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ABSOLUTE CONVERGENCE OF FOURIER INTEGRALS OF FUNCTION FROM THE CLASS $L_p(R^k)(1\le p\le 2)$

Abstract

This paper summarizes the well-known results of Tichmarch in relation to the absolute convergence of Fourier integrals for the function of one variable.

Let $f(x)\in L_p(R^k)$ and

$$f(x)\sim \int_{R^k}\hat{f}(u)e^{i(x,u)}du, \tag{1}$$

$$(x,u)=x_1u_1+\cdots+x_ku_k,\quad du=du_1\ldots du_k,$$

where

$$\hat{f}(u)=\frac{1}{(2\pi)^k}\int_{R^k}f(x)e^{-i(x,u)}dx,\quad dx=dx_1\ldots dx_k$$

its Fourier transform.

In this work the known results by Titchmarch [1] are generalized, which concern to absolute convergence of Fourier transformations of function of one variable for function of many variables from class $L_p(R^k)$ $(k\ge 1)$.

Denote by $f(x;t)$ the mean spherical of function $f(x)$ by the sphere with radius t and centre in point $x\in R^k$:

$$f(x,t)=\frac{\Gamma(k/2)}{2(\pi)^{k/2}}\int_{\Sigma}f(x_1+t\xi_1,\ldots,x_k+t\xi_k)d\sigma(\xi),$$

where $x=(x_1\ldots x_k)$, $\Sigma:\xi_1^2+\cdots+\xi_k^2=1$ is unique sphere; $d\sigma(\xi)$is element of volume by dimension $(k-1)$.

Fourier transform of function $f(x;t)$ is determined so:

$$f(x;t)=\int_{R^k}\hat{f}(u)\alpha_\mu(t|u|)e^{i(x,u)}du, \tag{2}$$

where $\alpha_\mu(t|u|)=2^\mu\Gamma(\mu+1)\cdot\dfrac{J_\mu(t|u|)}{(t|u|)^\mu}$, $\mu=\dfrac{k-2}{2}$ and $J_\mu(z)$ is Bessel function.

Function $\alpha_\mu(t)$ has following properties (see [3], p. 37).

Lemma 1. *For some $u>0$ and the corresponding it constant $b(u)>0$, it is valid the inequality $b(u)<1-\alpha_\mu(t)<2$ for $t>u$.*

In addition

$$1-\alpha_\mu(x)>\frac{t^2}{\pi^2(\mu+1)}\quad for\quad 0<t<\pi,$$

$$1-\alpha_\mu(t)<\left(\frac{t}{2}\right)^2\frac{1}{\mu}\quad for\quad t>0.$$

Suppose $\Phi(x,t)=f(x,t)-f(x)$ and

$$M_p\Phi(t)=\left\{\frac{1}{(2\pi)^k}\int_{R^k}|\Phi(x,t)|^p dx\right\}^{1/p},$$

where $x\in R^k$, $dx=dx_1dx_2\ldots dx_k$.

Let $\omega(t)$ be some positive function such that $\omega(t)\downarrow 0$ for $t\to 0$.

It is valid the following

Theorem 1. *If $M_1\Phi(t)=O(\omega(t))$ for $t\to 0$, then*

$$\left|\hat{f}(u)\right|=O\left(\omega\left(\frac{\delta}{|u|}\right)\right) \quad for \quad |u|\to\infty,$$

where $\delta>0$ is some number.

Proof. Keeping in mind the expansion (2) it is not difficult to show that Fourier transform of function $\Phi(x,t)$ is expression $\{\alpha_\mu(t|u|)-1\}\hat{f}(u)$.

