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ON RIEMANN-EARNSHAW INVARIANTS AND CHARACTERISTIC DIRECTONS IN HYPERBOLIC VARIATIONAL MODELS

Abstract

The author has considered the problem on the decomposability of the two-forms coming into a system of equations which describe two-dimensional extremal manifolds and investigated the problem on the exactness of the one-forms which are multipliers of this expansion. The solution of the first problem leads to the classification of variational models (hyperbolic, elliptic, degenerated) and helps to find characteristic directions (characteristic velocities) in the configuration space for hyperbolic models. The solution of the second problem helps us to find for each characteristic direction a family of Riemann-Earnshaw invariants having a functional degree of freedom. By this way, in the applications, the author has received families of Riemann-Earnshaw invariants and characteristic velocities for models of dynamics of an elastic pivot and for gas-dynamic models describing isothermal flows of ideal gas and adiabatic flows of polytropic gas. The characteristic velocities and Riemann-Earnshaw invariants are received for equations both in Lagrange and Euler coordinates.

For the descriptions of two-dimensional extremal manifolds in paper [1] were obtained systems of variational equations in the differentials of states space coordinates which have following form:

$$\begin{cases}\tilde{\sigma}_1 = 0\,,\\ \tilde{\sigma}_2 = 0\,,\end{cases} \qquad (*)$$

where $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are fields of linear independent two-forms in state space of two-dimensional manifolds. Such method of description is the most natural and has simple geometrical interpretation. The system of equations (*) single out at each point of state space the subspace of admissible bivectors (which describe infinitesimal elements of two-dimensional manifolds), i.e. those bivectors, which can represent the infinitesimal elements of integral (extremal) manifolds, passing through this point of state space. It is clear, that this subspace of available bivectors can be determined also by any pair of linear independent two-forms from subspace $\{\lambda_1\tilde{\sigma}_1 + \lambda_2\tilde{\sigma}_2\}$, where λ_1 and λ_2 are any real numeric functions of state space coordinates. If chosen pair of two-forms from $\{\lambda_1\tilde{\sigma}_1 + \lambda_2\tilde{\sigma}_2\}$ has special properties, then it promote to answer to some questions relying upon integral manifolds.

Now consecutively consider the following questions:

1) If forms $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are not simple, then under which conditions on coefficients of these forms in the space $\{\lambda_1\tilde{\sigma}_1 + \lambda_2\tilde{\sigma}_2\}$ there exist simple two-forms?
2) What are conditions of existence of exact one-form among multipliers of simple two-forms?

By resolving these questions there arises a problem of the constructive description of:

1) The whole set of fields of simple two-forms from the subspace $\{\lambda_1\tilde{\sigma}_1 + \lambda_2\tilde{\sigma}_2\}$;

2) the set of exact one-forms, which are multipliers of simple two-forms, and a set of potentials of these exact one-forms.

Below we shall give the answer to the first of above-mentioned questions for the system of variational equations in general case. However, the constructive description of a set of simple two-form fields and the answer to the second question in general case are not simple tasks. Therefore for these questions separate paper should be written. That is why here we shall consider a class of models for which it is not difficult to obtain objects that interest us and at the same time covering a number of models, which are used in dynamics of continuum.

Consider the system of variational equations in differentials of states space coordinates (see [1])

$$\begin{cases} dP_{23}\wedge dx^2 + dP_{13}\wedge dx^1 + \dfrac{\partial P_{12}}{\partial x^3}dx^1\wedge dx^2 = 0 \\ d\left(\dfrac{\partial P_{12}}{\partial P_{13}}\right)\wedge dx^2 - d\left(\dfrac{\partial P_{12}}{\partial P_{23}}\right)\wedge dx^1 = 0 \end{cases}, \tag{1}$$

where $P_{12} = P_{12}\left(P_{13}, P_{23}; x^1, x^2, x^3\right)$, and function $\dfrac{\partial P_{12}}{\partial x^3}$ is independent explicitly on x^3.

