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THE PROPERTIES OF A MAXIMAL SINGULAR OPERATOR IN TERMS OF MEAN OSCILLATION

Abstract

The estimation of image of a maximal singular integral operator via preimage in terms of mean oscillation was established. The theorems on bounded mapping of closed maximal singular operator into the spaces of mean oscillation BMO, VMO, $BMO_{\varphi,\theta}$ and etc. was proved.

1. Let $B(x,r):=\{y\in\mathbf{R}^n:|y-x|\le r\}$ be closed a ball in Euclidean space $\mathbf{R}^n$ of radius $r>0$ with center at the point $x\in\mathbf{R}^n$, f be locally summable in $\mathbf{R}^n$ function, i.e. $f\in L_{loc}(\mathbf{R}^n)$,

$$f_{B(x,r)}:=|B(x,r)|^{-1}\int\limits_{B(x,r)}f(t)dt,$$

$$OSC(f,B(x,r)):=|B(x,r)|^{-1}\int\limits_{B(x,r)}|f(t)-f_{B(x,r)}|dt,$$

where $|B(x,r)|$ defines volume of ball $B(x,r)$. $OSC(f,B(x,r))$ is called a mean oscillation of the function f in the ball $B(x,r)$.

Suppose (see [1], [2])

$$M_f(\delta):=\sup\{OSC(f,B(x,r)):r\le\delta,\ x\in\mathbf{R}^n\},\quad \delta>0.$$

It is easy to see that the function $M_f(\delta)$ is defined on interval $(0,+\infty)$, takes non-negative values and monotonically increased.

For locally bounded in $\mathbf{R}^n$ function f (i.e. $f\in L^\infty_{loc}(\mathbf{R}^n)$) introduce denotation

$$\omega_f(\delta):=ess\sup\{|f(x)-f(y)|:|x-y|\le\delta;\quad x,y\in\mathbf{R}^n\},\quad \delta>0.$$

It can be verified, that $\omega_f(\delta)$ is a halfadditive function, i.e. $\omega_f(\delta_1+\delta_2)\le$ $\le\omega_f(\delta_1)+\omega_f(\delta_2)$. From here it follows, that $\dfrac{\omega_f(\delta)}{\delta}$ almost decreases * on interval $(0,+\infty)$.

2. Consider the maximal singular integral operator

$$A^*f(x)=\tilde f(x)=\sup\{|A_\varepsilon f(x)|:\varepsilon>0\} \tag{1}$$

where

$$A_\varepsilon f(x)=\int\limits_{\mathbf{R}^n}\{K_\varepsilon(x-y)-K_{1,\varepsilon}(-y)\}f(y)dy,\quad K_\varepsilon(x)=K(x)\mathrm{X}_{\{|t|>\varepsilon\}}(x),$$

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* Function $h(x)$ is called almost decreasing (almost increasing) on $(0,+\infty)$, if exists such a constant $c>0$, that for any $x_1<x_2$ from $(0,+\infty)$

$$h(x_2)\le ch(x_1)\quad (h(x_1)\le ch(x_2)).$$

$$K_{1,\varepsilon}(x)=\begin{cases} K_\varepsilon(x) & \text{for} \quad \varepsilon>1, \\ K_1(x) & \text{for} \quad \varepsilon\le 1, \end{cases}$$

$$K(x)=|x|^{-n}\Omega(x), \quad \int_{S^{n-1}}\Omega(x)ds=0,$$

the function $\Omega(x)$ is homogeneous to the degree zero, S^{n-1} is a unit sphere in $\mathbf{R}^n$, $\sup\{|\Omega(x)-\Omega(y)|:|x-y|\le\delta;\ x,y\in S^{n-1}\}\le const\cdot\omega(\delta)$, $\omega(\delta)$ is monotonically increased on interval $(0,+\infty)$, $\dfrac{\omega(\delta)}{\delta}$ is almost decreasing,

$$\int_0^1 t^{-1}\omega(t)dt<+\infty,$$

$X_{\{|t|>\varepsilon\}}$ is a characteristic function of the set $\{t\in\mathbf{R}^n:|t|>\varepsilon\}$.