Consequently,

$$\{\alpha_\mu(t|u|)-1\}\hat{f}(u)=\frac{1}{(2\pi)^k}\int_{R^k}\Phi(x,t)e^{-i(x,u)}dx.$$

Hence we find

$$\left|\{\alpha_\mu(t|u|)-1\}\hat{f}(u)\right|\le C\int_{R^k}|\Phi(x,t)|dx=O(\omega(t)) \quad \text{for} \quad t\to 0.$$

Assuming $t=\dfrac{\delta}{|u|}$ $(0<\delta<\pi)$ and using Lemma 1 from the last inequality we obtain

$$\left|\hat{f}(u)\right|=O\left(\omega\left(\frac{\delta}{|u|}\right)\right) \quad \text{for} \quad |u|\to+\infty.$$

Theorem 2. *Let $1<p\le 2$ and $f(x)\in L_p(R^k)$. If $M_p\Phi(t)=O(\omega(t))$ for $t\to 0$ and*

$$\sum_{n=1}^{\infty}n^{-\beta/p'}\omega^\beta\left(\delta n^{-1/k}\right)<+\infty \tag{3}$$

for some $\beta>0$, then $\int_{R^k}\left|\hat{f}(u)\right|^\beta du<+\infty$.

Proof. Keeping in mind that $\{\alpha_\mu(t|u|)-1\}\hat{f}(u)$ is Fourier transform of function $\Phi(x,t)$ by virtue of Titchmarch theorem ([2], p. 128) we find

$$\left\{\int_{R^k}\left(\left|\alpha_\mu(t|u|)-1\right|\left|\hat{f}(u)\right|^{p'}\right)du\right\}^{1/p}\le C\left(\int_{R^k}|\Phi(x,t)|^p dx\right)^{1/p}=O(\omega(t)),$$

where $p>1$ and $pp'=p+p'$.

Hence it follows that

$$\int_{R^k}\left|\alpha_\mu(t|u|)\right|^{p'}\left|\hat{f}(u)\right|^{p'}du=O\left(\omega^{p'}(t)\right) \quad \text{for} \quad t\to 0.$$

It is easy to be persuaded in validity of following correlation

$$\int\limits_{n-1\le|u|^2\le 2n-1}\left|1-\alpha_\mu(t|u|)\right|^{p'}\left|\hat{f}(u)\right|^{p'}du=O\left(\omega^{p'}(t)\right)\quad\text{for}\quad t\to 0,\tag{4}$$

where $|u|^2=|u_1|^2+\cdots+|u_k|^2$.

Assuming $t=\frac{\delta}{\sqrt{n}}\left(0<\delta<\frac{\pi}{\sqrt{2}}\right)$ and using Lemma 1 from (4) we find

$$\int\limits_{n-1\le|u|^2\le 2n-1}\left|\hat{f}(u)\right|^{p'}du=O\left(\omega^{p'}\left(\frac{\delta}{\sqrt{n}}\right)\right)\quad\text{for}\quad n\to\infty.\tag{5}$$

Using Gelder inequality for $\frac{1}{q}+\frac{1}{q'}=I$, $q>1$ and assuming $\beta q=p'$ we find

$$\int\limits_{n-1\le|u|^2\le 2n-1}\left|\hat{f}(u)\right|^{\beta}du\le\left(\int\limits_{n-1\le|u|^2\le 2n-1}\left|\hat{f}(u)\right|^{p'}du\right)^{\beta/p'}\left(\int\limits_{n-1\le|u|^2\le 2n-1}du\right)^{1-\beta/p'}\le$$

$$\le C\cdot n^{k/2(1-\beta/p')}\left(\int\limits_{n-1\le|u|^2\le 2n-1}\left|\hat{f}(u)\right|^{p'}du\right)^{\beta/p'}\le C\cdot n^{k/2(1-\beta/p')}\omega^{\beta}\left(\frac{\delta}{\sqrt{n}}\right)$$

for $0<\beta<p'$.

