\left[\int\limits_{\cup Q^i(\alpha)\setminus e_\alpha} dx \left(\int\limits_{\cup Q^i(\alpha)\cap e_\alpha} \frac{\omega(x,y)}{|x-y|^{n+p\lambda}}dy\right)^{\frac{q}{p}}\right]^{\frac{r}{q}} \le$$

$$\le \frac{\aleph_n c\gamma^\delta}{1-c\gamma^\delta}\nu(e_\alpha) + \frac{c\aleph_n A^r}{\gamma^{2\beta r}}\left[\int\limits_{R^n\setminus e_{2\alpha}} dx \left(\int\limits_{e_\alpha} \frac{\omega(x,y)}{|x-y|^{n+p\lambda}}dy\right)^{\frac{q}{p}}\right]^{\frac{r}{q}}.$$

Therefore we have

$$\nu(e_{2\alpha}) \le \frac{c\gamma^\delta}{1-c\gamma^\delta}\nu(e_\alpha) + \frac{cA^r}{\gamma^{2\beta r}}\left[\int\limits_{R^n} dx \left(\int\limits_{\left\{\substack{y\in R^n:\\ \left||u(y)|-|u(x)|\right|\ge\alpha}\right\}} \frac{\omega(x,y)}{|x-y|^{n+p\lambda}}dy\right)^{\frac{q}{p}}\right]^{\frac{r}{q}}.$$

Integrating the latter inequality over $[0,\infty)$ we obtain

$$\int\limits_0^\infty \nu(e_{2\alpha})d\alpha^r \le \frac{c_1\gamma^\delta}{1-c\gamma^\delta}\nu(e_\alpha)d\alpha^r + \frac{cA^r}{\gamma^{2\beta r}}\left[\int\limits_0^\infty d\alpha^r\left[\int\limits_{R^n} dx \left(\int\limits_{\left||u(y)|-|u(x)|\right|\ge\alpha} \frac{\omega(x,y)}{|x-y|^{n+p\lambda}}dy\right)^{\frac{q}{p}}\right]\right]^{\frac{r}{q}} \tag{17}$$

Choose $\gamma: 0<\gamma\le\frac{1}{2^n}, c\gamma^\delta<1$ so that $\frac{1}{2^r}-\frac{c_1\gamma^d}{1-c\gamma^\delta}>0$.

Then from (17) with the help of Minkovsky inequality we find

$$\int\limits_0^\infty \nu(e_\alpha)d\alpha^r \le cA^r\left[\int\limits_{R^n} dx\left(\int\limits_{R_n}\frac{\left||u(x)|-|u(y)|\right|^p}{|x-y|^{n+p\lambda}}\omega(x,y)dy\right)^{\frac{q}{p}}\right]^{\frac{r}{q}},$$

Which with the identity $\int\limits_0^\infty \nu(e_\alpha)d\alpha^r = \|u\|^r_{L^r_\nu(R^n)}$ implies (7).

Theorem 1 has been proved.

Theorem 2. *Let all the assumptions with respect to numbers* p,q,r,λ *and functions* ν,ω *of Theorem 1 be fulfilled.*

Suppose, that the function u *belongs to the space* $L^r_\nu(R^n)\cap C(R^n)$.

Then (7) holds if the condition (6) takes place.

Proof. For $\alpha>0$ we denote $e_\alpha=\{x\in R^n: |u(x)|>\alpha\}$, as before. In differ to Theorem 1 here the set e_α can be unbounded.

Fix $\alpha;\alpha>0$. For any point $x\in e_\alpha$ there are the ball $Q^x_{\rho(x,\alpha)}$, for which

$$Q^x_{\rho(x,\alpha)}\setminus e_\alpha = \gamma\left|Q^x_{\rho(x,\alpha)}\right| \tag{18}$$

where $\gamma, 0<\gamma<\frac{1}{2^n}$, as in Theorem 1, will be chosen later.