Theorem 1. *Let $f\in L_{loc}(\mathbf{R}^n)$ and*

$$\int_1^\infty \omega\left(\frac{1}{t}\right)\frac{M_f(t)}{t}dt<+\infty.$$

Then the following inequality holds:

$$M_{\tilde f}(\delta)\le C\int_\delta^\infty \omega\left(\frac{\delta}{t}\right)\frac{M_f(t)}{t}dt \quad (\delta>0), \tag{2}$$

where C is a positive constant which depend only on Ω, n and ω.

Proof. Let ε and r be positive numbers and $2^{N-1}\le 4r<2,N$, where N is the integer number. Denote by $C_{N,\varepsilon}(f)$ following integral

$$C_{N,\varepsilon}(f):=\int_{\mathbf{R}^n}K(-y)f(y)H_{N,\varepsilon}(y)dy,$$

where

$$\begin{aligned}
&H_{N,\varepsilon}(y)=X_{\{2^{N-1}<|y|\le 1\}}(y) && \text{for} \quad \varepsilon\le 2^{N-1}\le 1, \\
&H_{N,\varepsilon}(y)=-X_{\{1<|y|\le 2^{N-1}\}}(y) && \text{for} \quad \varepsilon\le 1\le 2^{N-1}, \\
&H_{N,\varepsilon}(y)=-X_{\{\varepsilon<|y|\le 2^{N-1}\}}(y) && \text{for} \quad 1\le\varepsilon\le 2^{N-1}, \\
&H_{N,\varepsilon}(y)=X_{\{\varepsilon<|y|\le 1\}}(y) && \text{for} \quad 2^{N-1}\le\varepsilon\le 1, \\
&H_{N,\varepsilon}(y)\equiv 0 && \text{for} \quad \varepsilon\ge\max\{1,2^{N-1}\}
\end{aligned}$$

and $X_E(y)$ is a characteristic function of the set $E\subset\mathbf{R}^n$.

Consider the following cases:

a) $0<\varepsilon\le 2^{N-1}$, b) $2^{N-1}<\varepsilon<+\infty$. It can be verified that in case a)

$$A_\varepsilon f(x)-C_{N,\varepsilon}(f)=\int_{\varepsilon<|x-y|\le 2^{N-1}}K(x-y)f(y)dy+$$
$$+\int_{\mathbf{R}^n}\{K_{2^{N-1}}(x-y)-K_{2^{N-1}}(-y)\}f(y)dy,$$

and in case b)

$$A_\varepsilon f(x)-C_{N,\varepsilon}(f)=\int_{\mathbf{R}^n}\{K_\varepsilon(x-y)-K_\varepsilon(-y)\}f(y)dy.$$

In case a) we obtain

$$A_\varepsilon f(x)-C_{N,\varepsilon}(f)=\int\limits_{\varepsilon<|x-y|\le 2^{N-1}}K(x-y)\left(f(y)-f_{B(0,5r)}\right)dy+$$

$$+\sum_{k=N}^{\infty}\left\{\int\limits_{2^{k-1}<|x-y|\le 2^k}K(x-y)\left(f(y)-f_{B(0,2^{k+1})}\right)dy-\right.$$

$$\left.-\int\limits_{2^{k-1}<|y|\le 2^k}K(-y)\left(f(y)-f_{B(0,2^{k+1})}\right)dy\right\}=:g_{\varepsilon,N}(x)+$$

$$+\sum_{k=N}^{\infty}\{g_k(x)-g_k(0)\} \quad (3)$$

If case b) is occur, then there exist integer number $m\ge N$ such that $2^{m-1}<\varepsilon\le 2^m$. Therefore in case b) we have

$$A_\varepsilon f(x)-C_{N,\varepsilon}(f)=\int\limits_{\varepsilon<|x-y|\le 2^m}K(x-y)\left(f(y)-f_{B(0,2^{m+1})}\right)dy-$$

$$-\int\limits_{\varepsilon<|y|\le 2^m}K(-y)\left(f(y)-f_{B(0,2^{m+1})}\right)dy+\sum_{k=m+1}^{\infty}\{g_k(x)-g_k(0)\}= \quad (4)$$

$$=:\left(g^*_{\varepsilon,m}(x)-g^*_{\varepsilon,m}(0)\right)+\sum_{k=m+1}^{\infty}\{g_k(x)-g_k(0)\}$$

