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WEAK SOLVABILITY OF THE FIRST BOUNDARY VALUE PROBLEM FOR GILBARG-SERRIN EQUATION

Abstract

In present paper the Dirichlet problem for Gilbarg-Serrin equation is considered. The unique weak solvability of this problem in corresponding weighted Sobolev space is proved.

Let D be a bounded domain, placed in n- dimensional Eucledian space E_n of points $x=(x_1,\dots,x_n), n\geq 3, 0\in D$ with boundary ∂D. Consider Gilbarg-Serrin operator in D:

$$L=\Delta+\mu(r)\sum_{i,j=1}^{n}\frac{x_i x_j}{r^2}\frac{\partial^2}{\partial x_i \partial x_j},$$

where $r=|x|, b_1\leq\mu(r)\leq b_2, b_1>-1, b_2<\infty$.

It is obvious that with suppositions for $\mu(r)$ the operator L is uniformly elliptic in D. It is well-known that for big values of $\mu(r)$, i.e. when $\inf\limits_{x\in D}\mu(|x|)>n-2$, the qualitative properties of solutions of Gilbarg-Serrin equation essentially differ from the properties of harmonic functions (see [1-2]). In particular, the uniqueness of solution of Dirichlet problem in non-weighted Sobolev spaces doesn't take place. The aim of present paper is to prove the unique weak solvability of the first boundary value problem in weighted Sobolev spaces. Note, that in case of $\mu\equiv const$ the analogous results were obtained in papers [3-5].

Now we will introduce some denotations and definitions. By $L_{2,\gamma}(D)$ and $W_{2,\gamma}^1(D)$ we will denote Banach spaces of functions, defined on D, with finite norms

$$\|u\|_{2,\gamma(D)}=\left(\int_D r^{\gamma-2}u^2dx\right)^{\frac{1}{2}}$$

and

$$\|u\|_{W_{2,\gamma}^1(D)}=\left(\int_D\left(r^{\gamma-2}u^2+r^{\gamma}|\nabla u|^2\right)dx\right)^{\frac{1}{2}}$$

correspondingly. Here ∇u is a gradient vector of the function $u(x)$. By $\overset{\circ}{W}{}_{2,\gamma}^1(D)$ we denote subspace of $W_{2,\gamma}^1(D)$, the dense set of which is the aggregate of all infinitely differentiable functions with compact support in D, with finite corresponding norm. Let for $i,j=1,\dots,n$, $u_i=\frac{\partial u}{\partial x_i}$, $u_{ij}=\frac{\partial^2 u}{\partial x_i\partial x_j}$. Further, the record $C(\dots)$ means that positive constant C depends only on parameters in brackets. For function $\mu(r)$ we suppose the following conditions

$$b_0 \le \mu(r) \le b_2, \quad b_0 > 2n-3, \quad b_2 < \infty; \tag{1}$$

$$|\mu'(r)| \le \frac{C_1}{r}; \quad -\frac{C_2}{r^2} \le \mu''(r) \le 0; \quad r \in (0, diamD). \tag{2}$$

Lemma 1. *Let in bounded domain $D \subset E_n$ the coefficients of operator L be defined. If $\gamma = 1-n$, then for any function $u(x) \in \overset{\circ}{W}{}^1_{2,\gamma}(D)$ holds the following inequality:*

$$-\int_D r^\gamma u L u dx \ge C_3(\mu) \int_D r^\gamma |\nabla u|^2 dx. \tag{3}$$

Proof. It is enough to establish the estimate (3) for smooth functions from $\overset{\circ}{W}{}^1_{2,\gamma}(D)$. Further everywhere we will suppose $\gamma = 1-n$, if it is specially not defined. Let

$$\mathcal{L}u = \Delta u + \mu(r) \sum_{i,j=1}^n \left(\frac{x_i x_j}{r^2} u_j \right)_i.$$

It is easy to see that

$$Lu = \mathcal{L}u - \mu(r)(n-1) \sum_{i=1}^n \frac{x_i u_i}{r^2}.$$

Therefore

$$A = -\int_D r^\gamma u L u dx = -\int_D r^\gamma u \Delta u dx - \int_D r^\gamma \mu(r) u \sum_{i,j=1}^n \left(\frac{x_i x_j}{r^2} u_j \right)_i dx +$$

$$+(n-1) \int_D r^\gamma \mu(r) u \sum_{i=1}^n \frac{x_i u_i}{r^2} dx = i_1 + i_2 + i_3. \tag{4}$$

But on the other side

$$i_1 = \int_D r^\gamma |\nabla u|^2 dx + \gamma \sum_{i=1}^n \int_D r^{\gamma-2} x_i u \cdot u_i dx =$$

$$= \int_D r^\gamma |\nabla u|^2 dx - \frac{\gamma}{2}(n+\gamma-2) \int_D r^{\gamma-2} u^2 dx. \tag{5}$$