Introduce as above the function $F(t)$ by formula (9). Such $\varepsilon>0$ will be found that $F(\varepsilon)<0$. From $u\in L_v^r(R^n)$ it follows that there exists such constant $c>0$ that $v(e_\alpha)<\frac{c}{\alpha^r}$, then $v(Q_t^x\cap e_\alpha)<\frac{c}{\alpha^r}$ for any $t\in(0,\infty)$. By virtue of $v\in A_\infty$, there exist $c_1>0, c_2>0, 1\le\beta<\infty, 0<\delta\le 1$ such that

$$\frac{v(e_\alpha\cap Q_t^x)}{v(Q_t^x)}\ge c_1\left(\frac{|e_\alpha\cap Q_t^x|}{|Q_t^x|}\right)^\beta,\qquad \frac{v(Q_1^x)}{v(Q_t^x)}\le c_2\left(\frac{|Q_1^x|}{|Q_t^x|}\right)^\delta, \text{ where } 1<t<\infty,$$

hence

$$\frac{|e_\alpha\cap Q_t^x|}{|Q_t^x|}\le\left(\frac{c_2 c}{v(Q_1^x)c_1\alpha^r}\right)^{1/\beta} t^{-\frac{n\delta}{\beta}}.$$

Let $t=t_0\in(1,\infty)$ be such that the right-hand side $\le 1-\gamma$. Then

$$Q_{t_0}^x\setminus e_\alpha>\gamma|Q_{t_0}^x|,$$

so $F(t_0)>0$. With the help of Cauchy theorem for some $\xi\in(0,t_0)$, $F(\xi)=0$. Denote by $\rho(x,\alpha)$ the smallest of such numbers ξ. Then $F(\rho(x,\alpha))=0$ which yields (18).

So as $v(e_\alpha)$ is a finite quantity, then the sufficiently big ball $Q(\alpha)$ with center in zero will be found such that

$$v(e_{2\alpha}\cap Q(\alpha))>\frac{1}{2}v(e_{2\alpha}). \tag{19}$$

The set $e_{2\alpha}\cap Q(a)$ is bounded and the system of balls $\{Q_{\rho(x,\alpha)}^x\}$, $x\in e_{2\alpha}\cap Q(\alpha)$ make the covering for it. With the help of Bezikovich lemma from the system of balls $\{Q_{\rho(x,\alpha)}^x\}$, $x\in e_{2\alpha}\cap Q(\alpha)$ subcovering $\{Q^1(\alpha)\}$, $i\in N$ of finite multiplicity - $\aleph_n$ can be chosen.

Further, as in Theorem 1,

$$v(e_{2\alpha}\cap Q^i(\alpha))\le\frac{c\gamma^\delta}{1-c\gamma^\delta}v(e_\alpha\cap Q^i(\alpha))+\frac{cA^r}{\gamma^{2\beta r}}\left[\int_{Q^i(\alpha)\setminus e_\alpha}\left(\int_{Q^i(\alpha)\cap e_{2\alpha}}\frac{\omega(x,y)}{|x-y|^{n+p\lambda}}dy\right)^{\frac{q}{p}}\right]^{\frac{r}{q}}.$$

Summing by i all the latter inequalities, using (19) and the finite multiplicity of covering $\{Q^i(\alpha)\}$, $i\in N$ we have

$$v(e_{2\alpha})\le\frac{2\aleph_n c\gamma^\delta}{1-c\gamma^\delta}v(e_\alpha)+\frac{2c\aleph_n}{\gamma^{2\beta r}}\left[\int_{R^n}dx\left(\int_{\left\{\substack{y\in R^n,\\ ||u(y)|-|u(x)||\ge\alpha}\right\}}\frac{\omega(x,y)}{|x-y|^{n+p\lambda}}dy\right)^{\frac{q}{p}}\right]^{\frac{r}{q}}.$$

Choose γ: $0<\gamma\le\frac{1}{2^n}, c\gamma^\delta<1$ so that $\frac{1}{2^r}-\frac{2\aleph_n c\gamma^\delta}{1-c\gamma^\delta}>0$ and integrate the latter inequality on segment $[0,\infty)$. Then with the help of Minkovsky inequality as in Theorem 1, we obtain (7).

Theorem 2 has been proved.

