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SOLTANOV K.N., NOVRUZOV E.B.

THE FREE BOUNDARY VALUE PROBLEM FOR A NONLINEAR EQUATION OF PARABOLIC TYPE

Abstract

The present paper is devoted to the investigation of the free boundary problem for nonlinear parabolic equation. The existence of the solution of above-stated problem has been proved.

The aim of present paper is to investigate the free boundary problem for the equation

$$\frac{\partial u}{\partial t}-D\left(|u|^{p_0}Du+|Du|^{p_1}Du\right)=f(x,t). \tag{0.1}$$

In this connection smoothness properties of solution of this equation at some domain with a fixed boundary are also investigated. The necessity of obtaining of smooth solution is natural for free boundary problem. The noted necessity is more relevant in present case, because for equation (0.1) transformations that was used, for example, at papers [1], [2], which allow either get rid of non-linearity in the main part of equation, or reduce the problem to some variation problem, are not applicable. In particular, Kirchhoff transformation [1] which allows to get rid of the non-linearity in main part, in case of equation with first of non-linear members (0.1), in present case couldn't be useful.

Some results about smoothness of solution of similar equations are contained, for example, at papers [3], [4]. Note also, that the questions on solvability for equations of similar type for enough general conditions for non-linearity were investigated at papers [5]-[7].

§1. Investigation of initial boundary value problem.

Consider the following problem:

$$\frac{\partial u}{\partial t}-D\left(|u|^{p_0}Du+|Du|^{p_1}Du\right)=f(x,t)\quad (x,t)\in Q\equiv(0,T)\times\Omega \tag{1.1}$$

$$u(x,0)=u_0(x),\quad x\in\Omega\equiv(a,b) \tag{1.2}$$

$$u|_{\Gamma}=0\quad \Gamma=\partial\Omega\times[0,T), \tag{1.3}$$

where $p_0\ge 2, p_1\ge 2$ are some real numbers; $f(x,t), u_0(x)$ are some functions; $D=\frac{\partial}{\partial x}$, (a,b) is a bounded interval.

Now we introduce the following intersection of spaces of functions $v:Q\to R$:

$$P_1(Q)\equiv L_\nu\left(0,T;\mathring{S}_{1,p_0+p_*,2}(\Omega)\right)\cap L_\infty\left(0,T;\mathring{W}^1_{k+2}(\Omega)\right)\cap L_\mu\left(0,T;\mathring{S}_{1,p_*,p_1+2}(\Omega)\right)\cap$$
$$L_{\nu_1}\left(0,T;\mathring{S}^1_{1,p_1+k,2}(\Omega)\right)\cap W^1_2(Q)\cap L_{p_0+2}\left(0,T;S_{2,p_0,2}(\Omega)\right)\cap L_{p_1+2}\left(0,T;S^1_{1,p_1,2}(\Omega)\right)\cap$$
$$\cap\{v|v(x,0)=u_0(x)\}$$

where $\nu_1=k+p_1+2,\quad \nu=p_*+p_0+2,\quad \mu=p_1+p_*+2,\quad p_*=(4p_0-2)(k+2), k\ge p_1,$

(apropos of spaces $S^k_{\alpha,\beta,\gamma}$ see [8])

$$\overset{\circ}{S}_{1,\alpha,\beta}(\Omega)=\left\{u(x)\left|\int\limits_\Omega |u|^\alpha |Du|^\beta\, dx<+\infty,\ u|_{\partial\Omega}=0\right.\right\}$$

$$\overset{\circ}{S}_{2,\alpha,\beta}(\Omega)=\left\{u(x)\left|\int\limits_\Omega |u|^\alpha |D^2u|^\beta\, dx<+\infty,\ u|_{\partial\Omega}=0\right.\right\}$$

$$\underset{\circ}{S}{}^1_{1,\alpha,\beta}(\Omega)=\left\{u(x)\left|\int\limits_\Omega |Du|^\alpha |D^2u|^\beta\, dx<+\infty,\ u|_{\partial\Omega}=0\right.\right\}$$

$$\overset{\circ}{S}{}^1_{1,\alpha,\beta}(\Omega)=\left\{u(x)\left|\int\limits_\Omega |Du|^\alpha |D^2u|^\beta\, dx<+\infty,\ u|_{\partial\Omega}=0,\ Du|_{\partial\Omega}=0\right.\right\}$$

Definition 1. *The solution of problem (1.1)-(1.3) we will call the function $u(x,t)\in P_1(Q)$, which satisfies the equation (1.1) in the sense of the space $L_2(Q)$, i.e. for any $v(x,t)\in L_2(R)$ the following equality holds*

$$\int\limits_Q \frac{\partial u}{\partial t} v\,dxdt-\int\limits_Q D\left(|u|^{p_0}Du+|Du|^{p_1}Du\right)v\,dxdt=\int\limits_Q fv\,dxdt$$

(for that it is clear that equation (1.1) holds almost everywhere).