For simplification we introduce the following denotations $x^1 = y^1$, $x^2 = y^2$, $P_{13} = y^3$, $P_{23} = y^4$, $\dfrac{\partial^2 P_{12}}{\partial P_{13}^2} = A$, $\dfrac{\partial^2 P_{12}}{\partial P_{23}^2} = B$, $\dfrac{\partial^2 P_{12}}{\partial P_{13}\partial P_{23}} = C$, $\dfrac{\partial^2 P_{12}}{\partial P_{13}\partial x^1} = L$, $\dfrac{\partial^2 P_{12}}{\partial P_{23}\partial x^2} = M$, $\dfrac{\partial^2 P_{12}}{\partial x^3} = N$. With these denotations the system (1) is represented in the following form:

$$\begin{cases} \widetilde{\sigma}_1 \equiv dy^4\wedge dy^2 + dy^3\wedge dy^1 + N dy^1\wedge dy^2 = 0 \\ \widetilde{\sigma}_2 \equiv A dy^3\wedge dy^2 + C dy^4\wedge dy^2 - C dy^3\wedge dy^1 - \\ \qquad - B dy^4\wedge dy^1 - (L+M) dy^2\wedge dy^1 = 0 \end{cases} \tag{2}$$

Let us use É. Cartan criterion [2, p. 19] to find conditions on coefficients of forms $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ under which in the subspace $\{\lambda_1\widetilde{\sigma}_1 + \lambda_2\widetilde{\sigma}_2\}$ there are simple two-forms. According to this criterion for the simplicity of two-forms

$$\widetilde{a} \equiv \sum_{i<j} a_{ij}\, dy^i \wedge dy^j, \qquad (i,j = 1,2,3,4)$$

it is necessary and sufficient that the equality

$$a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0 \tag{3}$$

holds. Substituting coefficients of two-form $\lambda_1\widetilde{\sigma}_1 + \lambda_2\widetilde{\sigma}_2$ into equality (3), we obtain

$$\left(C^2 - AB\right)\lambda_2^2 - \lambda_1^2 = 0. \tag{4}$$

From the equality (4) it follows that: 1) in subspace $\{\lambda_1\widetilde{\sigma}_1 + \lambda_2\widetilde{\sigma}_2\}$ there exist simple two-forms if and only if $C^2 - AB \ge 0$; 2) two-form $\widetilde{\sigma}_1$ in the system of equation (2) is not simple; 3) two-form $\widetilde{\sigma}_2$ in the system of equations (2) will be simple if and only if $C^2 - AB = 0$ and in this case all simple two-forms from $\{\lambda_1\widetilde{\sigma}_1 + \lambda_2\widetilde{\sigma}_2\}$ will have form $\lambda_2\widetilde{\sigma}_2$; 4) when $C^2 - AB > 0$ there exist (accurate to arbitrariness, different from zero, numerical multipliers) two linear independent simple two-forms generated basis in

$\{\lambda_1\widetilde{\sigma}_1 + \lambda_2\widetilde{\sigma}_2\}$, and these simple two-forms can be obtained for the following two values of ratio λ_1 and λ_2

$$\frac{\lambda_1}{\lambda_2} = \pm\sqrt{C^2 - AB}\,.$$

Definition. *Variation models given by function* $P_{12} = P_{12}\left(P_{13}, P_{23}; x^1, x^2, x^3\right)$, *satisfying the condition* $C^2 - AB > 0$ $\left(C^2 - AB < 0\right)$ *we shall call hyperbolic (elliptic) models.*

The models with unfolding determinant surfaces $\left(C^2 - AB \equiv 0\right)$ apparently should be named degenerated, so as in mechanical interpretation they describe either dynamic of absolutely rigid one-dimensional continuum, or statical deformed state of one-dimensional continuum, all points of which have one and the same constant velocity.