We have

$$i_2 = \int_D r^\gamma \mu(r) \left(\sum_{i=1}^n \frac{x_i}{r} u_i \right)^2 + \gamma \sum_{i,j=1}^n r^{\gamma-4} \mu(r) u \cdot x_i^2 x_j u_j dx +$$

$$+ \sum_{i,j=1}^n \int_D r^{\gamma-3} \mu'(r) x_i^2 x_j u \cdot u_j dx = \int_D r^\gamma \mu(r) |\nabla_1 u|^2 dx -$$

$$-\frac{\gamma}{2}(n+\gamma-2) \int_D r^{\gamma-2} \mu(r) u^2 dx - \frac{1}{2}(n+2\gamma-1) \int_D r^{\gamma-1} \mu'(r) u^2 dx -$$

$$-\frac{1}{2} \int_D r^\gamma \mu''(r) u^2 dx, \tag{6}$$

where $|\nabla_1 u|^2 = \left(\sum_{i=1}^n \frac{x_i u_i}{r} \right)^2$.

Finally,

$$i_3 = -\frac{n-1}{2}\left[(n+\gamma-2)\int_D r^{\gamma-2}\mu(r)u^2dx + \int_D r^{\gamma-1}\mu'(r)u^2dx\right]. \tag{7}$$

Now taking into account (5)-(7) in (4), we obtain

$$A = \int_D r^\gamma|\nabla u|^2dx + \int_D r^\gamma\mu(r)|\nabla_1 u|^2dx - \frac{\gamma}{2}(n+\gamma-2)\int_D r^{\gamma-2}u^2dx -$$

$$-\frac{(\gamma+n-1)(n+\gamma-2)}{2}\int_D r^{\gamma-2}\mu(r)u^2dx + (\gamma+n-1)\int_D r^{\gamma-1}\mu'(r)u^2dx -$$

$$-\frac{1}{2}\int_D r^\gamma\mu''(r)u^2dx.$$

Using facts that $\gamma = 1-n$ and $\mu''(r)\le 0$ from the last equality we obtain

$$A \ge \int_D r^\gamma|\nabla u|^2dx + \int_D r^\gamma\mu(r)|\nabla_1 u|^2dx + \frac{1-n}{2}\int_D r^{\gamma-2}u^2dx. \tag{8}$$

Now we will prove the following auxiliary fact. Let $\gamma\in[1-n, 2-n)$, $u\in \overset{\circ}{W}{}^1_{2,\gamma}(D)$. Then

$$\int_D r^{\gamma-2}u^2dx \le \frac{4}{(2-\gamma-n)^2}\int_D r^\gamma|\nabla_1 u|^2dx. \tag{9}$$

It is obvious that it is enough to establish (9) for smooth functions from $\overset{\circ}{W}{}^1_{2,\gamma}(D)$. Consider integral

$$I = 2\int_D\sum_{i=1}^n r^{\gamma-2}x_i\cdot u\cdot u_i dx.$$

We have

$$I = (2-\gamma-n)\int_D r^{\gamma-2}u^2dx. \tag{10}$$

From the other side for any $\varepsilon>0$

$$I \le \varepsilon\int_D r^{\gamma-2}u^2dx + \frac{1}{\varepsilon}\int_D r^\gamma|\nabla_1 u|^2dx. \tag{11}$$

From (10)-(11) it follows that

$$(2-\gamma-n)\int_D r^{\gamma-2}u^2dx \le \varepsilon\int_D r^{\gamma-2}u^2dx + \frac{1}{\varepsilon}\int_D r^\gamma|\nabla_1 u|^2dx.$$

Supposing now $\varepsilon = \frac{2-\gamma-n}{2}$ we obtain the required estimation (9). In particular, for $\gamma = 1-n$ we obtain

$$\int_D r^{\gamma-2}u^2dx \le 4\int_D r^\gamma|\nabla_1 u|^2dx. \tag{12}$$

Taking into account (12) into (8) we conclude

$$A \ge \int_D r^\gamma|\nabla u|^2dx + \int_D r^\gamma\mu(r)|\nabla_1 u|^2 - 2(n-1)\int_D r^\gamma|\nabla_1 u|^2dx. \tag{13}$$

So as by condition (1) $b_0 > 2n-3$, then there exists $\delta = \delta(\mu) > 0$ such that $\inf\limits_{x\in D} \mu(|x|) \ge 2n-3+\delta$. Without loss of generality we will assume that $\delta < 1$. Further, from $|\nabla_1 u|^2 \le |\nabla u|^2$ it follows that

$$\int\limits_D r^\gamma |\nabla u|^2 dx \ge \delta \int\limits_D r^\gamma |\nabla u|^2 dx + (1-\delta)\int\limits_D r^\gamma |\nabla_1 u|^2 dx. \tag{14}$$

From (13)-(14) we obtain

$$A \ge \delta \int\limits_D r^\gamma |\nabla u|^2 dx + \left(1-\delta+\inf_{x\in D}\mu(|x|)-2n+2\right)\int\limits_D r^\gamma |\nabla_1 u|^2 dx \ge \delta \int\limits_D r^\gamma |\nabla u|^2 dx.$$

Therefore lemma is proved for $C_3 = \delta$.