Theorem 1. *Let $u_0(x)\in \overset{\circ}{W}{}^1_{k+2}(\Omega)\cap L_{p_*+2}(\Omega)$, $f(x,t)\in L_2\left(0,T;\overset{\circ}{W}{}^1_{k+2}(\Omega)\right)$. Then problem (1.1)-(1.3) is solvable in $P_1(Q)$.*

As it was indicated above the question on solvability of given problem was investigated in papers [5]-[7]. Theorem 1 is about the fact that for mentioned conditions the solution of problem is contained in $P_1(Q)$. In other words, the problem with some additional conditions has more smooth solution.

The proof of this theorem could be done by the scheme of the proof of the general theorem from papers [5,7]. The main thing is to construct corresponding operator generated in general sense coercive pair with operator generated by problem (1.1)-(1.3). For brief presentation we will not show the proof of this fact. We only describe the way of construction of the mentioned operator.

Introduce the following spaces:

$$P_2(Q)\equiv P_1(Q)\cap L_2\left(0,T;W_2^2(\Omega)\right)\cap L_{k+2}\left(0,T;\underset{\circ}{S}{}^1_{1,k,2}\right)\cap$$
$$\cap\left\{v\left||Dv|^{k/2}Dv_t\in L_2(Q)\right.\right\}\cap\left\{v\left|Dv_t\in L_2(Q)\right.\right\}$$
$$P'_2(Q)\equiv P_1(Q)\cap L_2\left(0,T;W_2^2(\Omega)\right)\cap L_{k+2}\left(0,T;\underset{\circ}{S}{}^1_{1,k,2}(\Omega)\right)\cap$$
$$\cap\left\{v\left|\sqrt{T-t}\,|Dv|^{k/2}Dv_t\in L_2(Q)\right.\right\}\cap\left\{v\left|\sqrt{T-t}\,Dv_t\in L_2(Q)\right.\right\}$$

Let $E:P_2(Q)\to W_2^1(Q)\subset L_2(Q)$ be a narrowing on Q of operator $\widetilde{E}:P'_2(Q_1)\to W_2^1(Q_1)$ generated by the problem:

$$c_1u_t-|Du|^kD^2u-c_2D^2u-c_3|u|^{p_*}u=h(x,t)\quad (x,t)\in Q_1\equiv[a,b]\times[0,T_1] \tag{1.4}$$
$$u(x,0)=u_0(x)\quad x\in\Omega \tag{1.5}$$
$$u|_{\Gamma_1}=0\quad \Gamma_1\equiv\partial\Omega\times[0,T_1), \tag{1.6}$$

where $T_1 > T$, k, p_*, c_1, c_2, c_3 are positive constants:

Theorem 2. *Let $h(x,t) \in W_2^1(Q_1)$, $Q_1 \equiv \Omega \times [0,T_1)$. Then for any $k > 0$ and $u_0(x) \in W_2^1(\Omega) \cap S^1_{1,k,2}(\Omega) \cap L_{p_*+2}(\Omega)$ there exists the solution $u(x,t) \in P'_2(Q_1)$ of the problem (1.4)-(1.6) and it is unique (in the sense analogous to definition 1).*

The proof of this theorem is similar to the proof of corresponding theorem from [9], where problem (1.4)-(1.6) was investigated in case when $c_1 = 1, c_2 = c_3 = 0$. Thus, we can determine the operator $E : P_2(Q) \to \overset{\circ}{W}{}_2^1(Q)$.

The further proof of theorem 1 could be done according to the scheme of the proof from [9].

§ 2. One result on smoothness.

At this paragraph we will show a theorem about the smoothness solution of problem (1.1)-(1.3). Videlicet, for some additional conditions we will prove, that solutions are more smooth and are contained in the following space:

$$P_3(Q) \equiv W_\infty^1(0,T;L_2(\Omega)) \cap P_1(Q) \cap \left\{ u \left| \frac{\partial}{\partial t}\left(|u|^{p_0/2} Du\right) \in L_2(Q) \right.\right\} \cap$$

$$\left\{ u \left| \frac{\partial}{\partial t}\left(|Du|^{p_1/2} Du\right) \in L_2(Q) \right.\right\}.$$

Or, the following theorem is true.

