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SHAHBAZOV A.I.

COMPACT WEIGHTED COMPOSITION OPERATORS INDUCED BY A FINITE GROUP OF TRANSFORMATIONS

Abstract

Let X be a compact Hausdorff space and let $C(X)$ denote the space of all continuous complex-valued functions on X equipped with the Sup-norm. In this work we will study compactness of operators on uniformly closed subspaces A of $C(X)$, which are induced by a finite number of continuous mappings $\omega_i : X \to X(i=1,...,n)$ of the form $T=\sum_{i=1}^{n} T_i$, where $T_i = u_i(f \circ \omega_i)$ are weighted composition operators on A and give some applications.

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1. Introduction.

In this paper we will consider operators induced by a finite number of continuous transformations $\omega_i : X \to X, i=1,2,...,m$; in other words we will study operators of the form $T=\sum_{i=1}^{m} T_i$, where $T_i = u_i(f \circ \omega_i)$,

Definition 1.1. *A closed subset $E \subset X$ is called a peak set with respect to A, if there exists a sequence $\{f_n\}, f_n \in A$, such that $\|f_n\| = f_n(x) = 1$ for all n and all $x \in E$, moreover, outside any neighborhood of the set E the sequence $\{f_n\}$ tends to 0 uniformly. A peak set consisting of only one point is called a peak point.*

Let A be a closed subspace of $C(X)$ and the set of all peak points with respect to A is denoted by Γ. Put $G = X \setminus \Gamma$. There is a correspondence between points $x \in X$ and functionals $\delta_x : f(x)$, which lie in the unit ball of conjugate space A^*. This induces on X the A^*- topology which is, generally speaking, stronger than the original one. Further, we shall always suppose that the original topology on G coincides with A^*- topology, G is everywhere dense in X and Γ is not empty. A typical example is given by the disc algebra, i.e., the algebra $A(D)$ of all analytic functions in the unit disc $D=\{z \in C : \|z\| < 1\}$ and continuous on its closure $\overline{D}$. Here $X=\overline{D}, \Gamma = \partial D = \overline{D} \setminus D$, $G = D$.

Consider the operators of the weighted composition type induced by a finite number of mappings, preserving G, i.e., the operator $T : A \to C(X)$ of the form $f(x) \mapsto \sum_{n=1}^{n} u_i(x) f(\omega_i(x))$, where $u_i \in C(X)$ are fixed functions and $\omega_i : X \mapsto X$ are continuous mappings such that $\omega_i(G) \subset G(i=1,...,m)$. Expecting for easy degenerate cases, we shall assume that $\omega_i \neq cons$ for all i.

The aim of this work is to clarify the compactness conditions of such operators. In particular, H.Kamowitz [2] gave a compactness criterion for the weighted composition operator (i.e., of the form $f \mapsto u(f \circ \omega)$) in the disc-algebra and described its spectrum when the operator is compact. Weighted composition operators have been studied on different function algebra's (see [1]-[10], etc.). In [5], [6] Kamowitz's results were extended to many other uniform spaces (including multidimensional analogues of disc-algebra) and in [2] by E.A.Gorin and A.I.Shahbazov an operator of the form T in the disc-algebra in case of two summands was considered. Here we shall give sufficient simple general compactness criteria for the operator T, which for $m=2$ the summands takes sufficiently transparent form, and we shall give some applications of compactness of operator T, when the mappings $\omega_i(i=1,...,m)$ are a finite group.

