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ON BOUNDEDNESS OF CAUCHY'S SINGULAR INTEGRAL IN A CLASS OF GENERALIZED ANALYTIC FUNCTIONS

Abstract

Zigmund type inequality for the Cauchy's singular integral is established in a class of generalized analytic functions.

In 1924 A.Zigmund (see [3]) established an upper estimate of a continuity module of Cauchy's singular integral by the continuity module of density in the case when a bound of the domain Γ is a circle, and the density is of Hиlder class. This result became the foundation of wide investigations in this direction and drew attention of many mathematicians.

In subsequent years, this inequality was extended to more general curves, and other singular integrals were studied. In this direction we are to note special merits of L.G.Magnaradze, A.A.Babaev, V.V.Salaev, P.M.Tamrazov and their followers (see bibl. in [5]).

By introduction the notion of generalized analytic functions by I.N.Vekua (see [1]) (also by L.Berg see [2]) it was found that many properties of Cauchy's singular integral are typical for Cauchy's singular integral in a class of generalized analytic functions.

In the paper, Zigmung type inequality is established for Chauchy's singular integral in a class of generalized analytic functions.

Consider a class of generalized analytic functions $U_{p,2}(A,B,G), p>2$ in I.N.Vekua sense, i.e. a class of regular solutions of the equation (see [1] p.146)

$$\partial F(z)+A(z)F(z)+B(z)\overline{F(z)}=0\,, \tag{1}$$

where $A,B\in L_p(G), p>2, \partial_{\bar z}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$.

Remind that (see [1] p.198) a generalized Chauchy type integral for the class $U_{p,2}(A,B,G)$ is the expression

$$F(z)=\frac{1}{2\pi i}\int_\Gamma \Omega_1(z,t)u(t)dt-\Omega_2(z,t)\overline{u}(t)\overline{d}t,\ z\in\Gamma \tag{2}$$

where $\Omega_1(z,t)$ and $\Omega_2(z,t)$ are normed kernels of the class $U_{p,2}(A,B,G), p>2$ is Jordan's rectifiable curve, $u\in L_1(\Gamma)$. Note, that $F\in U_{p,2}(A,B,G)$.

Integral (2) is called singular integral for $z\in\Gamma$ and is understood in the sense of Cauchy's principal value.

Let $f\in C(\overline{G})$. We denote by $\omega_\Gamma(f,\delta)$, the continuity module $f(z)$ on the contour Γ, and by $\omega_G(f,\delta)$ denote the continuity module $f(z)$ on the domain $\overline{G}$:

$$\begin{cases}\omega_\Gamma(f,\delta)=\max\limits_{S(t_1,t_2)\le\delta}|f(t_1)-f(t_2)|,\ 0<\delta<\frac{l}{2}, l \text{ length } \Gamma,\\ \omega_{\overline{G}}(f,\delta)=\max\limits_{\substack{|z_1-z_2|\le\delta\\ z_1,z_2\in\overline{G}}}|f(z_1)-f(z_2)|\end{cases} \tag{3}$$

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Let G be a bounded domain, and $\Gamma = \partial G$ is a k-curve, i.e. $\forall z_1, z_2 \in \Gamma \exists k > 0$; $S(z_1, z_2) \le k|z_1 - z_2|$.

(smooth and piecewise-smooth curves are k-curves).

Consider the singular integral:

$$F(t) = \frac{1}{2\pi i} \int_\Gamma \Omega_1(t,\tau) u(\tau) d\tau - \Omega_2(t,\tau) \overline{u(\tau)}\, \overline{d\tau}, t \in \Gamma. \qquad (4)$$

Theorem 1. *Let G be a bounded domain, $\partial G = \Gamma$ be a closed k- curve, $U \in C(\Gamma)$ (continuous in Γ). Then*

$$\|F\|_{C(\Gamma)} \le \frac{k}{\pi} \int_0^{e/2} \frac{\omega_\Gamma(u,\tau)}{\tau} d\tau + C_1 \|u\|_{C(\Gamma)}. \qquad (5)$$

Proof. As is known (see [1] p.179) the kernels $\Omega_1(z,t)$ and $\Omega_2(z,t)$ have the form:

$$\Omega_1(z,t) = \frac{1}{t-z} + \frac{m_1(z,t)}{|t-z|^\alpha}, \quad \Omega_2(z,t) = \frac{m_2(z,t)}{|t-z|^\alpha}, \alpha = \frac{2}{p}, \qquad (6)$$

where $m_1(z,t)$ and $m_2(z,t)$ are continuous and uniformly bounded in z and t:

$$|m_1(z,t)| \le C_2, \quad |m_2(z,t)| \le C_3, \quad z,t \in \overline{G}. \qquad (7)$$