Theorem 3. *Let p_0, p_1 be some real numbers, which satisfies the conditions $p_0 \ge 2, p_1 \ge 2$. Then for any functions $u_0 \in \overset{\circ}{W}{}_r^1(\Omega)$ and $f(x,t) \in \overset{\circ}{W}{}_l^1(Q)$ for $r = \max\{p_0 + p_1 + 2; 2p_0 + p_* + 2; k + 2\}$, $l = \max\{4; p_* + 2; k + 2\}$, the solution of problem (1.1)-(1.3) is contained in $P_3(Q)$.*

We show in short the proof of the last fact.

Let $u(x,t)$ be the solution of problem (1.1)-(1.3). Then almost everywhere the equality:

$$\frac{\partial u}{\partial t} - D\left(|u|^{p_0} Du + |Du|^{p_1} Du\right) = f(x,t) \tag{2.1}$$

holds.

Now taking into account that in this equality the right-hand side is contained in $\overset{\circ}{W}{}_l^1(Q)$, we obtain that at least after transformation on the set of measure zero, the left-hand side will also be contained in $\overset{\circ}{W}{}_l^1(Q)$:

$$\frac{\partial u}{\partial t} - D\left(|u|^{p_0} Du + |Du|^{p_1} Du\right) \in \overset{\circ}{W}{}_l^1(Q).$$

Then it is easy to see that for any $v \in L_2(Q)$ the following equality is valid:

$$\int_Q \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} - D\left(|u|^{p_0} Du + |Du|^{p_1} Du\right)\right) v dx dt = \int_Q \frac{\partial}{\partial t} f v dx dt .$$

As $u(x,t) \in P_1(Q)$, then the last equality for any $v \in \overset{\circ}{W}{}_2^1(Q)$ could be rewritten by the following way:

$$\int_Q \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) v dxdt - \int_Q \frac{\partial}{\partial t}\left(D\left(|Du|^{p_1} Du\right)\right) v dxdt - \int_Q \frac{\partial}{\partial t}\left(D\left(|Du|^{p_0} Du\right)\right) v dxdt = \int_Q \frac{\partial f}{\partial t} v dxdt \quad (2.2)$$

Note, that the proof of a general theorem of solvability was done by construction of approximate solutions. As it is seen from construction of the operator $E : P_2(Q) \to L_2(q)$ defined in §1 and the scheme of proof of mentioned theorem, the approximate solutions u_m belong to $P_2(Q)$.

Taking into account this fact and also (2.2) we obtain that for $m \uparrow \infty$ the equality

$$\int_Q \frac{\partial^2 u_m}{\partial t^2} v dxdt - \int_Q \frac{\partial}{\partial t}\left(D\left(|u|^{p_0} Du\right)\right) v dxdt - \int_Q \frac{\partial}{\partial t}\left(D\left(|Du|^{p_1} Du\right)\right) v dxdt = \int_Q \frac{\partial f}{\partial t} v dxd \quad (2.3)$$

holds.

By virtue of all said above, at this equality, except the first member the rest integrals are defined for arbitrary $v \in L_2\left(0,T; \overset{\circ}{W}{}_2^1(\Omega)\right)$. Then using standard conclusions and using the fact that instead of v we could take the element of a complete system we obtain that the first integral is also determined for $v \in L_2\left(0,T; \overset{\circ}{W}{}_2^1(\Omega)\right)$.

Further, as $\frac{\partial u_m}{\partial t} \in L_2\left(0,T; \overset{\circ}{W}{}_2^1(\Omega)\right)$, then from previous conclusions it follows that, in particular, instead of v in equation (2.3) could be $\frac{\partial u_m}{\partial t}$.

Now the proof of theorem 3 immediately follows from the next lemma.

Lemma 1. *Let conditions of theorem 3 hold. Then for approximate solutions* $u_m(t,x)$ *the following inclusions hold:*

$$u_{mt} \in L_\infty(0,T; L_2(\Omega))$$
$$|u_m|^{p_0/2} Du_{mt} \in L_2(Q)$$
$$|Du_m|^{p_1/2} Du_{mt} \in L_2(Q).$$

Proof. From previous to lemma conclusion it is seen that

$$\int_{Q_t} \frac{\partial^2 u_m}{\partial t^2} \frac{\partial u_m}{\partial t} dxdt - \int_{Q_t} D\left(|u_m|^{p_0} Du_{mt}\right) u_{mt} dxdt - p_0 \int_{Q_t} D\left(|u_m|^{p_0-2} u_m u_{mt} Du_m\right) u_{mt} dxdt -$$
$$-(p_1+1)\int_{Q_t} D\left(|Du_m|^{p_1} Du_{mt}\right) u_{mt} dxdt = \int_{Q_t} f_t u_{mt} dxdt .$$