2. Compact weighted composition operators induced by a finite number of mappings.

We will identify the unit ball of the conjugate space $C(X)^*$ of $C(X)$ with all complex Borel measures on X with variation ≤ 1, any point $x \in X$ will correspond to the δ-measure. Since the compactness of an operator is equivalent to the compactness of its conjugate so it can easily be shown that the compactness of the operator T is equivalent to continuity of mapping $x \mapsto T^*x = \sum_{i=1}^m u_i(x)\omega_i(x)$ acting on X with original topology into A^* with metric topology: if $z \mapsto \zeta$ in X with original topology, then $\sum_{i=1}^m u_i(z)\omega_i(z)$ must converge to $\sum_{i=1}^m u_i(\zeta)\omega_i(\zeta)$, with respect to A^* - topology. If $\zeta \in G$, then because of $\omega_i(\zeta) \in G$ and since the original topology coincides with A^*-topology on G we will have $\|\omega_i(z) - \omega_i(\zeta)\|_{A^*} \to 0$ when $z \to \zeta$ in the original topology, i.e., the above mentioned mapping $x \to T^*x$ for $x \in G$ is automatically continuous. For this reason we will investigate the continuity only on peak points.

Definition 2.1. *Let $\zeta \in \Gamma$ be a fixed point, we say that the indices i,j are equivalent with respect to $\zeta \in \Gamma$, if $\omega_i(\zeta) = \omega_j(\zeta) \in \Gamma$. Equivalence classes will be denoted by K. K_0 will denote those indices i such that $\omega_i(\zeta) \in G$. Indices i,j are called strongly equivalent, if $\|\omega_i(z) - \omega_j(\zeta)\|_{A^*} \to 0$ when $z \to \zeta$.* Equivalence classes of this kind will be denoted by L.

Lemma 2.2. *If the operator T is compact, then for any class K we have $\sum_{i\in K} u_i(\zeta) = 0$.*

Proof. Since $K \neq K_0$, then there exists a point $\xi \in \Gamma$ such that $\omega_i(\zeta) = \xi$ for all $i \in K$. Since ξ is a peak point, then we can find a sequence of functions $f_n \in A$ such that $\|f_n\| = f_n(\xi) = 1$ and outside any neighborhood of the point ξ the sequence tends to 0 uniformly as $n \to \infty$. We may assume that the sequence Tf_n converges uniformly. Since G is invariant with respect to the mappings ω_i and is everywhere dense in X then $\|Tf_n\| \to 0$ and this completes the proof.

The conditions $\omega_i(G) \subset G, (i=1,...,m)$ in Lemma 2.2 are essential

Theorem 2.3. *The operator T is compact, if for an arbitrary point $\zeta \in \Gamma$ we have $\sum_{i\notin K_0} u_i(\zeta)\omega_i(z) \to 0$ with respect to A^* - norm as $z \to \zeta$ (in original topology of X) and $\sum_{i\in K} u_i(\zeta) = 0$ for any class $K \neq K_0$.*

Proof. Since the original topology on G coincides with A^* - topology, so the compactness of operator T is equivalent to the following: for any point $\zeta \in \Gamma$, when

$z \to \zeta$ (converges with respect to original topology), the sum $\sum_{i=1}^{m} u_i(\zeta)[\omega_i(z) - \omega_i(\zeta)]$ converges to zero in A^*-topology. It is clear, that in the sum one can restrict indices $i \notin K_0$. If we divide the sum into corresponding classes, then the sufficiency will be obvious. The necessity of $\sum_{i \in K} u_i(\zeta) = 0$ for $K \neq K_0$ follows from Lemma 2.2 and if we consider the sum for $i \notin K_0$ the necessity of first condition will hold. So the theorem is proved.

Corollary. *If* $\sum_{i \in L} u_i(\zeta) = 0$ *for any subclass* L *and for any* $\zeta \in \Gamma$, *then the operator* T *is compact.*

The converse of the corollary is not valid in general true.

3. Weighted composition operators induced by two mappings.

For $m = 2$ the converse of the above corollary holds and in this case the criteria of compactness can easily be verified.

The following theorem generalizes the results of [1].