Theorem 3 (of Poincare's type). *Let* $1\le p\le q\le r<\infty$, $-\frac{n}{p}\le\lambda<\infty$ *be real numbers,* D *be the bounded domain in* R^n *with the property* K_ε *: there exists a constant* $\varepsilon\in(0,1]$ *such that*

$$\left|Q_t^x\cap D\right|\ge\varepsilon\left|Q_t^x\right| \tag{20}$$

for arbitrary point $x\in D$ *and any* t, $0<t<diamD$.

Suppose that the function $v(x)$ *belongs to* A_∞*-class.*

There exist the positive measurable function $\psi(x,y)$ *which belongs to* A_∞ *- class by each variables and such that the following conditions satisfied. There is, the* $A>0$ *such that for any ball* $Q=Q_t^x, x\in D, 0<t\le diamD$ *we have*

$$A^{\lambda,\psi}_{(p,q),r}:\qquad \frac{|Q|^{1/p+\lambda/n}v(Q)^{1/r}}{\psi(Q,Q)}\left\{\int_Q\left[\int_Q\left(\frac{\psi^p(x,y)}{\omega(x,y)}\right)^{\frac{1}{p-1}}dy\right]^{\frac{q(p-1)}{p(q-1)}}dx\right\}^{\frac{q-1}{q}}\le A \tag{21}$$

Then the inequality

$$\left(\int_D|u-\bar{u}|^r v(x)dx\right)^{t/r}\le c_0A\left[\int_D dx\left(\int_D\frac{|u(x)-u(y)|^p}{|x-y|^{n+p\lambda}}\omega(x,y)dy\right)^{\frac{q}{p}}\right]^{\frac{1}{q}} \tag{22}$$

holds for any function $u\in C(D)$, *where* $\bar{u}=\frac{1}{v(D)}\int_D u(x)v(x)dx$, $c_0>0$ *is constant, depending only on* n,p,q,r,ε *and on the constants from the* A_∞*- conditions for the functions* v,ψ.

Proof. Let

$$f(t)=|x\in D:u(x)>t|.$$

$f(t)$ is continuous from the right and nonincreasing function. There exists real number α, for which

$$f(a)\le\frac{1}{2}|D|, f(a-0)\ge\frac{1}{2}|D| \quad\text{where}\quad f(a-0):=\lim_{\varepsilon\to 0}f(a-\varepsilon). \tag{23}$$

If for any α, from the domain of values of function $u(x)$ the inequality $|x\in D:u(x)=\alpha|\le\frac{1}{4}|D|$ takes palce, then from (23) $\frac{1}{4}|D|\le f(a)\le\frac{1}{2}|D|$ follows. So $|x\in D:u<\alpha|=|D|-f(a)-|x\in D:u(x)\equiv a|\ge\ \ge\frac{1}{4}|D|$. Thus, according to continuity of function $u(x)$ in domain D the real number α exists such that

$$|x\in D:u(x)\equiv a|>\frac{1}{4}|D|$$

or

$$|x\in D:u(x)>a|\ge\frac{1}{4}|D|,$$

$$|x \in D : u(x) < a| \geq \frac{1}{4}|D|.$$

Define: $z_1(x) = u(x) - a;\ \ z_2(x) = a - u(x)$ in D and also $D_1 = \{x \in D : u > a\}$; $D_2 = \{x \in D : u < a\}$. By choose of a, the complements to D_1 and D_2 in D have the Lebesque measure $\geq \frac{1}{4}|D|$. We will obtain the upper estimates for the norms $\|z_1\|_{L_v^r(D_1)}, \|z_2\|_{L_v^r(D_2)}$ by the right-hand side of (22), and as a consequence the boundedness of $\|u - a\|_{L_v^r(D)}$ will follow. Further, with the help of inequality

$$\|u - \overline{u}\|_{L_v^r(D)} \leq 2\|u - a\|_{L_v^r(D)} \tag{24}$$

(22) will be obtained.