Integrating by parts the second-fourth integrals in the left-hand side of the last equality we have:

$$\frac{1}{2}\int_{Q_t} \frac{\partial}{\partial t}|u_t|^2 dxdt + \int_{Q_t} |u_m|^{p_0} |Du_{mt}|^2 dxdt + (p_1+1)\int_{Q_t} |Du_m|^{p_1} |Du_{mt}|^2 dxdt +$$
$$+ p_0 \int_{Q_t} |u_m|^{p_0-2} u_m u_{mt} Du_m Du_{mt} dxdt = \int_{Q_t} f_t u_t dxdt \quad (2.4)$$

Applying the Hölder and Young inequality we obtain that

$$p_0\left|\int\limits_{Q_t}|u_m|^{p_0-2}u_m u_{mt} Du_m Du_{mt}dxdt\right| \leq \left(p_0 c(\varepsilon_2)\|u_n\|^{p_0-2}_{L_\infty(Q_t)} + p_0 c(\varepsilon_1)\|u_m\|^{2p_0-2}_{L_\infty(Q_t)}\right)\times$$

$$\times \int\limits_{Q_t}|u_{mt}|^2 dxdt + p_0\varepsilon_2 \int\limits_{Q_t}|u_m|^{p_0}|Du_{mt}|^2 dxdt + p_0\varepsilon_1 \int\limits_{Q_t}|Du_m|^{p_1}|Du_{mt}|^2 dxdt$$

where $\varepsilon_1, \varepsilon_2$ are enough small.

Further, using the last inequality, equality (2.4) and also the condition for the function $f(x,t)$, after some simple estimations, we obtain the validity of Lemma 1. Lemma is proved.

As $\frac{\partial}{\partial t}\left(|u_m|^{p_0/2} Du_m\right) = |u_m|^{p_0/2} Du_{mt} + \frac{p_0}{2}|u_m|^{\frac{p_0}{2}-2} u_m \frac{\partial u_m}{\partial t} Du_m$, then by virtue of lemma 1 and the fact that $u_m \in P_2(Q)$ we have:

$$\frac{\partial}{\partial t}\left(|u_m|^{p_0/2} Du_m\right) \in L_2(Q).$$

Thus, $u_m \in P_3(Q)$. Consequently,

u_{mt} is bounded in $L_\infty(0,T;L_2(\Omega))$,

$|Du_m|^{p_1/2} Du_{mt}$ is bounded in $L_2(Q)$,

$\frac{\partial}{\partial t}\left(|u_m|^{p_0/2} Du_m\right)$ is bounded in $L_2(Q)$.

Consequently, there exists a subsequence of the sequence $\{u_m\}$ (which we again denote by $\{u_m\}$) that

$$u_{mt} \to u_t \quad *\text{-weak in } L_\infty(0,T;L_2(\Omega)) \text{ (see[10])}$$

$$|Du_m|^{p_1/2} Du_{mt} \to \chi \text{ in } L_2(Q)$$

$$\frac{\partial}{\partial t}\left(|u_m|^{p_0/2} Du_m\right) \to \eta \text{ in } L_2(Q).$$

Now making standard conclusions by using the known lemma (see [10], lemma 1.3) we have:

$$\frac{\partial}{\partial t}\left(|u_m|^{p_0/2} Du_m\right) \to \frac{\partial}{\partial t}\left(|u|^{p_0/2} Du\right)$$

$$|Du_m|^{p_1/2} Du_m \to |Du|^{p_1/2} Du_t.$$

Thus, the solution of problem (1.1-(1.3) is contained in $P_3(Q)$.

Theorem is proved.

§ 3. The free boundary problem.

The following problem is investigated:

$$\frac{\partial u}{\partial t} - D\left(|u|^{p_0} Du + |Du|^{p_1} Du\right) = f(x,t) \quad (x,t) \in Q(t) \tag{3.1}$$

$$u(x,t) = 0 \quad \text{for} \quad x < s_1(t) \text{ and } x > s_2(t) \tag{3.2}$$

$$u(x,0) = u_0(x) \geq c > 0 \quad \text{for} \quad x \in (s_1(0), s_2(0)) \tag{3.3}$$

where $s_i(t)$ are unknown boundaries of domain

$$Q(T) \in \{(x,t) \in R^2 | t \in (0,T), s_1(t) \leq x \leq s_2(t)\}.$$

More exactly, $u_0(x)$ is considered as narrowing on $(s_1(0),s_2(0))$ of some function $\tilde{u}_0(x)$, defined on interval $[a,b]\supset[s_1(0),s_2(0)]$ such that $D\tilde{u}_0(s_1(0),0)>0$, $D\tilde{u}_0(s_2(0),0)<0$. (It is supposed that $\tilde{u}_0(x)<c$ for $x\in[a,s_1(0)]\cup[s_2(0),b]$).