We suppose in the last inequality $n=2^{\nu}$ and sum the obtained inequality by $\nu=0,1,2,\ldots$. Then from the last inequality we find

$$\int\limits_{R^k}\left|\hat{f}(u)\right|^{\beta}du=O\left(\sum_{\nu=0}^{\infty}2^{\nu\frac{k}{2}\left(1-\frac{\beta}{p'}\right)}\omega^{\beta}\left(\delta\cdot 2^{-\nu/2}\right)\right).\tag{6}$$

Note that convergence of series $\sum_{\nu=0}^{\infty}2^{\nu\left(1-\frac{\beta}{p'}\right)\frac{k}{2}}\omega^{\beta}\left(\delta\cdot 2^{-\nu/2}\right)$ is equivalent (see [3], p. 39, corollary) for $h=k\left(1-\frac{\beta}{p'}\right)$ to convergence of following series

$$\sum_{n=1}^{\infty}n^{\frac{h}{k}-1}\omega^{\beta}\left(\delta\cdot n^{-1/k}\right)=\sum_{n=1}^{\infty}n^{-\frac{\beta}{p'}}\omega^{\beta}\left(\delta\cdot n^{-1/k}\right).$$

Therefore, by virtue of condition (2) from the last equality and from (6) we conclude $\int\limits_{R^k}\left|\hat{f}(u)\right|^{\beta}du<\infty$ for $0<\beta<p'$.

For $\beta=p'$ convergence of integral $\int\limits_{R^k}\left|\hat{f}(u)\right|^{p'}du$ follows from Titchmarsh inequality ([2], p. 128).

Thus, if we have proved that if $1<p\le 2$, then for fulfillment of condition (3) integral $\int\limits_{R^k}\left|\hat{f}(u)\right|^{\beta}du$ converges for some $\beta>0$.

For finish of proof of Theorem 2, we must only consider the case $p=1$. In this case condition (3) has a view

$$\sum_{n=0}^{\infty}\omega^{\beta}\left(\delta\cdot n^{-1/k}\right)<\infty .$$

Taking into account Theorem 1 we find

$$\int_{n-1\le|u|^2\le 2n-1}\left|\hat{f}(u)\right|^{\beta}du=O\left(\int_{n-1\le|u|^2\le 2n-1}\omega^{\beta}\left(\frac{\delta}{|u|}\right)du\right)=O\left(n^{k/2}\omega^{\beta}\left(\frac{\delta}{\sqrt{n}}\right)\right).$$

Supposing $n=2^{\nu}$ and summing by $\nu=0,1,2,3,\ldots$ we find

$$\int_{R^k}\left|\hat{f}(u)\right|^{\beta}du=\left(\sum_{\nu=0}^{\infty}2^{\nu k/2}\omega^{\beta}\left(\delta\cdot 2^{-\nu/2}\right)\right).$$

But convergence of series $\sum_{\nu=0}^{\infty}2^{\nu\frac{k}{2}\left(1-\frac{\beta}{p'}\right)}\omega^{\beta}\left(\delta\cdot 2^{-\nu/2}\right)$ is equivalent to convergence of series $\sum_{n=1}^{\infty}n^{\frac{h}{k}-1}\omega^{\beta}\left(\delta\cdot n^{-1/k}\right)$ for $h=k$, that is, convergence of series $\sum_{n=1}^{\infty}\omega^{\beta}\left(\delta\cdot n^{-1/k}\right)$, which validity follows from the condition of the theorem.

Thus, integral $\int_{R^k}\left|\hat{f}(u)\right|^{\beta}du<+\infty$ in case $p=1$.

Proof of Theorem 2 has been completely finished.

Theorem 3. *Let $1<p\le 2$ and $f\in L_p\left(R^k\right)$. If $M_p\Phi(t)=O(\omega(t))$ for $t\to 0$ and $\int_1^{\infty}t^{k\left(1-\frac{\beta}{p}\right)-1}\omega^{\beta}\left(\frac{1}{t}\right)dt<+\infty$ for some $\beta>0$, then*

$$\int_{R^k}\left|\hat{f}(u)\right|^{\beta}du<+\infty .$$

Proof. Keeping in mind that $\omega(t)$ is positive and $\omega(t)\downarrow 0$ for $t\to 0$, we find

$$n^{-\beta/p'}\omega^{\beta}\left(\delta\cdot n^{-1/k}\right)\le C\int_{n-1}^{n}\tau^{-\beta/p'}\omega^{\beta}\left(\delta\cdot\tau^{-1/k}\right)d\tau .$$