Now we shall prove (24). We have

$$\|u - \overline{u}\|_{L_v^r(D)} \leq \|u - a\|_{L_v^r(D)} + \|a - u\|_{L_v^r(D)} \tag{25}$$

where

$\|a - \overline{u}\|_{L_v^r(D)} = \|a - \overline{u}\| v(D)^{1/r}$. By H□lder inequality

$$|a - \overline{u}| = \frac{1}{v(D)}\left|\int_D (u - a) v dx\right| \leq \|u - a\|_{L_v^r(D)} v(D)^{-1/r}.$$

Then by (25) it follows (24).

Let's prove the upper estimations of the norms $\|z_1\|_{L_v^r(D_1)}, \|z_2\|_{L_v^r(D_2)}$.

Let $\alpha > 0$ be arbitrary fixed number from the domain of values of the function $z_1(x)$ and $\gamma, 0 < \gamma < \frac{\varepsilon}{4}$ be the sufficient small number, which later will be precisely. Denote $e_\alpha = \{x \in D : z_1(x) > \alpha\}$. By choose of $a, |D \setminus e_\alpha| \geq |D \setminus D_1| \geq \frac{1}{4}|D|$.

For any $x \in D \cap e_{2\alpha}$ the such $\rho(x,\alpha),\ 0 < \rho(x,\alpha) < diam D$ will be found that

$$\left|D \cap Q^x_{\rho(x,\alpha)} \setminus e_\alpha\right| = \gamma\left|Q^x_{\rho(x,\alpha)}\right|. \tag{26}$$

The function

$$F(t) = \frac{|D \cap Q^x_t \setminus e_\alpha|}{|Q^x_t|} - \gamma$$

is continuous on $[0; d];\ d = diam D$. The set $e_{2\alpha}$ is open, because such $\varepsilon > 0$ will be found that $F(\varepsilon) < 0$. As it was pointed out above, $|D \setminus e_\alpha| \geq \frac{1}{4}|D|$. Then with the help of (20),

$$F(d) = \frac{|D \cap Q^x_d \setminus e_\alpha|}{|Q^x_d|} - \gamma \geq \frac{1}{4}\frac{|D|}{|Q^x_d|} - \gamma \geq \frac{\varepsilon}{4} - \gamma \geq 0.$$

By Cauchy theorem on continuous functions $\exists \rho(x,\alpha), 0 < \rho(x,\alpha) \leq d$ and $F(\rho(x,\alpha)) = 0$, therefore (26) is valid.

With the help of Bezikovich lemma from the system of balls $\{Q^{x}_{\rho(x,\alpha)}\}$, $x \in e_{2\alpha}$ choose the subcovering $\{Q^{i}(\alpha)\}$, $i \in N$ of the finite multiplicity $\aleph_n$. Then for any ball $Q^{i}(\alpha)$

$$\left|D \cap Q^{i}(\alpha) \setminus e_{\alpha}\right| = \gamma\left|Q^{i}(\alpha)\right| \tag{27}$$

takes place.

Further, as in Theorem 1, if $\left|e_{2\alpha} \cap Q^{i}(\alpha)\right| < \gamma\left|Q^{i}(\alpha)\right|$, then from $\nu \in A_{\infty}$ it follows

$$\nu\left(e_{2\alpha} \cap Q^{i}(\alpha)\right) \le c\gamma^{\beta}\nu\left(Q^{i}(\alpha)\right). \tag{28}$$

So as the center of the ball $Q^{i}(\alpha)$ is in D then from K_{ε}-condition for the domain we obtain $\left|Q^{i}(\alpha) \cap D\right| \ge \varepsilon\left|Q^{i}(\alpha)\right|$. Then $\exists\beta \ge 1, \exists c \ge 1$, also, that

$$\frac{\nu\left(Q^{i}(\alpha) \cap D\right)}{\nu\left(Q^{i}(\alpha)\right)} \ge \frac{\varepsilon^{\beta}}{c}. \tag{29}$$

With the help of this inequality and (28) we find

$$\nu\left(Q^{i}(\alpha) \cap e_{2\alpha}\right) \le \frac{c}{\varepsilon^{\beta}}\gamma^{\delta}\nu\left(Q^{i}(\alpha) \cap D\right). \tag{30}$$