Taking into account (6) in (4), we obtain

$$F(t) = \frac{1}{2\pi i} \int_\Gamma \frac{u(\tau) d\tau}{\tau - t} + \frac{1}{2\pi i} \int_\Gamma \frac{m_1(\tau,t) u(\tau) d\tau}{|\tau - t|^\alpha} - \frac{1}{2\pi i} \int_\Gamma \frac{m_2(\tau,t) \overline{u(\tau)}\, \overline{d\tau}}{|\tau - t|^\alpha} = \qquad (8)$$
$$= J_1 + J_2 + J_3$$

Consider $J_1 : \tilde{u}(t) = \frac{1}{2\pi i} \int_\Gamma \frac{u(\tau) d\tau}{\tau - t}$.

We have:

$$\tilde{u}(t) = \frac{1}{2\pi i} \int_\Gamma \frac{(u(\tau) - u(t)) + u(t)}{\tau - t} d\tau = \frac{1}{2\pi i} \int_\Gamma \frac{(u(\tau) - u(t))}{\tau - t} d\tau + \frac{u(t)}{2\pi i} \int_\Gamma \frac{d\tau}{\tau - t}. \qquad (9)$$

Let t_0 be a point of the curve Γ such that $S(t_0, t) = \frac{l}{2}$ and let the point t_0 be the beginning reference. Denote the arc $\gamma_1 = (t_0; t)$ with origin at the point t_0 and the end t, and $\gamma_2 = \Gamma | \gamma_1$.

Then

$$\tilde{u}(t) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{(u(\tau) - u(t)) d\tau}{\tau - t} + \frac{1}{2\pi i} \int_{\gamma_2} \frac{(u(\tau) - u(t)) d\tau}{\tau - t} + \frac{1}{2} u(t). \qquad (10)$$

Further, we have

$$|\tilde{u}(t)| \le \frac{1}{2\pi} \int_{\gamma_1} \frac{|u(\tau) - u(t)||d\tau|}{|\tau - t|} + \frac{1}{2\pi} \int_{\gamma_2} \frac{|u(\tau) - u(t)|}{|\tau - t|} |d\tau| + \frac{1}{2} \|u\|_{C(\Gamma)} \le$$

$$\le \frac{k}{2\pi} \int_0^{l/2} \frac{\omega_\Gamma(u, s(\tau,t))}{s(\tau,t)} ds + \frac{k}{2\pi} \int_{\frac{l}{2}}^{l} \frac{\omega_\Gamma(u, s(\tau,t))}{s(\tau,t)} ds + \frac{1}{2} \|u\|_{C(\Gamma)} =$$

$$= \frac{k}{2\pi} \int_0^{l/2} \frac{\omega_\Gamma(u, (s_1 - s))}{s_1 - s} ds + \frac{k}{2\pi} \int_{\frac{l}{2}}^{l} \frac{\omega_\Gamma(u, (s_1 - s))}{s_1 - s} ds + \frac{1}{2} \|u\|_{C(\Gamma)} \le$$

$$\leq \frac{k}{\pi}\int_0^{l/2}\frac{\omega_\Gamma(u,\tau)}{\tau}d\tau + \frac{1}{2}\|u\|_{C(\Gamma)} \qquad (11)$$

(We took into account that Γ is a k-curve, and $(\tau - t) \geq \frac{s(\tau,t)}{k}, t = t(s_1)$).

We have

$$|J_2 + J_3| \leq \frac{1}{2\pi}\int_\Gamma \frac{|m_1||u(\tau)||d\tau|}{|\tau - t|^\alpha} + \frac{1}{2\pi}\int_\Gamma \frac{|m_2||\overline{u(\tau)}||\overline{d\tau}|}{|\tau - t|^\alpha} =$$

$$= \frac{1}{\pi}\int_\Gamma \frac{(|m_1| + |m_2|)|u(\tau)|}{|\tau - t|^\alpha}|d\tau| \leq \frac{1}{2}(C_2 + C_3)\|u\|_{C(\Gamma)}\int_\Gamma \frac{|d\tau|}{|\tau - t|^\alpha} \leq$$

$$\leq \frac{k}{\pi}(C_2 + C_3)\|u\|_{C(\Gamma)}\int_0^l \frac{ds}{s^\alpha} \leq C_4\|u\|_{C(\Gamma)} \qquad \left(\alpha = \frac{2}{p} < 1\right) \qquad (12)$$

Taking (11) and (12) into account we have

$$\|F\|_{C(\Gamma)} \leq \frac{\pi}{k}\int_0^{l/2}\frac{\omega_\Gamma(u,\tau)}{\tau}d\tau + C\|u\|_{C(\Gamma)}.$$

The theorem has been proved.