For the finding simple two-forms from subspace $\{\lambda_1\sigma_1 + \lambda_2\sigma_2\}$ we will use the following identity

$$\begin{aligned}
&\widetilde{a} \wedge \widetilde{b} \overset{def}{=} \left(a_1dy^1 + dy^2 + a_3dy^3 + a_4dy^4\right) \wedge \left(b_1dy^1 + b_2dy^2 + dy^3 + b_4dy^4\right) \equiv \\
&\equiv \lambda_1\left(dy^4 \wedge dy^2 + dy^3 \wedge dy^1 + Ndy^1 \wedge dy^2\right) + \lambda_2\left(Ady^3 \wedge dy^2 + Cdy^4 \wedge dy^2 - \right.\\
&\left. - Cdy^3 \wedge dy^1 - Bdy^4 \wedge dy^1 - (L+M)dy^2 \wedge dy^1\right) + \left(a_1b_2 - b_1 - \lambda_1N - \lambda_2(L+M)\right)\times \\
&\times dy^1 \wedge dy^2 + \left(a_1 - a_3b_1 + \lambda_1 - C\lambda_2\right)dy^1 \wedge dy^3 + \left(a_1b_4 - a_4b_1 - B\lambda_2\right)dy^1 \wedge dy^4 + \\
&= \left(1 - a_3b_2 + A\lambda_2\right)dy^2 \wedge dy^3 + \left(b_4 - a_4b_2 + \lambda_1 + C\lambda_2\right)dy^2 \wedge dy^4 + \left(a_3b_4 - a_4\right)dy^3 \wedge dy^4,
\end{aligned} \tag{5}$$

where coefficients of one-forms $\widetilde{a}$ and $\widetilde{b}$ should satisfy the condition of linear independence of these forms.

Remark 1. As in considered question we are interested in the form $\widetilde{a} \wedge \widetilde{b}$ with accuracy up to arbitrary numeric multiplier, so at each form $\widetilde{a}$ and $\widetilde{b}$ one of coefficients chosen equal to unit.

Identity (5) gives the conditions for coefficients of forms $\widetilde{a}$ and $\widetilde{b}$ and λ_1 and λ_2. Excluding from these conditions λ_1 and λ_2, we obtain conditions for coefficients of forms $\widetilde{a}$ and $\widetilde{b}$

$$\begin{cases}
a_3b_4 - a_4 = 0 \\
a_1b_2 - b_1 + \dfrac{1}{2}N\left(a_1 - a_3b_1 + b_4 - a_4b_2\right) - (L+M)\dfrac{1}{B}\left(a_1b_4 - a_4b_1\right) = 0 \\
A\left(a_1b_4 - a_4b_1\right) + B\left(1 - a_3b_2\right) = 0 \\
B\left(\left(a_1 - a_3b_1\right) - \left(b_4 - a_4b_2\right)\right) - 2C\left(a_1b_4 - a_4b_1\right) = 0
\end{cases} \tag{6}$$

So, for determination the coefficients of forms $\widetilde{a}$ and $\widetilde{b}$ we obtain system of four algebraic equations of the second order with six unknowns. Further, in present paper we will consider the models for which conditions $L = M = N \equiv 0$ holds. In this case, substituting expressions b_1 and a_4, obtained from the first two equations of system (6), into third and fourth, and taking into account condition $\left(1 - a_3b_2\right) \neq 0$ (in opposite case, taking first two equations of system (6) we obtain a linear dependence of forms $\widetilde{a}$ and $\widetilde{b}$), we get a simple system of equations for a_1 and b_4

$$\begin{cases} a_1 - b_4 = -\dfrac{2C}{A} \\ a_1 b_4 = -\dfrac{B}{A} \end{cases} \tag{7}$$

which have two real solutions for $C^2 - AB > 0$

$$a_1^{\pm} = \frac{-C \pm \sqrt{C^2 - AB}}{A}, \quad b_4^{\pm} = \frac{C \pm \sqrt{C^2 - AB}}{A}.$$

Returning to the first two equations of system (6), we find expressions for $\widetilde{a}$ and $\widetilde{b}$

$$\widetilde{a}_{\pm} = a_1^{\pm} dy^1 + dy^2 + a_3 dy^3 + b_4^{\pm} a_3 dy^4, \quad \widetilde{b}_{\pm} = a_1^{\pm} b_2 dy^1 + b_2 dy^2 + dy^3 + b_4^{\pm} dy^4,$$

where b_2 and a_3 are arbitrary functions of coordinates (y^1, y^2, y^3, y^4), receiving any values not from parabola $a_3 b_2 = 1$ (parabola of linear dependence of forms $\widetilde{a}$ and $\widetilde{b}$). In present paper we will use only that class of forms for which $a_3 = b_2 \equiv 0$.