Summing this inequality by $n=2,3,4\ldots$ we obtain

$$\sum_{n=2}^{\infty}n^{-\beta/p'}\omega^{\beta}\left(\delta\cdot n^{-1/k}\right)\le C\int_{1}^{+\infty}\tau^{-\beta/p'}\omega^{\beta}\left(\delta\cdot\tau^{-1/k}\right)d\tau .$$

Having substituted of variable by formula $\tau=(\delta t)^k$, $d\tau=k\delta(\delta t)^{k-1}dt$ we find that

$$\int_{1}^{\infty}\tau^{-\beta/p'}\omega^{\beta}\left(\delta\cdot\tau^{-1/k}\right)d\tau=C\int_{1/\delta}^{+\infty}t^{(1-\beta/p')k-1}\omega^{\beta}\left(\frac{1}{t}\right)dt .$$

Hence we find

$$\sum_{n=2}^{\infty}n^{-\beta/p'}\omega^{\beta}\left(\delta\cdot n^{-1/k}\right)\le C\int_{1/\delta}^{+\infty}t^{(1-\beta/p')k-1}\omega^{\beta}\left(\frac{1}{t}\right)dt=$$

$$=C\int_{1/\delta}^{1}t^{(1-\beta/p')k-1}\omega^{\beta}\left(\frac{1}{t}\right)dt+\int_{1}^{+\infty}t^{(1-\beta/p')k-1}\omega^{\beta}\left(\frac{1}{t}\right)dt .$$

Integral $\int_{1/\beta}^{1} t^{(1-\beta/p')k-1}\omega^{\beta}\left(\frac{1}{t}\right)dt$ for the fixed $\delta>0$ converges. Thus, convergence of integral $\int_{1}^{+\infty} t^{(1-\beta/p')k-1}\omega^{\beta}\left(\frac{1}{t}\right)dt$ implies convergence of the series $\sum_{n=1}^{\infty} n^{-\beta/p'}\omega^{\beta}\left(\delta\cdot n^{-1/k}\right)$.

Then by virtue of theorem we have

$$\int_{R^k}\left|\hat{f}(x)\right|^{\beta}dx<+\infty$$

for some $\beta>0$.

Corollary. *If* $\omega(t)=O(t^{\alpha})$ $(\alpha>0)$, *then* $\int_{R^k}\left|\hat{f}(x)\right|^{\beta}dx<+\infty$ *for* $\beta>\frac{kp}{kp+p\alpha-k}$.

For $k=1$ hence if follows the well-known theorem by Titchmarch on the absolute convergence of Fourier transforms of function of one variable [1].

For $k=2$ from the proved theorems some results of [5], [6] follow.

Now let proven some meaning the reversibility of Theorem 2.

For that first we prove following

Lemma. *Let* $\delta>0$, $r>0$ *some number* $\varphi(x)\geq 0$ $\left(x\in R^k\right)$ *is some positive function determined in* R^k. *Then following two corellations are equivalent*

1⁰. $$n^{-r}\int_{|x|\leq\sqrt{n}}|x|^{2r}\varphi(x)dx+\int_{|x|\geq\sqrt{n}}\varphi(x)dx=O\left(\omega^{r}\left(\delta\cdot n^{-1/2}\right)\right)\quad(n\to\infty),\tag{7}$$

2⁰. $$\int_{R^k}\varphi(x)\left|1-\alpha_{\mu}(x|t|)\right|^{r}dx=O\left(\omega^{r}(t)\right)\quad(t\to 0).\tag{8}$$

Proof. Assume (7) is valid. For $t>0$ choose the natural number n such that

$$n=\left[\frac{\delta^2}{t^2}\right]\leq\frac{\delta^2}{t^2}<n+1\quad(0<\delta<\pi).$$