It is obvious, that

$$\nu\left(D \cap Q^{i}(\alpha)\right) = \nu\left(Q^{i}(\alpha) \cap e_{\alpha}\right) + \nu\left(Q^{i}(\alpha) \setminus e_{\alpha} \cap D\right).$$

Taking into account (27) and $\nu \in A_{\infty}$, we conclude that the second summand doesn't exceed $c\gamma^{\delta}\nu\left(Q^{i}(\alpha)\right)$ and the latter by virtue of (29) is less than $\frac{c}{\varepsilon^{\beta}}\nu\left(Q^{i}(\alpha) \cap D\right)$. So, we get

$$\nu\left(D \cap Q^{i}(\alpha)\right) \le \nu\left(Q^{i}(\alpha) \cap e_{\alpha}\right) + \frac{c}{\varepsilon^{\beta}}\gamma^{\delta}\nu\left(D \cap Q^{i}(\alpha)\right),$$

i.e. $\left(1 - \frac{c}{\varepsilon^{\beta}}\gamma^{\delta}\right)\nu\left(Q^{i}(\alpha) \cap D\right) \le \nu\left(Q^{i}(\alpha) \cap e_{\alpha}\right)$;

Using the latter in (30) we have

$$\nu\left(Q^{i}(\alpha) \cap e_{2\alpha}\right) \le \frac{c\gamma^{\delta}}{\varepsilon^{\beta} - c\gamma^{d}}\nu\left(Q^{i}(\alpha) \cap e_{\alpha}\right). \tag{31}$$

If $\left|e_{2\alpha} \cap Q^{i}(\alpha)\right| > \gamma\left|Q^{i}(\alpha)\right|$, then by (27) and $\psi \in A_{\infty}$ we obtain

$$\int\limits_{D \cap Q^{i}(\alpha) \setminus e_{\alpha}} \left(\int\limits_{Q^{i}(\alpha) \cap e_{2\alpha}} \psi(x,y)dy \right) dx \ge \left(c\gamma^{\beta}\right)^{2} \psi\left(Q^{i}(\alpha), Q^{i}(\alpha)\right).$$

Further, as in Theorem 1, with the help of Hölder inequality

$$1 \le \left(\frac{1}{\left(c\gamma^{\beta}\right)^{2} \psi\left(Q^{i}(\alpha), Q^{i}(\alpha)\right)} \right) \left[\int\limits_{D \cap Q^{i}(\alpha)} dx \left(\int\limits_{Q^{i}(\alpha) \cap e_{2\alpha}} \omega(x,y)dy \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} \times$$

$$\times\left\{\int_{D\cap Q^i(\alpha)\setminus e_\alpha}\left[\int_{Q^i(\alpha)\cap e_{2\alpha}}\left(\frac{\psi^p}{\omega}\right)^{\frac{1}{p-1}}dy\right]^{\frac{q(p-1)}{p(q-1)}}dx\right\}^{\frac{q-1}{q}}$$

Since $|x-y|\le c_0|Q^i(\alpha)|^{1/n}$ for $x\in D\cap Q^i(\alpha)\setminus e_\alpha$, $y\in Q^i(\alpha)\cap e_{2\alpha}$ the latter inequality implice

$$v(Q^i(\alpha)\cap e_{2\alpha})\le v(Q^i(\alpha))\left(\frac{\left|Q^i(\alpha)^{\frac{n+p\lambda}{pn}}\right|}{c\gamma^{2\beta}\psi(Q^i(\alpha),Q^i(\alpha))}\right)^r\times$$

$$\times\left\{\int_{Q^i(\alpha)}dx\left[\int_{Q^i(\alpha)}\left(\frac{\psi^p}{\omega}\right)^{\frac{1}{p-1}}dy\right]^{\frac{q(p-1)}{p(q-1)}}\right\}^{\frac{q-1}{q}r}\left[\int_{D\cap Q^i(\alpha)\setminus e_\alpha}dx\left(\int_{Q^i(\alpha)\cap e_{2\alpha}}\frac{\omega(x,y)}{|x-y|^{n+p\lambda}}dy\right)^{\frac{q}{p}}\right]^{\frac{r}{q}}$$