The constructed two-forms $\hat{a}_+ \wedge \widetilde{b}_+$ and $\hat{a}_- \wedge \widetilde{b}_-$ are linear independent, so as

$$\left(\hat{a}_+ \wedge \widetilde{b}_+\right) \wedge \left(\hat{a}_- \wedge \widetilde{b}_-\right) = (1 - a_3 b_2)^2 \left(b_4^+ - b_4^-\right)\left(a_1^+ - a_1^-\right) dy^1 \wedge dy^2 \wedge dy^3 \wedge dy^4 =$$

$$= 4(1 - a_3 b_2)^2 \frac{C^2 - AB}{A^2} dy^1 \wedge dy^2 \wedge dy^3 \wedge dy^4 \neq 0$$

The objects known as a Riemann-Earnshaw invariants appear by resolving question on the proportionality of one-form $\widetilde{b}$ to exact one-form and by finding a potential of this form. For this, in general case, it is necessary to verify the equality $\widetilde{b} \wedge d\widetilde{b} = 0$, which put conditions on coefficients of forms $\widetilde{b}$. However, in considered case $b_2 \equiv 0$, this equality identically holds and decision of question reduces to finding an integrant multiplier. The integrant multiplier $\overline{\lambda}_{\pm}(y^3, y^4)$ must be a solution of equation

$$\frac{\partial}{\partial y^4} \overline{\lambda}_{\pm} = \frac{\partial}{\partial y^3}\left(b_4^{\pm} \overline{\lambda}_{\pm}\right). \tag{8}$$

If the function $b_4^{\pm}$ depends only on the argument y^3, then it is convenient to make substitution in equation (8) $b_4^{\pm} \overline{\lambda}_{\pm} = \lambda_{\pm}$ and then we obtain equation for the function $\lambda(y^3, y^4)$

$$b_4^{\pm} \frac{\partial}{\partial y^3} \lambda_{\pm} = \frac{\partial}{\partial y^4} \lambda_{\pm}, \tag{9}$$

which has the solution $\lambda_{\pm}(y^3, y^4) = \varphi_{\pm}\left(y^4 + m^{\pm}(y^3)\right)$, where $m^{\pm}(y^3)$ is the antiderivative of function $\left(b_4^{\pm}(y^3)\right)^{-1}$, and $\varphi_{\pm}(\cdot)$ are arbitrary differentiable functions.

Functions $R^{\pm}(y^3, y^4)$ determined by the equation

$$dR^{\pm}(y^3, y^4) = \overline{\lambda}_{\pm} \widetilde{b}_{\pm} \tag{10}$$

are called Riemann- Earnshaw invariants in continuum dynamics, and functions $a_1^{\pm}$, which give characteristic directions of system of equations (1), are called characteristic

$$\begin{cases} a_1 - b_4 = -\dfrac{2C}{A} \\ a_1 b_4 = -\dfrac{B}{A} \end{cases} \tag{7}$$

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$$a_1^{\pm} = \frac{-C \pm \sqrt{C^2 - AB}}{A}, \quad b_4^{\pm} = \frac{C \pm \sqrt{C^2 - AB}}{A}.$$

Returning to the first two equations of system (6), we find expressions for $\widetilde{a}$ and $\widetilde{b}$

$$\widetilde{a}_{\pm} = a_1^{\pm} dy^1 + dy^2 + a_3 dy^3 + b_4^{\pm} a_3 dy^4, \quad \widetilde{b}_{\pm} = a_1^{\pm} b_2 dy^1 + b_2 dy^2 + dy^3 + b_4^{\pm} dy^4,$$

where b_2 and a_3 are arbitrary functions of coordinates $\left(y^1, y^2, y^3, y^4\right)$, receiving any values not from parabola $a_3 b_2 = 1$ (parabola of linear dependence of forms $\widetilde{a}$ and $\widetilde{b}$). In present paper we will use only that class of forms for which $a_3 = b_2 \equiv 0$.