Further, with the help of (3) and (4) we obtain

$$OSC\left(\tilde f,B(0,r)\right)=\frac{1}{|B(0,r)|}\int\limits_{B(0,r)}\left|\tilde f(x)-\tilde f_{B(0,r)}\right|dx\le$$

$$\le\frac{2}{|B(0,r)|}\int\limits_{B(0,r)}\left|\tilde f(x)-\sup_{\varepsilon>0}\left|C_{N,\varepsilon}(f)\right|\right|dx\le$$

$$\le\frac{2}{|B(0,r)|}\int\limits_{B(0,r)}\sup_{0<\varepsilon\le 2^{N-1}}\left|g_{\varepsilon,N}(x)\right|dx+$$

$$+\frac{2}{|B(0,r)|}\int\limits_{B(0,r)}\sum_{k=N}^{\infty}\left|g_k(x)-g_k(0)\right|dx+$$

$$+\frac{2}{|B(0,r)|}\int\limits_{B(0,r)}\sup_{\varepsilon>2^{N-1}}\left(\left|g^*_{\varepsilon,m}(x)-g^*_{\varepsilon,m}(0)\right|+\sum_{k=m+1}^{\infty}\left|g_k(x)-g_k(0)\right|\right)dx. \quad (5)$$

Estimating each of members in the right-hand side of inequality (5) separately with the help of theorem on boundedness of a maximal singular integral operator

$$T^*f(x)=\sup_{\varepsilon>0}\left|\int\limits_{\mathbb{R}^n}K_\varepsilon(x-y)f(y)dy\right|$$

in the space $L^2(R^n)$ (see [3]), theorem of John-Nirenberg [4], Lemmas 1 and 2 from [5], and also applying some elementary transformations, related to with passage from the sums to integrals, we obtain

$$OSC\left(\tilde f,B(0,r)\right)\le C\int\limits_r^{\infty}\omega\left(\frac{r}{t}\right)\frac{M_f(t)}{t}dt,\quad r>0,$$

where constant $C>0$ depends only on Ω,n and ω.

Further, applying previous reasoning to function $f_{x_0}(x)=f(x_0+x)$ we obtain that

$$OSC(\widetilde{f},B(x_0,r))\leq C\int_r^\infty \omega\left(\frac{r}{t}\right)\frac{M_f(t)}{t}dt,$$

where $x_0\in\mathbf{R}^n$ is an arbitrary point. Therefore we obtain the required statement. Theorem is proved.

3. By Φ we denote a class of all positive monotonically increased on $(0,+\infty)$ functions $\varphi(t)$ such that $\varphi(+0)=0$. Function $\varphi(t)\equiv 1$ by definition we will also consider as element of class Φ.

Denote by Z^θ $(1\leq\theta\leq\infty)$ the aggregate of all $\varphi\in\Phi$ such, that $\varphi(t)t^{-\frac{1}{\theta}}$ almost increases on $(0,+\infty)$ and holds following condition

$$\delta^{1+\frac{1}{\theta}}\int_\delta^\infty\frac{\varphi(t)}{t^{2+\frac{1}{\theta}}}dt=O(\varphi(\delta)),\quad \delta\in(0,+\infty).$$

By Z_1^θ $(1\leq\theta\leq\infty)$ we denote the aggregate of all functions $\varphi\in\Phi$, which satisfies the condition

$$\delta^{\frac{1}{\theta}}\int_0^\delta\frac{\varphi(t)}{t^{1+\frac{1}{\theta}}}dt=O(\varphi(\delta)),\quad \delta\in(0,+\infty).$$

Let $\varphi\in\Phi_{1+\frac{1}{\theta}}:=\left\{\varphi\in\Phi:\varphi(x)x^{-1-\frac{1}{\theta}} \text{ is almost decreases}\right\}$, $1\leq\theta\leq\infty$. Denote by $BMO_{\varphi,\theta}$ the class of all functions $f\in L_{loc}(\mathbf{R}^n)$ such that $\|f\|_{BMO_{\varphi,\theta}}<+\infty$, where

$$\|f\|_{BMO_{\varphi,\theta}}:=\begin{cases}\left(\int_0^\infty\left(\frac{M_f(t)}{\varphi(t)}\right)^\theta dt\right)^{1/\theta} & \text{for } 1\leq\theta<\infty\\ \sup\left\{\frac{M_f(t)}{\varphi(t)}:t>0\right\} & \text{for } \theta=\infty.\end{cases}$$