From here by $(v,\omega)\in A^{\lambda,\psi}_{(p;q),r}$ we conclude

$$v(Q^i(\alpha)\cap e_{2\alpha})\le\frac{cA^r}{\gamma^{2\beta r}}\left[\int_{D\cap Q^i(\alpha)\setminus e_\alpha}dx\left(\int_{Q^i(\alpha)\cap e_{2\alpha}}\frac{\omega(x,y)}{|x-y|^{n+p\lambda}}dy\right)^{\frac{q}{p}}\right]^{\frac{r}{q}} \qquad (32)$$

From (31) and (32) it follows that

$$v(e_{2\alpha}\cap Q^i(\alpha))\le\frac{c\gamma^\delta}{\varepsilon^\beta-c\gamma^\delta}v(Q^i(\alpha)\cap e_\alpha)+\frac{cA^r}{\gamma^{2\beta r}}\left[\int_{D\cap Q^i(\alpha)\setminus e_\alpha}dx\left(\int_{Q^i(\alpha)\cap e_{2\alpha}}\frac{\omega(x,y)}{|x-y|^{n+p\lambda}}dy\right)^{\frac{q}{p}}\right]^{\frac{r}{q}} \qquad (33)$$

If in the second summand in (33) we will substitute integration (for fixed $x\in D\cap Q^i(\alpha)$) over $Q^i(\alpha)\cap e_{2\alpha}$ by $\{y\in D_1\cap Q^i(\alpha):|z_1(y)-z_1(x)|>\alpha\}$, then the right-hand side won't decrease. On the other hand, $\{y\in D_1\cap Q^i(\alpha):|z_1(y)-z_1(x)|>\alpha\}=$ $=\{y\in D_1\cap Q^i(\alpha):|u(y)-u(x)|>\alpha\}$,so

$$v(e_{2\alpha}\cap Q^i(\alpha))\le\frac{c\gamma^\delta}{\varepsilon^\beta-c\gamma^\delta}v(e_{2\alpha}\cap Q^i(\alpha))+\frac{cA^r}{\gamma^{2\beta r}}\left[\int_{D\cap Q^i(\alpha)}dx\left(\int_{\left\{\substack{y\in Q^i(\alpha)\cap D:\\ |u(y)-u(x)|>\alpha}\right\}}\frac{\omega(x,y)}{|x-y|^{n+p\lambda}}dy\right)^{\frac{q}{p}}\right]^{\frac{r}{q}}.$$

Summing all the inequalities by i and taking into account the finite multiplicity of the cover $\{Q^i(\alpha)\}, i\in N$ for any α we find

$$\nu(e_{2\alpha})\le\frac{\aleph_n c\gamma^\delta}{\varepsilon^\beta-c\gamma^\delta}\nu(e_\alpha)+\frac{\aleph_n cA^r}{\gamma^{2\beta r}}\left[\int_D dx\left(\int_{\{y\in D|u(y)-u(x)>\alpha\}}\frac{\omega(x,y)}{|x-y|^{n+p\lambda}}dy\right)^{\frac{q}{p}}\right]^{\frac{r}{q}}.$$

Choose $\gamma:\ c\gamma^\delta<\varepsilon^\beta,\ 0<\gamma<\frac{\varepsilon}{4}$ so that $\frac{1}{2^r}-\frac{\aleph_n c\gamma^\delta}{\varepsilon^\beta-c\gamma^\delta}>0$. Then by similar way as in Theorem 1, integrating the latter inequality over segment $\left[0,\left(\sup_D u-a\right)/2\right]$, by Minkovsky inequality we obtain

$$\|z_1\|_{L^r_\nu(D_1)}\le cA\left[\int_D dx\left(\int_D\omega(x,y)\frac{|u(x)-u(y)|^p}{|x-y|^{n+p\lambda}}dy\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}. \tag{34}$$

The analogous inequality takes place also for function $z_2(x)$:

$$\|z_2(x)\|_{L^r_\nu(D_2)}\le cA\left[\int_D dx\left(\int_D\omega(x,y)\frac{|u(x)-u(y)|^p}{|x-y|^{n+p\lambda}}dy\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}. \tag{35}$$

It is obvious that

$$\|u-a\|_{L^r_\nu(D)}\le\|z_1\|_{L^r_\nu(D_1)}+\|z_2\|_{L^r_\nu(D_2)}.$$

Then

$$\|u-a\|_{L^r_\nu(D)}\le c_0A\left[\int_D dx\left(\int_D\omega(x,y)\frac{|u(x)-u(y)|^p}{|x-y|^{n+p\lambda}}dy\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}.$$

Further, taking into account (24) we obtain (22).