Moreover, on a free boundary it is supposed the validity of condition of Stefan condition type (almost everywhere)

$$s'_i(t)=-\left(c_1+c_2|Du|^{p_1}\right)Du(s_i(t),t)\quad i=1,2 \tag{3.4}$$

where $c_1=c^{p_0-1}$, $c_2=\frac{1}{c}$, and also of the following condition

$$u(s_i(t),t)=c\,. \tag{3.5}$$

Assume now that there exists the function $u(x,t)$ almost everywhere satisfying (3.1)-(3.5).

Then obtained solution we can consider as the solution of equation with the discontinuity coefficients. More exactly, as the solution of the following equation:

$$\frac{\partial u}{\partial t}-D\left(a_1(x,t)|u|^{p_0}Du+a_2(x,t)|Du|^{p_1}Du\right)=f \tag{3.1'}$$

$$a_i(x,t)=\begin{cases}1 & \text{for } (x,t)\in Q(T)\\ 0 & \text{for } x<s_1(T),\quad x>s_2(t)\end{cases}$$

From the general theory of quasilinear equations, in divergent case it is known that by virtue of stability of boundary value problems regarding to variation of coefficients and free members of equations, the solution of diffraction problems to which belongs problem (3.1), (3.3), (3.5) could be established as limits of «good» solutions of equations with smooth able coefficients $(a_{1m}\to a_1, a_{2m}\to a_2)$ and right-hand sides of f_m which approximates function f (see [11]). Using this fact we show the conclusion of condition (3.4) according to scheme from [11]. For the solution of equation (3.1') (i.e. equations (3.1') with smooth coefficients a_{im}) the following equations are equivalent:

$$\int_Q\frac{\partial u}{\partial t}\varphi dxdt-\int_Q D\left(a_{1m}|u|^{p_0}Du+a_{2m}|Du|^{p_1}Du\right)\varphi dxdt=\int_Q f_m\varphi dxdt \tag*{(3.1'')$_m$}$$

$$-\int_Q u\frac{\partial\varphi}{\partial t}dxdt+\int_Q\left(a_{1m}|u|^{p_0}Du+a_{2m}|Du|^{p_1}Du\right)D\varphi dxdt=\int_Q\varphi f_m dxdt \tag*{(3.1''')$_m$}$$

where φ is such that $\operatorname{supp}\varphi\in Q=[a,b]\times[0,T]$, $\varphi\in\overset{\circ}{C}{}^{\infty}(Q)$.

Then, using supposition on convergence of coefficients and right-hand sides to the corresponding coefficients and right-hand side, obtained equivalent equalities are true for equation (3.1') as well. From here it follows that

$$\int_0^T\int_{s_1(t)}^{s_2(t)}\frac{\partial u}{\partial t}\varphi dxdt-\int_0^T\int_{s_1(t)}^{s_2(t)}D\left(|u|^{p_0}Du+|Du|^{p_1}Du\right)\varphi dxdt=\int_0^T\int_{s_1(t)}^{s_2(t)}f\varphi dxdt \tag{3.1''}$$

$$-\int_0^T\int_{s_1(t)}^{s_2(t)}u\frac{\partial\varphi}{\partial t}\varphi dxdt+\int_0^T\int_{s_1(t)}^{s_2(t)}\left(|u|^{p_0}Du+|Du|^{p_1}Du\right)D\varphi dxdt=\int_0^T\int_{s_1(t)}^{s_2(t)}f\varphi dxdt \tag{3.1'''}$$

Further, transforming equation (3.1'') we obtain:

$$-\int_0^T\int_{s_1(t)}^{s_2(t)} u\frac{\partial\varphi}{\partial t}dxdt+\int_0^T\int_{s_1(t)}^{s_2(t)}\left(|u|^{p_0}Du+|Du|^{p_1}Du\right)D\varphi dxdt+$$

$$+\int_0^T\int_{s_1(t)}^{s_2(t)}\frac{\partial}{\partial t}(u\varphi)dxdt-\int_0^T\int_{s_1(t)}^{s_2(t)}D\left(\left(|u|^{p_0}Du+|Du|^{p_1}Du\right)\varphi\right)dxdt=\int_0^T\int_{s_1(t)}^{s_2(t)}f\varphi dxdt$$

The last equality is equivalent to the following:

$$-\int_0^T\int_{s_2(t)}^{s_2(t)} u\frac{\partial\varphi}{\partial t}dxdt+\int_0^T\int_{s_1(t)}^{s_2(t)}\left(|u|^{p_0}Du+|Du|^{p_1}Du\right)D\varphi dxdt-$$

$$-\int_0^T\left(|u|^{p_0}Du\varphi\big|_{s_2(t)}-|u|^{p_0}Du\varphi\big|_{s_1(t)}+|Du|^{p_1}Du\varphi\big|_{s_2(t)}-|Du|^{p_1}Du\varphi\big|_{s_1(t)}\right)dt+$$

$$+\int_0^T\frac{\partial}{\partial t}\left(\int_{s_1(t)}^{s_2(t)}\varphi udx\right)dt-\int_0^T s'_2(t)u(s_2(t),t)\varphi(s_2(t),t)dt+\int_0^T s'_1(t)u(s_1(t),t)\varphi(s_1(t),t)dt=\int_0^T\int_{s_1(t)}^{s_2(t)}f\varphi dxdt$$