Now we will introduce bilinear form for $u, \vartheta \in \overset{\circ}{W}{}^1_{2,\gamma}(D)$

$$B(u,\vartheta) = \sum_{i=1}^n r^\gamma u_i \vartheta_i dx + \gamma \int\limits_D r^{\gamma-2} u\vartheta dx - \gamma \sum_{i=1}^n \int\limits_D r^{\gamma-2} x_i \vartheta_i u dx +$$
$$+ \sum_{i,j=1}^n \int\limits_D r^{\gamma-2} x_i x_j \mu(r) u_i \vartheta_j dx + \int\limits_D r^{\gamma-1}\mu'(r) u\cdot\vartheta\, dx -$$
$$- \int\limits_D r^\gamma \mu''(r) u\vartheta\, dx - \sum_{i=1}^n \int\limits_D r^{\gamma-1}\mu'(r) x_i u \vartheta_i dx.$$

Lemma 2. *Bilinear form $B(u,\vartheta)$ is bounded and coercive in $\overset{\circ}{W}{}^1_{2,\gamma}(D)$, i.e. there exist constants $C_4(\mu,n)$ and $C_5(\mu,n)$ such that for any $u, \vartheta \in \overset{\circ}{W}{}^1_{2,\gamma}(D)$*

$$|B(u,\vartheta)| \le C_4 \|u\|\cdot\|\vartheta\|, \tag{15}$$
$$B(u,\vartheta) \ge C_5 \|u\|^2, \tag{16}$$

where $\|\cdot\| = \|\cdot\|_{W^1_{2,\gamma}(D)}$.

Proof. We have

$$\left|\sum_{i=1}^n \int\limits_D r^\gamma u_i \vartheta_i dx\right| \le \sqrt{\int\limits_D r^\gamma |\nabla u|^2 dx \int\limits_D r^\gamma |\nabla \vartheta|^2 dx} \le \|u\|\cdot\|\vartheta\|;$$
$$\left|\gamma \int\limits_D r^{\gamma-2} u\vartheta dx\right| \le |\gamma| \sqrt{\int\limits_D r^{\gamma-2} u^2 dx \int\limits_D r^{\gamma-2}\vartheta^2 dx} \le |\gamma| \|u\| \|\vartheta\|;$$
$$\left|\gamma \sum_{i=1}^n \int\limits_D r^{\gamma-2} x_i \vartheta_i u dx\right| \le |\gamma| \sqrt{\int\limits_D r^\gamma |\nabla\vartheta|^2 dx \int\limits_D r^{\gamma-2} u^2 dx} \le |\gamma|\cdot\|u\|\cdot\|\vartheta\|;$$
$$\left|\sum_{i,j=1}^n r^{\gamma-2} x_i x_j \mu(r) u_i \vartheta_i\right| dx \le n b_2 \|u\|\cdot\|\vartheta\|;$$
$$\left|\int\limits_D r^{\gamma-2}\mu'(r) u\vartheta\, dx\right| \le c_1 \|u\|\cdot\|\vartheta\|; \quad \left|\int\limits_D r^\gamma \mu''(r) u\cdot\vartheta\, dx\right| \le c_2 \|u\|\cdot\|\vartheta\|;$$
$$\sum_{i=1}^n \int\limits_D r^{\gamma-1}\mu'(r) x_i \vartheta_i u dx \le c_1 \|u\|\cdot\|\vartheta\|.$$

From here follows (15) with $C_4 = 1 + 2|\gamma| + nb_2 + nC_1 + C_2$. From the other side

$$B=(u,\vartheta)=-\int_D r^{\gamma}uLudx.$$

Therefore taking into account Lemma 1 and estimation (12), we obtain

$$B(u,\vartheta)\geq\frac{C_3}{2}\int_D r^{\gamma}|\nabla u|^2dx+\frac{C_3}{8}\int_D r^{\gamma-2}u^2dx\geq\frac{C_3}{8}\|u\|^2$$

and estimation (16) is achieved for $C_5=\frac{C_3}{8}$.

Lemma is proved.