Theorem 3 has been proved.

Note. *If all conditions of Theorem 2 relative to the function $u(x)$ will be taken for $u(x)-u(0)$, then we obtain the following*

Statement. *Let all denotations and assumptions of Theorem 2 be fulfilled with respect to real numbers p,q,r,λ and functions ν,ω. Moreover if $u(x)-u(0)\in L^q_\nu(R_n)\cap C(R^n)$, then for validity of the inequality*

$$\left(\int_{R^n}|u(x)-u(0)|^r\nu(x)dx\right)^{1/r}\le c_0A\left[\int_{R^n}dx\left(\int_{R^n}\frac{|u(x)-u(y)|^p}{|x-y|^{n+p\lambda}}\omega(x,y)dy\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}$$

the condition (6) is sufficient.

From Theorem 3 we obtain.

Corollary 1. *Let all assumptions of Theorems 3 be fulfilled with respect to $\varepsilon,p,q,r,\lambda$, domain D and the weight functions ν,ω. Then positive constant c_0 which* *depends only on* n,r,ε *and on the constants from* A_∞*-condition for functions* v,ψ *can be found such that for any function* $u\in C(D)$ *the inequality*

$$\|u\|_{L_v^r(D)}\le c_0\left\{v(D)^{\left(\frac{1}{r}-1\right)}\|u\|_{L_v^1(D)}+A\left[\int_D dx\left(\int_D\frac{|u(x)-u(y)|^p}{|x-y|^{n+p\lambda}}\omega(x,y)dy\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}\right\} \quad (36)$$

holds.

For proof note that (36) is obtained by application of Minkovsky inequality to

$$u=(u-\bar{u})+\bar{u}$$

and the results of Theorem 3 to function $u-\bar{u}$.

By corollary 1 in one-dimensional case we receive the next

Corollary 2. *Let* $1<p<\infty,\ p'=\dfrac{p}{p-1},\ 0<a\le\infty,\quad v:[0,a)\to(0,\infty)$ *be positive and non-increasing function for which*

$$\left(\frac{1}{a}\int_0^a v^{p'}(\tau)d\tau\right)^{1/p'}\le cv(0) \quad (37)$$

with some positive constant c.

Further, let $W[0,a)\to(0,\infty)$ *be such positive function that*

$$\sup_{0<h\le a}h^{-1-1/p}\left(\int_0^h v(s)ds\right)^{1/p}\left(\int_0^h W_{(s)}^{1-p'}\alpha\varepsilon\right)^{1/p'}<\infty \quad (38)$$

Then the estimation

$$\left(\int_0^a|f|^p vdx\right)^{1/p}\le c_0\left[\left(v(0)\int_0^a|f|^p dx\right)^{1/p}+\left(\int_0^a\int_0^a|f(x)-f(y)|^p W(|x-y|)dxdy\right)^{1/p}\right] \quad (39)$$

takes place for any function $f\in L_v^p(0,a),\ v\equiv 1$, *where* c_0 *doesn't depend on function* f.

The similar result was obtained in [2].

For proof note that if the condition (38) is true, then the condition (21) according to Corollary 1 will be fulfilled for $n=1,\lambda=-\dfrac{1}{p},r=q=p$ and $\psi(x,y)\equiv 1$, $\omega(x,y)=w(|x-y|)$. The fulfillment of the condition (37) let us estimate the first summand in the right-hand side of (36) as it takes place in (39). Now statement of the Corollary 2 is obtained by using of Corollary 1.

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Received June 3, 1999; Revised September 31, 1999.
Translated by Soltanova S.M.