The constructed two-forms $\hat{a}_+ \wedge \widetilde{b}_+$ and $\hat{a}_- \wedge \widetilde{b}_-$ are linear independent, so as

$$\left(\hat{a}_+ \wedge \widetilde{b}_+\right) \wedge \left(\hat{a}_- \wedge \widetilde{b}_-\right) = \left(1 - a_3 b_2\right)^2 \left(b_4^+ - b_4^-\right)\left(a_1^+ - a_1^-\right) dy^1 \wedge dy^2 \wedge dy^3 \wedge dy^4 =$$

$$= 4\left(1 - a_3 b_2\right)^2 \frac{C^2 - AB}{A^2} dy^1 \wedge dy^2 \wedge dy^3 \wedge dy^4 \neq 0$$

The objects known as a Riemann-Earnshaw invariants appear by resolving question on the proportionality of one-form $\widetilde{b}$ to exact one-form and by finding a potential of this form. For this, in general case, it is necessary to verify the equality $\widetilde{b} \wedge d\widetilde{b} = 0$, which put conditions on coefficients of forms $\widetilde{b}$. However, in considered case $b_2 \equiv 0$, this equality identically holds and decision of question reduces to finding an integrant multiplier. The integrant multiplier $\overline{\lambda}_{\pm}\left(y^3, y^4\right)$ must be a solution of equation

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If the function $b_4^{\pm}$ depends only on the argument y^3, then it is convenient to make substitution in equation (8) $b_4^{\pm} \overline{\lambda}_{\pm} = \lambda_{\pm}$ and then we obtain equation for the function $\lambda\left(y^3, y^4\right)$

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Functions $R^{\pm}\left(y^3, y^4\right)$ determined by the equation

$$dR^{\pm}\left(y^3, y^4\right) = \overline{\lambda}_{\pm} \widetilde{b}_{\pm} \tag{10}$$

are called Riemann- Earnshaw invariants in continuum dynamics, and functions $a_1^{\pm}$, which give characteristic directions of system of equations (1), are called characteristic velocities. In the case when the function $b_4^{\pm}$ depends only on the argument y^3, it is easy to find functions $R^{\pm}(y^3, y^4)$ from equation (10)

$$R^{\pm}(y^3, y^4) = \Phi_{\pm}(y^4 + m^{\pm}(y^3)),$$

where $\Phi_{\pm}(\cdot)$ are arbitrary twice differential functions. And so, in considered class of hyperbolic systems of equations (1) for each of two characteristic directions, given by functions $a_1^{\pm}$, there exists family of Reimann-Earnshaw invariants having functional power of freedom. In applications, as far as the author knows, there use only one function $\Phi(\eta) \equiv \eta$, through the system of equations

$$\begin{cases} (a_1^+ dy^1 + dy^2) \wedge dR^+(y^3, y^4) = 0 \\ (a_1^- dy^1 + dy^2) \wedge dR^-(y^3, y^4) = 0 \end{cases} \tag{11}$$

is equivalent to system (2) for any invariants $R^{\pm}(y^3, y^4)$ from mentioned- above families, if $\overline{\lambda}_{\pm}(y^3, y^4) \neq 0$.

The Jacobian of the system of functions

$$\begin{cases} R^+ = R^+(y^3, y^4) \\ R^- = R^-(y^3, y^4) \end{cases}$$

is equal to $\overline{\lambda}_+ \overline{\lambda}_-(b_4^- - b_4^+)$, therefore for those $R^{\pm}$, generated by $\lambda_{\pm} \neq 0$, it is possible to express (at least locally) y^3 and y^4 by R^+ and R^-.