Using Lemma 1 we find

$$\int_{|x|\leq\sqrt{n+1}}\varphi(x)\left|1-\alpha_{\mu}(|x|t)\right|^{r}dx\leq\frac{t^{2r}}{4^{r}(\mu+1)^{r}}\int_{|x|\leq\sqrt{n+1}}|x|^{2r}\varphi(x)dx\geq$$
$$\leq Ct^{2r}n^{r}\omega^{r}\left(\delta(n+1)^{-1/2}\right)\leq C\omega^{r}(t).\tag{9}$$

Further taking into account Lemma 1 and condition (7)

$$\int_{|x|\geq\sqrt{n-1}}\varphi(x)\left|1-\alpha_{\mu}(|x|t)\right|^{r}dx=2^{r}\int_{|x|\geq\sqrt{n+1}}\varphi(x)dx=$$
$$=O\left(\omega^{r}\left(\delta(n+1)^{-1/2}\right)\right)=O\left(\omega^{r}(t)\right).\tag{10}$$

From (9) and (10) we conclude that

$$\int_{R^k}\varphi(x)\left|1-\alpha_{\mu}(|x|t)\right|^{r}dx=O\left(\omega^{r}(t)\right)\quad(t\to 0).$$

Now let prove that from validity of (8) the correlation (7) follows.

For the given n and for $\delta>0$ choose t such

$$t=\min\left\{\frac{\delta}{\sqrt{n}},\frac{\pi}{\sqrt{n}}\right\}.$$

By virtue of Lemma 1 we have $\left(t|x|<\pi\right)$

$$\int_{|x|\le\sqrt{n}} \varphi(x)\left|1-\alpha_\mu(|x|t)\right|^r dx \le \frac{t^{2r}}{\pi^{2r}(\mu+1)^r} \int_{|x|\le\sqrt{n}} |n|^{2r}\varphi(x)dx.$$

Hence we find

$$\begin{aligned} n^{-r} \int_{|x|\le\sqrt{n}} |x|^{2r}\varphi(x)dx = Cn^{-r}t^{-2r} \int_{|x|\le\sqrt{n}} \varphi(x)\left|1-\alpha_\mu(t|x|)\right|^r dx \le \\ \le C\int_{R^k}\varphi(x)\left|1-\alpha_\mu(t|x|)\right|^r dx \le C\omega^r(t) \le C\omega^r\left(\delta\cdot n^{-1/2}\right). \end{aligned} \tag{11}$$

Consequently,

$$n^{-r}\int_{|x|\le\sqrt{n}} |x|^{2r}\varphi(x)dx = O\left(\omega^r\left(\delta\cdot n^{-1/2}\right)\right) \quad (n\to\infty).$$

Further, using again Lemma 1 we find

$$\int_{|x|\ge\sqrt{n}} \varphi(x)\left|1-\alpha_\mu(t|x|)\right|^r dx \le b\int_{|x|\ge\sqrt{n}} \varphi(x)dx \quad (b>0).$$

Hence we obtain

$$\begin{aligned} \int_{|x|\ge\sqrt{n}} \varphi(x)dx = C\int_{|x|\ge\sqrt{n}} \varphi(x)\left|1-\alpha_\mu(t|x|)\right|^r dx \le \\ \le C\int_{R^k}\varphi(x)\left|1-\alpha_\mu(t|x|)\right|^r dx \le C\omega^r(t) \le C\omega^r\left(\delta\cdot n^{-1/2}\right). \end{aligned} \tag{12}$$

From (11) and (12) we conclude the validity of (7).

Now using Lemma 2 let prove the following theorem

Theorem 4. *If*

$$n^{-2}\int_{|x|\le\sqrt{n}} |x|^4|F(x)|^2dx + \int_{|x|\ge\sqrt{n}} |F(x)|^2dx = O\left(\omega^2\left(\delta\cdot n^{-1/2}\right)\right), \quad n\to\infty,$$

then $M_2\Phi(t) = O(\omega(t)) \quad (t\to 0)$.