Theorem 3.1. *The operator* $u_1 f \circ \omega_1 + u_2 f \circ \omega_2$ *is compact if, and only if, when for any point* $\zeta \in \Gamma$ *the following conditions hold: if indices are strongly equivalent, then* $u_1(\zeta) + u_2(\zeta) = 0$; *if, they are not strongly equivalent, but* $\omega_i(\zeta) \in \Gamma$ *then* $u_i(\zeta) = 0$.

Proof. The sufficiency is clear from the corollary of Theorem 2.3.

We assume that the operator is compact. Let $\zeta \in \Gamma$, and $\omega_1(\zeta) \in \Gamma$, $\omega_2(\zeta) \neq \omega_1(\zeta)$. According to Theorem 1 we have $u_1(\zeta)\omega_1(\zeta) + u_2(\zeta)\omega_2(\zeta) = 0$. Since $\omega_1(\zeta)$ is a peak point, then we have $u_1(\zeta) = 0$. Now, if we assume, that $\omega_1(\zeta) = \omega_2(\zeta)$ (i.e., they are equivalent), then from Theorem 2.3 we have $u_1(\zeta) + u_2(\zeta) = 0$. If they are not strongly equivalent, then $\|\omega_1(z_n) - \omega_2(z_n)\| \geq \delta > 0$ for some sequence $z_n \to \zeta$. Hence using Theorem 2.3, we get $u_1(\zeta)\omega_1(z_n) + u_2(\zeta)\omega_2(z_n) \to 0$, so we have $u_1(\zeta)[\omega_1(z_n) - \omega_2(z_n)] \to 0$, from this, we have $u_1(\zeta) = u_2(\zeta) = 0$. The theorem is proved.

Now we consider the disc-algebra, as the algebra of all continuous functions on the half-plane $\operatorname{Im}\lambda \geq 0$ and analytic in $\operatorname{Im}\lambda > 0$, such that they have limit as $|\lambda| \to \infty$. Let ω be a continuous mapping from $\operatorname{Im}\lambda \geq 0$ into itself, which is holomorhic for $\operatorname{Im}\lambda > 0$ and have limit as $|\lambda| \to \infty$. Then from Theorem 3.1 we have the following corollary.

Corollary. *Let* ω_1, ω_2 *be such that they map* $\operatorname{Im}\lambda \geq 0$ *into itself and have the above mentioned conditions, and* $\operatorname{Im}\omega_i(\lambda) > 0$, *for all* λ *and* $\omega_i(\infty) = \infty$. *Then the operator* $f \mapsto u_1 f \circ \omega_1 + u_2 f \circ \omega_2$ $(u_1, u_2$ *are the elements of the disc-algebra) is compact iff, we have the following conditions: either* $u_1(\infty) = u_2(\infty) = 0$, *or* $u_1(\infty) + u_2(\infty) = 0$ *and* $\left|\dfrac{\omega_1(\lambda) - \omega_2(\lambda)}{\omega_1(\lambda) - \overline{\omega_2(\lambda)}}\right| \to 0$ *when* $|\lambda| \to \infty$.

4. Weighted composition operators induced by a finite group of transformations.

In this section we consider the operator $T : A \to C(X)$, such that $(Tf)(x) = \sum_{i=1}^{N} u_i(x) f(\omega_i(x)), (u_i \in C(X))$, where ω_i's are invertible and form a finite group of transformations on X, i.e., $\omega_i : X \to X$. W will investigate the compactness of this operator under the conditions that $\omega_i(G)\subset G$ (for all i), where $G=X\setminus\Gamma$ and Γ is the peak point set of A and the original topology on G coincides with A^*-topology. The importance of this kind of operators is that they are applicable in solving and existence of equations of the form $\sum_{i=1}^{N} u_i(x)f(\omega_i(x))=g(x)$, i.e. differential-functional equations, which contain both the argument and its shift.