Considering P.M.Tamrazov's results (see [5] p.121) we get:

$$\|F\|_{C(\overline{G})} \leq \frac{k}{\pi}\int_0^{l/2}\frac{\omega_\Gamma(u,\tau)d\tau}{\tau} + C\|u\|_{C(\Gamma)}.$$

Now, estimate the continuity module $F(z)$ by $\Gamma = \partial G$. Let $z_1, z_2 \in \Gamma$, Γ be a k-curve, $k \geq 1$ and $u \in C(\Gamma)$, we have

$$F(z_1) - F(z_2) = \left[\frac{1}{2\pi i}\int_\Gamma \frac{u(t)dt}{t - z_1} - \frac{1}{2\pi i}\int_\Gamma \frac{u(t)dt}{t - z_2}\right] + \left[\frac{1}{2\pi i}\int_\Gamma \frac{m_1(z_1,t)u(t)dt}{|t - z_1|^\alpha} - \right.$$

$$\left. - \frac{1}{2\pi i}\int_\Gamma \frac{m_1(z_2,t)u(t)dt}{|t - z_1|^\alpha}\right] - \left[\frac{1}{2\pi i}\int_\Gamma \frac{m_2(z_1,t)\overline{u(t)}\overline{dt}}{|t - z_1|^\alpha} - \frac{1}{2\pi i}\int_\Gamma \frac{m_2(z_2,t)\overline{u(t)}\overline{dt}}{|t - z_2|^\alpha}\right] = \qquad (13)$$

$$= J_1 + J_2 + J_3.$$

The continuity module of the integral $\tilde{u}(z) = \frac{1}{2\pi i}\int_\Gamma \frac{u(t)dt}{t - z}$ is studied in detail in [5] (p.121), it was established that

$$\omega_\Gamma(\tilde{u},\delta) \leq C\left(\int_0^\delta \frac{\omega_\Gamma(u,t)dt}{t} + \delta\int_\delta^l \frac{\omega_\Gamma(u,t)dt}{t^2}\right), t \in [0,l] \qquad (14)$$

and

$$\omega_{\overline{G}}(\tilde{u},\delta) \leq C\left(\int_0^\delta \frac{\omega_\Gamma(u,t)dt}{t} + \delta\int_\delta^l \frac{\omega_\Gamma(u,t)dt}{t^2}\right).$$

Consider the behavior J_2 and J_3 from (13).

Denote

$$g_1(z) = \frac{1}{2\pi i}\int_\Gamma \frac{m_1(z,t)u(t)dt}{|t - z|^\alpha} \text{ and estimate } J_2:$$

$$|J_2|=|g_1(z_1)-g_1(z_2)|=\left|\frac{1}{2\pi i}\int_\Gamma \frac{\left(m_1(z_1,t)|t-z_2|^\alpha - m_2(z_2,t)|t-z_1|^\alpha\right)u(t)dt}{|t-z_1|^\alpha |t-z_2|^\alpha}\right| \le$$

$$\le \frac{1}{2\pi}\max(|m_1|,|m_2|)\|u\|_{C(\Gamma)}\int_\Gamma \frac{\left(|t-z_2|^\alpha + |t-z_1|^\alpha |dt|\right)}{|t-z_1|^\alpha |t-z_2|^\alpha}$$

Since Γ is a k-curve, then the values $|t-z_1|$ and $|t-z_2|$ are equivalent, therefore $|t-z_1|\le C_4|t-z_1|$:

$$|J_2|\le C_4\|u\|_{C(\Gamma)}\int_\Gamma \frac{|t-z_1|^\alpha}{|t-z_1|^{2\alpha}}|dt| \le C_4\|u\|_{C(\Gamma)}\int_\Gamma \frac{|dt|}{|t-z_1|^\alpha}=$$

$$=C_4\|u\|_{C(\Gamma)}\int_0^l \frac{ds}{s^\alpha}\le C_5\|u\|l^{1-\alpha}\le C_6\|u\|_{C(\Gamma)}|z_1-z_2|^{1-\alpha}= \tag{15}$$

$$=C_6\|u\|_{C(\Gamma)}\delta^{1-\alpha}=C_6\|u\|_{C(\Gamma)}\delta^{\frac{p-2}{p}}$$

Hence, it follows that

$$\omega_\Gamma(g_1,\delta)\le C_6\|u\|_{C(\Gamma)}\delta^{\frac{p-2}{p}} \tag{16}$$