Proof. By virtue of Plansherel theorem

$$\int_{R^k}|F(u)|^2\left|1-\alpha_\mu(t|u|)\right|^2du = \int_{R^k}|\Phi(x,t)|^2dx. \tag{13}$$

Taking into account the condition of this theorem and Lemma 2 we find

$$\int_{R^k}|F(u)|^2\left|1-\alpha_\mu(t|u|)\right|^2du = O\left(\omega^2(t)\right) \quad \text{for } t\to 0. \tag{14}$$

Then from correlations (13) and (14)

$$M^2\Phi(t) = O(\omega(t)) \quad (t\to 0).$$

Now we will consider the spherical means of order $q\ (q>0)$ of function $f\ (x\in R^k)$.

For $q>0$ function

$$f_q(x,t) = \frac{2t^{-2q-k+2}}{B(q,k/2)}\int_0^t \left(t^2-s^2\right)^{q-1}s^{k-1}f(x,s)ds$$

is called a spherical mean of order q of function f, where $B(\alpha;\beta)$ is Beth function (see [4]).

It is not difficult to show that

$$\frac{2t^{-2q-k+2}}{B(q,k/2)}\int_0^t \left(t^2-s^2\right)^{q-1}s^{k-1}ds = 1$$

(for that it is sufficient to introduce change of the variable by formula $S = t\cos\varphi$).

Then we have

$$f_q(x,t)-f(x)=\frac{2t^{-2q-k+2}}{B(q,k/2)}\int_0^t\left(t^2-s^2\right)^{q-1}s^{k-1}\left[f(x,s)-f(x)\right]ds. \qquad (15)$$

So as Fourier transform of function $f(x,t)-f(x)$ is equal to $\{\alpha_\mu(s|u|)-1\}\hat{f}(u)$, from (15) we conclude that Fourier transform of function $f_q(x,t)-f(x)$ is following function:

$$\frac{2t^{-2q-k+2}}{B(q,k/2)}\int_0^t\left(t^2-s^2\right)^{q-1}s^{k-1}\{\alpha_\mu(s|u|)-1\}\hat{f}(u)ds.$$

Denote $\Phi_q(x,t)=f_q(x,t)-f(x)$.

Prove the analogy of Theorems 1 and 2 for spherical mean of order q and of function $f(x)$ $\left(x\in R^k\right)$.

Theorem 5. *If $M_1\Phi_q(t)=O(\omega(t))$ for $t\to 0$, then for some $\delta>0$ the correlation is valid*

$$\left|\hat{f}(u)\right|=O\left(\omega\left(\frac{\delta}{|u|}\right)\right) \quad for \quad |u|\to\infty.$$

Proof. We have

$$\frac{2t^{-2q-k+2}}{B(q,k/2)}\int_0^t\left(t^2-s^2\right)^{q-1}s^{k-1}\{\alpha_\mu(s|u|)-1\}ds\circ\hat{f}(u)=\frac{1}{(2\pi)^k}\int_{R^k}\Phi_q(q,t)e^{-i(x,u)}dx.$$

Hence it follows that

$$\frac{\left|\hat{f}(u)\right|}{t^{2q+k-2}}\left|\int_0^t\left(t^2-s^2\right)^{q-1}s^{k-1}\{1-\alpha_M(S|u|)\}ds\right|\le C\int_{R^k}\left|\Phi_q(x,t)\right|dx.$$

Assuming $t=\frac{\delta}{|u|}$ $(0<\delta<\pi)$ and using Lemma 1 we find

$$|u|^{2q+k-2}\left|\hat{f}(u)\right|\left|\int_0^{\delta/|u|}\left(\frac{\delta^2}{|u|^2}-s^2\right)^{q-1}s^{k-1}(s|u|)^2ds\right|=O\left(\omega\left(\frac{\delta}{|u|}\right)\right).$$

Hence it follows that

$$|u|^{k+2}\left|\hat{f}(u)\right|\left|\int_0^{\delta/|u|}\left(\delta^2-s^2|u|^2\right)^{q-1}s^{k+1}(s|u|)^2ds\right|=O\left(\omega\left(\frac{\delta}{|u|}\right)\right).$$