Now, taking into account that $\operatorname{supp}\varphi\in Q$, by virtue of arbitrariness of φ by the help of equation ($3.1'''$) we obtain:

$$\int_0^T\left(|u|^{p_0}+|Du|^{p_1}\right)Du\varphi\big|_{s_i(t)}dt=-\int_0^T s'_i(t)u(s_i(t),t)\varphi(s_i(t),t)dt \qquad (i=1,2).$$

It is clear, that if ∂D is continuous, then almost everywhere it holds the equality:

$$s'_i(t)u(s_i(t),t)=-\left(|u|^{p_0}+|Du|^{p_1}\right)Du(s_i(t),t).$$

The last equality is nothing else as condition (3.4). So, it is remained to show the existence of enough smooth solution $u(x,t)$ of equation (1.1) such that with curves $s_i(t)$ defined from (3.4), (3.5), the triple $\{u(x,t),s_1(t),s_2(t)\}$ satisfies the conditions (3.1)-(3.5).

For this consider the following problem:

$$\frac{\partial u}{\partial t}-D\left(|u|^{p_0}Du+|Du|^{p_1}Du\right)=f(x,t)\quad (x,t)\in Q \tag{3.6}$$

$$u(x,0)=\tilde{u}_0(x)>0 \qquad x\in(a,b) \tag{3.7}$$

$$u|_\Gamma=0 \qquad \Gamma=\partial\Omega\times[0,T) \tag{3.8}$$

(as it was mentioned $u_0(x)$ is the narrowing of some function $\tilde{u}_0(x)>0$ defined on the interval $[a,b]$).

In previous paragraphs it was proved the existence of definite smooth solution of problem (3.6)-(3.8). However, for further investigations it is necessary to prove the following lemma

Lemma 2. *With conditions of theorem 3 for the solution of problem (3.6)-(3.8) the following inclusions:*

$$u\in C\left(\Omega;C^\gamma(0,T)\right)\quad \gamma<1 \tag{3.9}$$

$$Du\in C\left(0,T;C^\delta(\Omega)\right)\quad \delta<1 \tag{3.10}$$

hold.

Firstly we prove the following two statements.

Statement 1. *Let $u(x,t)\in P_3(Q)$ be the solution of problem (3.6)-(3.8). Then it is valid the following inclusion: $|Du|^{p_1}Du\in L_\infty\left(0,T;W_2^1(\Omega)\right)$.*

Proof. Rewrite equation (3.6) in the following form:

$$(p_1+1)|Du|^{p_1}D^2u+|u|^{p_0}D^2u=f-u_t-p_0|u|^{p_0-2}u|Du|^2.$$

From the results of previous paragraphs and condition on $f(x,t)$ we have:

$$\left[(p_1+1)|Du|^{p_1}+|u|^{p_0}\right]D^2u=f-u_t-p_0|u|^{p_0-2}u|Du|^2\in L_\infty(0,T;L_2(\Omega)).$$

Consequently, $|Du|^{p_1}D^2u\in L_\infty(0,T;L_2(\Omega))$.

Taking into account that $|Du|^{p_1}Du\in L_\infty(0,T;L_2(\Omega))$ (see definition $P_1(Q)$) we obtain the validity of statement 1.

Statement 2. *With conditions of lemma 2 the following inclusions are valid:*

$$u\in C\left(\Omega;C^{\frac{1}{p_0+2}}(0,T)\right).$$

Proof. As $u\in P_3(Q)$, then, in particular,

$$|u|^{p_0/2}u\in L_2\left(\Omega;W_2^1(0,T)\right)$$
$$\frac{\partial}{\partial x}\left(|u|^{p_0/2}u\right)\in L_2\left(\Omega;W_2^1(0,T)\right).$$

From here according to known lemma (see [10])

$$|u|^{p_0/2}u\in C\left(\Omega;W_2^1(0,T)\right).$$

On the base of generalization of Kondrashov's theorem (see [12])

$$C\left(\Omega;W_2^1(0,T)\right)\subset C\left(\Omega;C^{1/2}(0,T)\right).$$

I.e.

$$|u|^{p_0/2}u\in C\left(\Omega;C^{1/2}(0,T)\right).$$

By virtue of

$$|u(x,t_2)-u(x,t_1)|^{\frac{p_0}{2}+1}\le const\left(|u|^{p_0/2}u(x,t_2)-|u|^{p_0/2}u(x,t_1)\right)\le const|t_2-t_1|^{1/2}$$

we have:

$$|u(x,t_2)-u(x,t_1)|\le const|t_2-t_1|^{\frac{1}{p_0+2}}\forall t_1,t_2\subset[0,T],x\in\Omega.$$

Consequently, $u(x,t)\in C\left(\Omega;C^{\frac{1}{p_0+2}}(0,T)\right)$.