If we consider the class $BMO_{\varphi,\theta}$ as the subset in a factor-space $L_{loc}(\mathbf{R}^n)/\{\text{constants}\}$, then $\|\bullet\|_{BMO_{\varphi,\theta}}$ is a norm in $BMO_{\varphi,\theta}$. The space $BMO_{\varphi,\theta}$ is a Banach one in the introduced norm.

If $\theta=\infty$ and $\varphi(t)\equiv 1$, then $BMO_{\varphi,\theta}=BMO$, where BMO is a class of all local summable functions, which have bounded mean oscillation (BMO is a short term «Bounded Mean Oscillation»). The class BMO for the first time was introduced in [4].

Note that space $BMO_\varphi:=BMO_{\varphi,\infty}$ was introduced in [6]. Space $BMO_{\varphi,\theta}$ for the first time was introduced in paper [7].

By $H_{\varphi,\theta}$ we denote a class of all local bounded in $\mathbf{R}^n$ functions such that $\|f\|_{H_{\varphi,\theta}}<+\infty$, where

$$\|f\|_{H_{\varphi,\theta}} := \begin{cases} \left(\int_0^\infty \left(\frac{\omega_f(t)}{\varphi(t)}\right)^\theta dt\right)^{1/\theta} & \text{for } 1 \le \theta < \infty, \\ \sup\left\{\frac{\omega_f(t)}{\varphi(t)} : t > 0\right\} & \text{for } \theta = \infty. \end{cases}$$

The class $H_{\varphi,\theta}$ is considered as a subset in factor-space $L_{loc}^\infty(\mathbf{R}^n)/\{\text{constants}\}$. Then $\|\cdot\|_{H_{\varphi,\theta}}$ is a norm in $H_{\varphi,\theta}$ and space $H_{\varphi,\theta}$ is a Banach one in this norm.

If we denote by $H_\varphi := H_{\varphi,\infty}$, then it is easy to see, that

$$H_\varphi = \left\{ f \in L_{loc}^\infty(\mathbf{R}^n) : \|f\|_{H_\varphi} = \operatorname*{ess\,sup}_{x,y \in \mathbf{R}^n} \frac{|f(x) - f(y)|}{\varphi(|x-y|)} < \infty \right\}.$$

In particular, $H_{\delta^\alpha}(0 < \alpha \le 1)$ is a class of functions satisfying the Hölder condition with exponent α. We can show that if $\varphi(+0) = 0$, then each function $f \in H_\varphi$ can be on the set of measure zero such, that it will be continuous in $\mathbf{R}^n$.

Consider also a class, which was introduced in [2]:

$$VMO := \{f \in BMO : \lim_{\delta \to 0} M_f(\delta) = 0\},$$

and for $f \in VMO$ we suppose $\|f\|_{VMO} := \|f\|_{BMO}$. It can be verified that VMO is a closed subspace in BMO (VMO is a short term of «Vanishing Mean Oscillation»).

In paper [7] was proved

Theorem 2. *Let* $\varphi \in Z^\theta$, $1 \le \theta \le \infty$. *Then* $BMO_{\varphi,\theta} = H_{\varphi,\theta}$ *and for any function* $f(x)$

$$\|f\|_{BMO_{\varphi,\theta}} \le 2\|f\|_{H_{\varphi,\theta}} \le C\|f\|_{BMO_{\varphi,\theta}},$$

where the constant $C > 0$ *doesn't depend on* f.

4. The statements on actions of operator $A^* f = \tilde{f}$ in the mean oscillation were proved with the help of Theorem 1, some of which are represented below

Theorem 3. *Let* $\varphi \in \Phi_1$ *and*

$$\int_\delta^\infty \omega\left(\frac{\delta}{x}\right)\frac{\varphi(x)}{x}dx = O(\varphi(\delta)) \quad (\delta > 0). \tag{6}$$

Then if $f \in BMO_\varphi$, *then* $A^* f \in BMO_\varphi$ *and*

$$\|A^* f\|_{BMO_\varphi} \le C\|f\|_{BMO_\varphi},$$

where the constant $C > 0$ *doesn't depends on* f.

