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CARDINALITY HOMOGENEOUS TOPOLOGICAL SPACES WITH SPECIAL PROPERTIES

Abstract

In this paper cardinality homogeneous topological spaces with special properties are considered.

Let X be a cardinality homogeneous topological space (that is for any open subspace $\Omega \subseteq X$ we have $|\Omega| = |X|$) containing such a disjunctive system of subsets $\{X_t\}_{t\in T}$, that $|X_t| = |X|$ for each $t \in T$ and for every open set $\Omega \subseteq X$ there exists $X_{t(\Omega)} \subseteq \Omega$. Call this system an open containing system of space X. Denote the class of all such spaces by P. It is clear that if $X \in P$ and $\bigcup_t X_t \subseteq X' \subseteq X$, then $X' \in P$, too. It's obvious, if $\{X_t\}_{t\in T}$ is an open containing system of the space X, then the set $\bigcup_t X_t$ is dense in X. It is clear, any system $\{X'_t\}_{t\in T}$ such that $X'_t \subseteq X_t$ and $|X'_t| = |X|$ for each $t \in T$ will be also an open containing system of the space X.

Suggestion 1. *Let (X,r) be a space of the class P. If $r' < r$, then $(X,r') \in P$, too.*

Suggestion 2. *Let X be a non-one-point T_0- space of the class P, $\{X_t\}_{t\in T}$ is its open containing system. Then $Jnt\, X_t = \varnothing$ for every $t \in T$.*

Proof. Suppose, $JntX_{t_0} \neq \varnothing$ for some $t_0 \in T$. Let us take two different points x_1, $x_2 \in Jnt\, X_{t_0}$. So as X is a T_0- space, then one of these points has a neighbourhood not containing the other point. Let, for example, the point x_1 can be separated from the point x_2 by some neighbourhood V_{x_1}. Then the open set $V_{x_1} \bigcap Jnt\, X_{t_0}$ doesn't contain any set of the system $\{X_t\}_{t\in T}$.

The following example shows, that $Jnt\, \overline{X}_t$ can be also not empty.

Suggestion 3. *The Zarissky space belongs to the class P.*

Proof. Let X be an infinite set of m cardinality with respect to the Zarissky topology. So as $m \cdot m = m$, then there exists a disjunctive system of subsets $\{X_t\}_{t\in T}$, where $|T| = m$ such that $\bigcup_t X_t = X$ and $|X_t| = m$ for each $t \in T$. Let us take an open subset Ω in X. So as the set $X \setminus \Omega$ is finite, then there exists X_{t_0}, where $t_0 \in T$, such that $X_{t_0} \subset \Omega$. So, the system $\{X_t\}_{t\in T}$ is an open containing system of the space X. It is obvious, that $\overline{X}_t = X$ for each $t \in T$.

Suggestion 4. *Let Ω be an open subspace of a space X of the class P. Then $\Omega \in P$.*

Suggestion 5. *Let* $X \in P$ *and* $\widetilde{X}$ *be an extension of the space* X *such that* $|\widetilde{X}| = |X|$. *Then* $\widetilde{X} \in P$.

From suggestions 4 and 5 it follows the following

Suggestion 6. *Let* Ω *be an open subspace of a space* X *of the class* P *and* $\Omega \subseteq \Omega' \subseteq \overline{\Omega}$. *Then* $\Omega' \in P$.

Suggestion 7. *The Tikhonov product of any system of spaces of the class* P *is a space of the class* P.

Suggestion 8. *Let* $X \in P$, $\{X_t\}_{t \in T}$ *be an open containing system of the space* X *and* f *be a one-to-one mapping* X *onto* Y *such that for any open in* Y *set* G *there exists* $X_{t(G)} \in \{X_t\}_{t \in T}$ *provided that* $fX_{t(G)} \subseteq G$. *Then* $Y \in P$.

Proof. It is easy to note that the system $\{fX_t\}_{t \in T}$ will be an open containing system of the space Y.

Corollary 1. *Let* $X \in P$ *and* f *be a one-to-one continuous mapping* X *onto* Y. *Then* $Y \in P$.

Corollary 2. *Let* $Y \in P$, $\{Y_t\}_{t \in T}$ *be an open containing system of the space* Y, *and* f *be a one-to-one mapping* X *onto* Y, *such that for any open in* X *subset* Ω *there exists* $Y_{t(\Omega)} \in \{Y_t\}_{t \in T}$, *provided that* $Y_{t(\Omega)} \subseteq f\Omega$. *Then* $X \in P$.

Proof. It is clear, that mapping f^{-1} satisfies the conditions of suggestion 8.

Corollary 3. *Let* $Y \in P$, *and* f *be a one-to-one open mapping* X *onto* Y. *Then* $X \in P$.

Theorem 1. *Let* X *be a topological space with a countable* π*- base whose every element contains equipotent to* X *a nowhere dense set. Then* $X \in P$.