Make change of the variable by formula $s|u|\delta\cos\varphi$. Then

$$ds=-\frac{\delta}{|u|}\sin\varphi d\varphi \quad \text{and} \quad 0\le\varphi\le\frac{\pi}{2}.$$

We have

$$|u|^{k+2}\left|\hat{f}(u)\right|\left|\int_0^{\pi/2}\delta^{2q-2}\sin^{2q-2}\varphi\cdot\frac{\delta^{k+1}\cos^{k+1}\varphi}{|u|^{k+1}}\frac{\delta}{|u|}\sin\varphi d\varphi\right|=O\left(\omega\left(\frac{\delta}{|u|}\right)\right).$$

After some simplification from the last correlation we find

$$\left|\hat{f}(u)\right|\int_0^{\pi/2}\sin^{2q-2}\varphi\cdot\cos^{k+1}\varphi d\varphi=O\left(\omega\left(\frac{\delta}{|u|}\right)\right). \quad (16)$$

Taking into account

$$\int_0^{\pi/2}\sin^{2q-2}\varphi\cdot\cos^{k+1}\varphi d\varphi=B\left(q,\frac{k}{2}+1\right).$$

From (16) we conclude that

$$\left|\hat{f}(u)\right|=O\left(\omega\left(\frac{\delta}{|u|}\right)\right) \quad \text{for} \quad |u|\to\infty.$$

The theorem has been proved.

Theorem 6. *Let* $1<p\le 2$ *and* $f(x)\in L_p\left(R^k\right)\left(x\in R^k\right)$.

If $\left\|f^q(x,t)-f(x)\right\|_p=O(\omega(t))$ $t\to 0$ *and* $\sum_{n=1}^{\infty}n^{-\beta/p'}\omega^{\beta}\left(\delta n^{-1/k}\right)<+\infty$ *for some* $\beta>0$, *then*

$$\int_{R^k}\left|\hat{f}(x)\right|^{\beta}dx<+\infty.$$

Proof. By virtue of Titchmarch lemma we have:

$$\left\{\int_{R^k}\left|\frac{2t^{-2q-k+2}}{B(q,k/2)}\int_0^t\left(t^2-s^2\right)^{q-1}s^{k-1}\{\alpha_\mu(|u|s)-1\}\hat{f}(u)ds\right|^{p'}du\right\}^{1/p'}\le$$

$$\le C\left(\int_{R^k}\left|f_q(x,t)-f(x)\right|^p dx\right)^{1/p}=O(\omega(t)) \quad \text{for} \quad t\to 0,$$

where $\frac{1}{p}+\frac{1}{p'}=1$ and $p>1$.

From this correlation it follows that

$$\int_{n-1\le|u|^2\le 2n-1}\left|t^{2q-k+2}\int_0^t\left(t^2-s^2\right)^{q-1}s^{k-1}\{\alpha_\mu(|u|s)-1\}\hat{f}(u)ds\right|^{p'}=O\left(\omega^{p'}(t)\right),$$

where $|u|^2=|u_1|^2+\cdots+|u_k|^2$, $du=du_1\ldots du_k$.

Assuming $t=\frac{\delta}{|u|}$ $(0<\delta<\pi)$ and using Lemma 1 we find

$$\int_{n-1\le|u|^2\le 2n-1}|u|^k\left|\int_0^{\delta/|u|}\left(\delta^2-|u|^2s^2\right)^{q-1}s^{k-1}ds\right|^{p'}\left|\hat{f}(u)\right|^{p'}du=O\left(\omega^{p'}\left(\delta\cdot n^{-1/2}\right)\right).$$

Having change of the variable by formula $s|u|=\delta\cos\varphi$ as in the foregoing theorem, after some simplification we find

$$\int_{n-1\le|u|^2\le 2n-1}\left|\hat{f}(u)\right|^{p'}du=O\left(\omega^{p'}\left(\delta\cdot n^{-1/2}\right)\right).$$

Proof of the theorem is finished as it was in theorem 2. Analogously the analogies of Theorems 3 and 4 are proved for the spherical means of order q $(q>0)$.

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