If we introduce denotations

$$g_2(z)=\frac{1}{2\pi i}\int_\Gamma \frac{m_2(z,t)u(t)dt}{|t-z|^\alpha}, \text{ then}$$

similarly we shall have:

$$\omega_\Gamma(g_2,\delta)\le C_7\|u\|_{C(\Gamma)}\delta^{\frac{p-2}{p}}. \tag{17}$$

Considering (14), (16), (17) we have:

$$\omega_\Gamma(F,\delta)\le C\left[\int_0^\delta \frac{\omega_\Gamma(u,t)dt}{t}+\delta\int_\delta^l \frac{\omega_\Gamma(u,t)dt}{t^2}\right]+C_1\|u\|_{C(\Gamma)}\delta^{\frac{p-2}{p}}.$$

Thus we shall prove

Theorem 2. *Let G be an one-connected bounded domain, $\partial G=\Gamma$ is a k-curve, $k\ge 1$, $u\in C(\Gamma)$.*

The following estimate holds:

$$\omega_\Gamma(F,\delta)\le C\left[\int_0^\delta \frac{\omega_\Gamma(u,t)dt}{t}+\delta\int_\delta^l \frac{\omega_\Gamma(u,t)dt}{t}dt\right]+C_1\|u\|_{C(\Gamma)}\delta^{\frac{p-2}{p}}. \tag{18}$$

Considering the result of paper [5] (p.121) we get the estimate (18) for $\omega_{\overline{G}}(F,\delta)$ with the same constants C and C_1.

As an application of the inequality (18) we construct a class of functions that are invariant with respect to the operator $F(t)$.

We say that the function $\varphi(t)\left(0<t<\frac{l}{2}\right)$ belongs to the class Φ if:

a) $\varphi(t)$ is continuous in $\left[0;\frac{l}{2}\right]$ and is monotonically increasing in $\left(0;\frac{l}{2}\right]$;

b) $\frac{\varphi(t)}{t}$ is decreasing in $\left(0;\frac{l}{2}\right]$, $\varphi(t)\neq 0$ on $\left(0;\frac{l}{2}\right]$ and $\lim\limits_{t\to +0}\varphi(t)=0$.

If we introduce to H_φ the norm:

$$H_\varphi = \{U(t):\omega(u,\delta)=O(\varphi(\delta)),\ \varphi\in\Phi\}.$$

H_φ turns into the Banach space. Following [3] we shall consider that $\varphi(t)\in\Phi H$, if $Z\varphi(\delta)=O(\varphi(\delta))$, where $\varphi\in\Phi$ and Z is the Zigmund's operator definable by the equality:

$$Z\varphi(\delta)=\int_0^\delta \frac{\varphi(\tau)dt}{\tau}+\delta\int_0^{l/2}\frac{\varphi(\tau)dt}{\tau^2}. \tag{19}$$

Note some properties of the operator Z:

a) if $\varphi\in\Phi$, then $Z\varphi\in\Phi$;

b) if $\varphi(\delta)=O(\psi(\delta))$ $\varphi(\delta)\le c\psi(\delta)$, then $Z\varphi(\delta)=O(\psi(\delta))$;

c) if $\varphi(\delta)=o(\psi(\delta))$, then $Z\varphi(\delta)=o(Z\psi(\delta))$.

By means of the inequality (18) we prove

Theorem 3. *Let $\varphi(t)\in\Phi H$. If $u\in H_\varphi$, then $F\in H_\varphi$ the estimate $\|F\|_{H_\varphi}\le c\|u\|_{H_\varphi}$ holds where the constant C does not depend on u.*

By using the inequality (18) and the properties of the operator $Z\varphi$ we have:

$$\omega_\Gamma(F,\delta)\le c\left[\int_0^\delta\frac{\omega_\Gamma(u,\tau)d\tau}{\tau}+\delta\int_\delta^{l/2}\frac{\omega_\Gamma(u,\tau)d\tau}{\tau}\right]+C_1\|u\|_{C(\Gamma)}\delta^{\frac{p-2}{p}}=$$

$$=CZ\omega(\varphi,\delta)+C_1\|u\|_{C(\Gamma)}\delta^{\frac{p-2}{p}}.$$

Since $\varphi\in\Phi H$ then there exists $\alpha\in(0,1)$, that $\frac{\varphi(t)}{t^{1-\alpha}}$ is almost decreasing. Choose p such that $1-\alpha=\frac{p-2}{p}$ or $p=\frac{2}{\alpha}>2$. Whence $\delta^{\frac{p-2}{p}}=O(\varphi(\delta))$, i.e. $\omega(F,\delta)=o(\varphi(\delta))$, $F(t)\in H_\varphi$.

Boundness of $F(t)$ is proved similarly.

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