The statement is proved.

Note, that as $|Du|^{p_1}Du\in L_\infty\left(0,T;W_2^1(\Omega)\right)\cap W_2^1(0,T;L(\Omega))$ (see statement 1 and definition of $P_3(Q)$), then from Condrachov theorem it follows that $Du\in L_\infty\left(0,T;C^{\frac{1}{2(p_1+1)}}(\Omega)\right)$. Moreover, it is know the following lemma.

Lemma (see [11]). *Let $\Omega\subset R^n$ satisfy to cone condition, function $u(x,t)$ in $Q=\Omega\times(0,T)$ satisfies the Hölder condition by t with exponent α and Hölder constant μ_1 and have derivatives $D_i u$, which for any t from $[0,T]$ satisfies the Hölder condition by x with exponent β and Hölder constant μ_2. Then derivatives $D_i u$ satisfy in Q the Hölder condition by t with exponent $\widetilde{\delta}=\alpha\beta/(1+\beta)$ and Hölder constant $\widetilde{\mu}$, defined by only $\alpha,\beta,\mu_1,\mu_2,d$ and solid angle at the top of cone.*

The proof of lemma allows to say that the statement of lemma remains valid also in case, if we require for $D_t u$ to satisfy Hölder condition for almost all t with the only difference that $D_t u$ satisfies in Q Hölder condition after corresponding determination on the set of measure zero.

Taking into account all said-above by virtue of statement 2 and last lemma we obtain

$$Du \in C\left(0,T;C^{\delta}(\Omega)\right).$$

Lemma 2 is proved.

By virtue of choosing of $\tilde{u}_0(x)$, more exactly, that $D\tilde{u}_0(s_i(0),0) \neq 0$, taking into account the smoothness of function $u(x,t)$ (see lemma 2), we obtain (on the base theorem on explicit function) the existence of some locally-unique curves $s_i(t)$ coming out from points $(s_i(0),0)$ on which condition $u(x,t)=C$ holds. It is easy to see then, that narrowing on $Q(T)$ the solution of problem (3.6)-(3.8) is required the solution of (3.1)-(3.5). As $Du(s_1(0),0)>0$, $Du(s_2(0),0)<0$ we obtain that this curves by virtue of condition (3.4) are corresponding locally non-decreasing and non-increasing up to some point (x_i,t_i), where $Du=0$. (It is obvious, if it exists). i.e. the following lemma is valid.

Lemma 3. *With condition of theorem 3 there exists point $(t_i, x(t_i))$ up to which the functions $s_i(t)$ are non-decreasing and non-increasing correspondingly.*

So, let $Du(x_i,t_i)=0$. Then, generally speaking, from point (x_i,t_i) could come out ad lib many curves on which $u(x,t)=C$. By virtue of continuity of the function $u(x,t)$ these curves are continuous. We choose the most last in the passing by clock-wise direction around the point (x_1,t_1). Two variants are possible:

1) whatever point $(\tilde{x},\tilde{t})$ of curve we will choose its continuation will lay above the line $t=\tilde{t}$ (fig.1)
2) supposition 1 is violated.

1) Obtained after continuation curve we again denote by $s_1(t)$. As Du everywhere satisfies the inclusion (3.10) then it is clear that at all points of this part of curve Du is defined. Note immediately that Du on the expanded curve $s_1(t)$ also could not be less than zero by virtue of fact that from the left-hand side of $s_1(t)$ $u>C$ and $u(s_1(t),t)=C$.

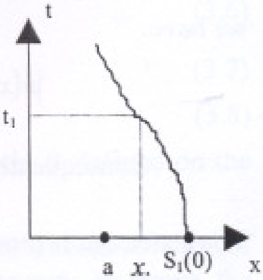

Fig.1.