Proof. Denote this π - base by $\{\mathcal{B}_n\}_{n=1}^{\infty}$. In $\mathcal{B}_1$ there exists a nowhere dense set equipotent to X and we call it A_1. Let us take the second element $\mathcal{B}_2$ of our π -base. So as A_1 is nowhere dense, then in $\mathcal{B}_2$ there exists an open set $\mathcal{B}_2'$ such that $\mathcal{B}_2' \cap A_1 = \varnothing$. By the condition of the theorem in $\mathcal{B}_2'$ there exists a nowhere dense set, which is equipotent to X and we call it A_2. It is clear, that $A_1 \cap A_2 = \varnothing$ and $A_1 \subset \mathcal{B}_1$, $A_2 \subset \mathcal{B}_2$. Let us assume now that for k elements $\{\mathcal{B}_n\}_{n=1}^{k}$ of our π - base there exists nowhere dense sets $\{A_n\}_{n=1}^{k}$ which are equipotent to X and such that $A_n \subset \mathcal{B}_n$ for $n = 1,\dots,k$ and $A_{n_1} \cap A_{n_2} = \varnothing$, when $n_1 \neq n_2$, $1 \le n_1$, $n_2 \le k$. Let us take $k+1$-th element $\mathcal{B}_{k+1}$ of our π - base. So as the set $\bigcup_{n=1}^{k} A_n$ is nowhere dense, then there exists an open set $\mathcal{B}_{k+1}' \subset \mathcal{B}_{k+1}$, such that $\mathcal{B}_{k+1}' \cap \left(\bigcup_{n=1}^{k} A_n \right) = \varnothing$. By the condition of the theorem there exists a nowhere dense set A_{k+1}, which is equipotent to X, such that $A_{k+1} \subset \mathcal{B}_{k+1}'$. It is clear, that $A_n \subset \mathcal{B}_n$, $|A_n| = |X|$ for $n = 1,\dots,k+1$ and $A_{n_1} \cap A_{n_2} = \varnothing$, when $n_1 \neq n_2$, $1 \le n_1$, $n_2 \le k+1$. Therefore, the system $\{A_n\}_{n=1}^{\infty}$ will be an open containing system of the space X.

Theorem 2. *The cube I^r for $r \geq 1$, and also the cubes D^r and F^r for $r \geq \aleph_0$ belong to the class P.*

Proof. Let Ω be an open subset of the segment $I = [0,1]$. The segment $[0,1]$ contains the Cantor perfect set. In analogy with that in Ω we can find a nowhere dense set which is equipotent to the segment $[0,1]$. From Theorem 1 it follows, that the segment $[0,1]$ is a space of the class P. Now we apply suggestion 7. Further, the Cantor discontinuum is the product of countable number of two point discrete spaces D_n, where $n = 1,2,\ldots$. Let $\{n_k\}_{k=1}^{\infty}$ be an infinite sequence of the set of positive integers $\mathcal{N}$, such that the set $\mathcal{N} \setminus \{n_k\}_{k=1}^{\infty}$ is infinite. In the Cantor discontinuum $D^{\aleph_0} = \prod_n D_n$ one can find a subset which is the Tikhonov product of two point discrete spaces D_n in n_k-th places and one-element sets in other places. Let us denote it by $\left\{A_{\{n_k\}}\right\}$. It is clear, that $\left|A_{\{n_k\}}\right| = c$ and this set is nowhere dense in $D^{\aleph_0}$.

It is easy to see that for each open set $\Omega \subseteq D^{\aleph_0}$ a nowhere dense set $A_{\{n_k\}}(\Omega)$ can be constructed by above done method, provided that $A_{\{n_k\}}(\Omega) \subset \Omega$ and $\left|A_{\{n_k\}}(\Omega)\right| = c$. So, $D^{\aleph_0} \in P$. From suggestion 7 it follows, that $D^r \in P$ for $r \geq \aleph_0$. So as there exists a one-to-one continuous mapping of D^r onto F^r using Corollary 1 we obtain $F^r \in P$ for $r \geq \aleph_0$.

Theorem 3. *The Baire space $B(r)$, where $r \geq \aleph_0$ belongs to the class P.*

Proof. Let A be some set of cardinality $r \geq \aleph_0$. It is clear, that power of all finite sequences of the set A is equal to r. Moreover, the set A is easy represented in the form of a disjunctive union of its subsets $\{A_i\}_{i \in I}$, such that $|A_i| = r$ and $|I| = r$. Denote by φ some biaktion between the set of all finite sequences of the set A and the family of subsets $\{A_i\}_{i \in I}$. Now let $(a_1,\ldots,a_n)$ be some finite sequence of elements of the set A and $\varphi(a_1,\ldots,a_n) = A_{i(a_1,\ldots,a_n)}$. Let us complement each countable sequence from the set $A_{i(a_1,\ldots,a_n)}$ by the first terms $a_1,\ldots,a_n$. We obtain the set of countable sequences, which is equipotent to the Baire space $B(r)$. Denote it by $B_{i(a_1,\ldots,a_n)}$. It is clear, that $B_{i(a_1,\ldots,a_n)}$ will be contained in the spherical neighbourhood of the point $(a_1,\ldots,a_n,\ldots)$ with radius $\frac{1}{n}$. Therefore, the system $\{B_{i(a_1,\ldots,a_n)}\}$, where $(a_1,\ldots,a_n)$ are all finite sequences from A, is an open containing system of the space $B(r)$ and it means that $B(r) \in P$.

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