Corollary. *The operator* A^* *boundedly acts in the space BMO.*

The last statement is immediately obtained from Theorem 3 by virtue of the fact that if $\varphi(\delta) \equiv 1$, then condition (6) holds and in this case

$$BMO_\varphi = BMO.$$

Theorem 4. *The operator* A^* *boundedly maps in the space VMO.*

Proof. It is enough to show that if $\lim_{\delta\to 0} M_f(\delta)=0$, then $\lim_{\delta\to 0} M_{\tilde f}(\delta)=0$. By virtue of inequality (2) for $0<\delta\le 1$ we have

$$M_{\tilde f}(\delta)\le C\int_{\delta}^{\infty}\omega\left(\frac{\delta}{t}\right)\frac{M_f(t)}{t}dt=C\int_{\delta}^{\sqrt{\delta}}\omega\left(\frac{\delta}{t}\right)\frac{M_f(t)}{t}dt+$$

$$+C\int_{\sqrt{\delta}}^{1}\omega\left(\frac{\delta}{t}\right)\frac{M_f(t)}{t}dt+C\int_{1}^{\infty}\omega\left(\frac{\delta}{t}\right)\frac{M_f(t)}{t}dt\le$$

$$\le CM_f\left(\sqrt{\delta}\right)\int_{\delta}^{\infty}\omega\left(\frac{\delta}{t}\right)\frac{1}{t}dt+CM_f(1)\int_{\sqrt{\delta}}^{1}\omega\left(\frac{\delta}{t}\right)\frac{1}{t}dt+$$

$$+C\|f\|_{BMO}\int_{1}^{\infty}\omega\left(\frac{\delta}{t}\right)\frac{1}{t}dt=CM_f\left(\sqrt{\delta}\right)\int_{0}^{1}\frac{\omega(x)}{x}dx+$$

$$+CM_f(1)\int_{\delta}^{\sqrt{\delta}}\frac{\omega(x)}{x}dx+C\|f\|_{BMO}\int_{0}^{\delta}\frac{\omega(x)}{x}dx.$$

From here by the help of Theorem 3 we obtain that, if $f\in VMO$, then $A^*f\in VMO$ and

$$\|A^*f\|_{VMO}\le C\|f\|_{VMO},$$

where $C>0$ doesn't depends on f.

Theorem is proved.

By the help of Theorems 2 and 3 could prove the following theorem.

Theorem 5. *If conditions (6) hold and*

$$\int_{0}^{\delta}\frac{\varphi(t)}{t}dt=O(\varphi(\delta))\quad(\delta>0),$$

then operator A^* *boundedly acts in* H_φ.

Theorem 6. *Let* $\varphi\in Z_1^\theta, 1\le\theta\le\infty, \omega(\delta)\equiv\delta$. *Then operator* A^* *boundedly acts in space* $BMO_{\varphi,\theta}$.

Theorem 7. *Let* $\varphi\in Z^\theta\cap Z_1^\theta, 1\le\theta\le\infty, \omega(\delta)\equiv\delta$. *Then operator* A^* *boundedly acts in space* $H_{\varphi,\theta}$.

The proofs of last theorems are analogous to the proofs of theorems 4.1 and 4.2 from [7] correspondingly.

Remark. In paper [8] was considered a maximal singular operator

$$B^*f(x)=\sup_{\varepsilon>0}\left|\int_{\mathbf{R}^n}\{K_\varepsilon(x-y)-K_1(-y)\}f(y)dy\right|$$

and was proved theorem which show that operator B^* boundedly acts from $L^\infty(\mathbf{R}^n)$ to BMO. Moreover that if $n=1, K(x)=\frac{1}{x}(x\ne 0)$ and $f_0(x)=\text{sign}\,x\in L^\infty(\mathbf{R}^1)$, then we can say that $B^*f_0(x)$ doesn't belong to $BMO(\mathbf{R}^1)$.

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