We will call the weak solution of Dirichlet problem

$$Lu=g+\sum_{i=1}^{n}\frac{\partial f^i}{dx_i},\quad u|_{\partial D}=0,\tag{17}$$

where $g\in L_{2,\gamma+2}(D), f^i\in L_{2,\gamma}(D), i=1,\ldots,n$;

the function $u(x)\in\mathring{W}^1_{2,\gamma}(D)$ such that for any function $\vartheta(x)\in\mathring{W}^1_{2,\gamma}(D)$ holds the following integral identity

$$B(u,\vartheta)=-\int_D r^{\gamma}g\vartheta dx+\sum_{i=1}^{n}\int_D r^{\gamma}\vartheta_i f^i dx+\gamma\sum_{i=1}^{n}\int_D r^{\gamma-2}x_i\vartheta f^i dx.\tag{18}$$

Theorem 1. *The Dirichlet problem (17) has a unique weak solution $u(x)\in\mathring{W}^1_{2,\gamma}(D)$ for all $g\in L_{2,\gamma+2}(D), f^i\in L_{2,\gamma}(D), i=1,\ldots,n$.*

Proof. Now we will show that expression in right-hand side of equality (18) (we denote it by $F(\vartheta)$) is a linear bounded functional on $\mathring{W}^1_{2,\gamma}(D)$. It is obvious, that it is enough to show that $F(\vartheta)$ is bounded. We have

$$|F(\vartheta)|\leq\left|\int_D r^{\gamma}g\vartheta dxn\right|+\left|\sum_{i=1}^{n}\int_D r^{\gamma}\vartheta_i f^i dx\right|+|\gamma|\sum_{i=1}^{n}\int_D\left|r^{\gamma-2}x_i\vartheta f^i dx\right|\leq$$

$$\leq\sqrt{\int_D r^{\gamma+2}g^2dx}\|\vartheta\|+\sum_{i=1}^{n}\sqrt{r^{\gamma}\left(f^i\right)^2dx}\cdot\|\vartheta\|+|\gamma|\sqrt{\int_D r^{\gamma}\left(f^i\right)^2dx}\|\vartheta\|=$$

$$\left(\|g\|_{L_{2,\gamma+2}(D)}+(1+|\gamma|)\sum_{i=1}^{n}\left\|f^i\right\|_{L_{2,\gamma}(D)}\right)\|\vartheta\|.$$

Now we apply Lemma 1 and theorem of Lax and Milgram and we prove the theorem (see [6]).

Theorem 2. *For the weak solution $u(x)$ of Dirichlet problem (17) the following estimate holds*

$$\|u\|_{W^1_{2,\gamma}(D)}\leq C_6(\mu,n)\left(\|g\|_{L_{2,\gamma+2}(D)}+\sum_{i=1}^{n}\left\|f^i\right\|_{L_{2,\gamma}(D)}\right).\tag{19}$$

Proof. We suppose in (18) $\vartheta=u$. We have for any $\sigma>0$

$$|F(u)|<\frac{\sigma}{2}\|u\|^2+\frac{1}{2\sigma}\|g\|^2_{L_{2,\gamma+2}(D)}+\frac{\sigma}{2}\|u\|^2+\frac{1}{2\sigma}\sum_{i=1}^{n}\left\|f^i\right\|^2_{L_{2,\gamma}(D)}+\frac{|\gamma|\sigma}{2}\|u\|^2+$$

$$+\frac{|\gamma|}{2\sigma}\sum_{i=1}^{n}\left\|f^i\right\|^2_{L_{2,\gamma}(D)}=\sigma\left(1+\frac{|\gamma|}{2}\right)\|u\|^2+\frac{1}{2\sigma}\|g\|^2_{L_{2,\gamma+2}(D)}+\frac{1+|\gamma|}{2\sigma}\sum_{i=1}^{n}\left\|f^i\right\|^2_{L_{2,\gamma}(D)}.\tag{20}$$

From the other hand, according to Lemma 1 and estimate (12)

$$B(u,u) \geq \frac{C_3}{8}\|u\|^2. \tag{21}$$

We choose and fix $\sigma = \frac{C_3}{8(2+|\gamma|)}$. Then from (20)-(21) we obtain

$$\frac{C_3}{16}\|u\|^2 \leq \frac{4(2+|\gamma|)}{C_3}\|g\|^2_{L_{2,\gamma+2}(D)} + \frac{4(1+|\gamma|)(2+|\gamma|)}{C_3}\sum_{i=1}^{n}\|f^i\|^2_{L_{2,\gamma}(D)}.$$

From here follows the required estimate (19) with $C_6 = \frac{64(1+|\gamma|)(2+|\gamma|)}{C_3^2}$. Theorem is proved.

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