According to these facts, we will prove that the curve $s_1(t)$ doesn't increase. Suppose the contrary, i.e. we suppose that at some interval $[t'',t''']$ $s_1(t)$ increases. Then by virtue of Lebesque theorem [13], $s_1(t)$ have almost everywhere a derivative. The mentioned derivative will almost everywhere more or equal to zero. From the other hand, as $Du(s_1(t),t)\geq 0$, then $s'_1(t)\leq 0$ by virtue of condition (3.4). Consequently, $s'_1(t)=0$ almost everywhere on $[t'',t''']$. It is known (see [13]), that if derivative is absolute continuous monotone function equal to zero everywhere, then this function is constant. Thus, if $s_1(t)$-absolute continuous function then mentioned theorem contradicts with supposition that $s_1(t)$ is increasing function. In other words, if we prove the absolute continuity of $s_1(t)$, then from here it will follow that there is no such interval $[t'',t''']$ where $s_1(t)$ increases.

It is known, that if for two summable functions v_1 and v_2 equality $v'_1 = v'_2$ holds in the distribution sense, then $v_1 - v_2 = const$ (almost everywhere). Let $v_1 = s_1$, and v_2 is a reconstructed by derivative $v'_2 = \left(|Du|^{p_1} Du\right)(s_1(t),t)$ function. As $|Du|^{p_1} Du$ is a continuous function, then function v_2 is absolutely continuous. We have mentioned that $v_1 - v_2 = const$ (almost everywhere). From here follows the validity of this equation everywhere, so as s_1 and v_2 are continuous functions. But v_2 is an absolutely continuous function. Consequently, $s_1 = const + v_2$ is obviously continuos function.

Thus, we obtain that $s_1(t)$ could not be an increasing function. So, $s_1(t)$ is a non-increasing function.

2) Now we will assume that at point (x_1,t_1) the supposition 1) is violated (fig.2). Then, as at the neighborhood of the point (x_1,t_1) $u > C$ from the left-hand side from expansion of the curve $s_1(t)$ from the point (x_1,t_1), we obtain: $Du(s_1(t),t) \leq 0$ on this part of the curve $s_1(t)$. From here, applying again condition (3.4) and reasons similar to described above, we have: $s_1(t)$ non-decreasing function beginning from some $t' < t_1$.

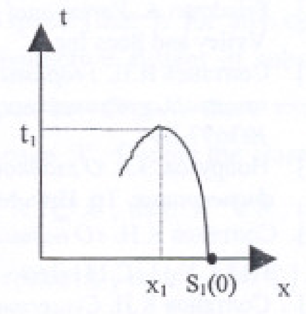

Fig. 2.

Consider the further behavior of $s_1(t)$.

It is obvious, that $s_1(t)$ can intersect the axis OX only at the point $(s_1(0),0)$, as $\tilde{u}_0 < C$ at the interval $[a, s_1(0))$. But it cannot be true as from $(s_1(0),0)$ comes out only one curve. Therefore, $s_1(t)$, have it finite or infinite number of «peak points» of the type of point (x_1,t_1) in case 2) (fig.2), will decrease.

The same reasons applied to point $(s_1(0),0)$ could be useful for the proof of the fact that from point $(s_2(0),0)$ came out some continuous curve, which have almost everywhere derivative. But unlike the curve $s_1(t)$ the curve came out from point $(s_2(0),0)$ doesn't decrease at any neighborhood of mentioned point. Further, curve $s_2(t)$ distant from curve $s_1(t)$ (fig.3) making, possibly, «peak» points.

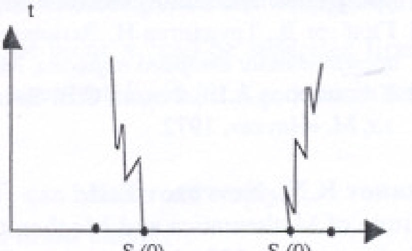

Fig.3

Thus, the following Lemma is true.

Lemma 4. *Curve $s_1(t)(s_2(t))$ is global non-increasing (non-decreasing). However, $s_1(t)(s_2(t))$ locally may have the parts of increasing (decreasing) non-intersected with line $t=0$.*

From described reasons it follows the validity of the following theorem.

Theorem 4. *Let* $p_2 \geq 2$, $p_1 \geq 2$. *Then for any functions* $f \in \overset{\circ}{W}{}^1_l(Q(T))$ *and* $\widetilde{u}_0(x) \in \overset{\circ}{W}{}^1_r(\Omega)$ *for* $r = \max\{p_0 + p_1 + 2, 2p_0 + p_* + 2, k + 2\}$, $l = \max\{4, p_* + 2, k + 2\}$ *the problem (3.1)-(3.5) is solvable in* $P_3(Q) \times W_2^1(0,T) \times W_2^1(0,T)$ *(i.e.* $s_i(t) \in W_2^1(0,T)$, $u(x,t) \in P_3(Q(T))$*).*

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Soltanov K.N., Novruzov E.B.
Institute of Mathematics and Mechanics of AS Azerbaijan.
9, F. Agayev str., 370141, Baku, Azerbaijan.
Tel.: 39-47-20.

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