When the set $Y=\{\omega_i\}$ is a finite group, then for any $x\in X$ corresponds a finite set $\{\omega_i(x)\}(\omega_i\in Y)$, hence the above mentioned equations can be represented as a finite system of equations without shift arguments. For example when ω is a 2-cycle mapping $(\omega\circ\omega=id)$, then in the equation

$$u_0(x)f(x)+u_1(x)f(\omega(x))=g(x) \tag{1}$$

(where u_0,u_1,g are given continuous functions and f is unknown) by changing $x\mapsto\omega(x)$, and putting $f(x)=f_0(x), f(\omega(x))=f_1(x)$ we have for f_0,f_1 the following system of equations without shift-argument

$$\begin{cases} u_0(x)f_0(x)+u_1(x)f_1(x)=g(x) \\ u_1(\omega(x))f_0(x)+u_0(\omega(x))f_1(x)=g(\omega(x)) \end{cases} \tag{2}$$

The conditions for the existence of (2) can easily be written, but they would not be necessary for equation (1), so additional conditions must be investigated.

Let $Y=<\omega>$ be a cyclic group induced by the mapping ω, such that $|Y|=N$: in other words if, we denote $\omega_1=\omega, \omega_2=\omega\circ\omega_1,\ldots,\omega_N=\omega\circ\omega_{N-1}$, then $Y=\{\omega_1,\ldots,\omega_N\}$, where $\omega_N=\text{id}$ (identity).

For peak points Γ of A we introduce the notion of equivalence degree with respect to Y.

Definition 4.1. *For any point $\zeta\in\Gamma$ the least positive integer N_ζ satisfying the equality $\omega_{N_\zeta}(\zeta)=\zeta$ is called equivalent degree for ζ. It is clear that such number exists (because for any $\zeta\in\Gamma$ the equality $\omega_N(\zeta)=\zeta$ hold, and also $1\le N_\zeta\le N$). If we denote by K_j all peak points such that their equivalent degrees are j, then the peak set Γ has a disjoint union of classes K_j, i.e., $\Gamma=\bigcup_{j=1}^{N}K_j$, where $K_i\bigcap K_j=\varnothing$ for any $1\le i,j\le N$, which $i\ne j$.*

It is clear that if, $|i-j|$, where $1\le i,j\le N$ is divisible by N_ζ, then for $\zeta\in\Gamma$, we have $\omega_i(\zeta)=\omega_j(\zeta)$.

Since at the point ζ the group $\{\omega_i(\zeta)\}_{i=1}^{N_\zeta}$ is a subgroup of $Y=<\omega(\zeta)>$, then N is divisible by N_ζ, so the following lemma holds:

Lemma 4.2. *At any point $\zeta\in\Gamma$ indices are separated into N_ζ (denoted by $K_1,K_2,\ldots,K_{N_\zeta}$) equivalence classes, such that each power is equal to $N:N_\zeta$.*

Corollary. *If N is a prime number, then all indices are equivalent.*

Using the above mentioned facts for the operator T induced by the group $Y=<\omega>$ we have the following compactness criteria:

Theorem 4.3. *The operator T is compact iff, for any point $\zeta \in K_{N_\zeta}$ we have $\sum_{i=1}^{N} u_i(\zeta)\omega_i(z) \to 0$ with respect to A^*-norm, as $z \to \zeta$ (with respect to original topology on X) and $\sum_{i \in A} u_i(\zeta) = 0$, where $A = K_1, ..., K_{N_\zeta}$.*

In particular, if N is a prime number, then from Theorem 4.3 and the corollary of Lemma 4.2 we have the following simple criterion of compactness.

Theorem 4.4. *If N is a prime number, then the finite sum weighted composition operator T induced by the group $Y = <\omega> = \{\omega_1, ..., \omega_N\}$ is compact iff, for any peak point $\zeta \in \Gamma$ we have $u_i(\zeta) = 0$, for any $i = 1, ..., N$.*

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Shahbazov A.I.
Institute of Mathematics and Mechanics of AS Azerbaijan.
9, F. Agayev str., 370141, Baku, Azerbaijan.
Tel.: 39-47-20.

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