Consider the model of dynamics of longitudinal displacements of an elastic pivot. The equation of determinant surface of this model $P_{12} = P_{12}(P_{13}, P_{23})$, after introduction of denotations $P_{12} = h$, $P_{13} = \sigma$, $P_{23} = q$ has the form

$$h = \frac{1}{2\rho_0} q^2 - \sigma_0 f\left(\frac{\sigma}{\sigma_0}\right), \quad \left(\sigma_0 f''\left(\frac{\sigma}{\sigma_0}\right) > 0\right),$$

and system of equations (1) for $x^1 = t$, $x^2 = \xi$, will be rewritten in the form:

$$\begin{cases} dq \wedge d\xi + d\sigma \wedge dt = 0 \\ d(h_\sigma) \wedge d\xi - d(h_q) \wedge dt = 0 \end{cases}$$

The physical sense of introduced symbols and the method of receiving of equations are shown in [1]. For this model we have

$$A = -\frac{1}{\sigma_0} f''\left(\frac{\sigma}{\sigma_0}\right), \quad B = \frac{1}{\rho_0}, \quad C = 0,$$

therefore from (7) we obtain

$$a_1 = b_4, \quad a_1^{\pm} = \pm\left(\frac{\rho_0}{\sigma_0} f''\left(\frac{\sigma}{\sigma_0}\right)\right)^{-1/2}$$

and consequently, the solution of equation (9) is

$$\lambda_{\pm}(q, \sigma) = \varphi_{\pm}\left(\frac{1}{(\rho_0 \sigma_0)^{1/2}} q \pm \int^{\eta} (f''(z))^{1/2} dz\right)$$

where $\eta=\dfrac{\sigma}{\sigma_0}$, and $\varphi_\pm(\cdot)$ are arbitrary differentiable functions. Riemann-Earnshaw invariants are determined from the system of equations

$$\begin{cases} R_\sigma^\pm=\pm\left(\dfrac{\rho_0}{\sigma_0}\right)^{1/2}\left(f''\left(\dfrac{\sigma}{\sigma_0}\right)\right)^{1/2}\varphi_\pm\left(\dfrac{1}{(\rho_0\sigma_0)^{1/2}}q\pm\int^{\eta}(f''(z))^{1/2}dz\right) \\ R_q^\pm=\varphi_\pm\left(\dfrac{1}{(\rho_0\sigma_0)^{1/2}}q\pm\int^{\eta}(f''(z))^{1/2}dz\right) \end{cases}$$

which have a solution

$$R^\pm(q,\sigma)=(\rho_0\sigma_0)^{1/2}\Phi_\pm\left(\frac{1}{(\rho_0\sigma_0)^{1/2}}q\pm\int^{\eta}(f''(z))^{1/2}dz\right),$$

where $\Phi_\pm(\cdot)$ are arbitrary twice differentiable functions. For example, for $f(\eta)=\exp\eta$, we obtain a model with state equations $q=\rho_0\mathrm{v}$, $\sigma=\sigma_0\ln\dfrac{dx}{d\xi}$, where v is velocity of pivot's element in viewer's space and X is a coordinate of viewer's space. The second equation could be written in the form $\sigma=\sigma_0\ln(1+u_\xi)$, when we chose those coordinates of viewers space, which has elements of pivot in non-deformed state, as Lagrangian coordinates, where u_ξ describes relative prolongation of elements of pivot (deformation tensor). The characteristic velocities and Riemann-Earnshaw invariants are

$$a_1^\pm=\pm\left(\frac{\sigma_0}{\rho_0}\right)^{1/2}\exp\left(-\frac{1}{2}\frac{\sigma}{\sigma_0}\right),$$

$$R^\pm=(\rho_0\sigma_0)^{1/2}\Phi_\pm\left(\frac{1}{(\rho_0\sigma_0)^{1/2}}q\pm2\exp\left(\frac{1}{2}\frac{\sigma}{\sigma_0}\right)\right).$$

Consider gas-dynamic models, more exactly, the model of isothermic flows of ideal gas and the model of adiabatic flows of polytropic gases. The equation of determinant surfaces of these models are (see [1])

$$h=\frac{1}{2\rho_0}q^2+\frac{\rho_0}{m}kT\ln\left(\frac{P}{P_0}\right)+h_0,$$

$$h=\frac{1}{2\rho_0}q^2+\frac{\gamma}{\gamma-1}BP_0\left(\frac{P}{P_0}\right)^{\frac{\gamma-1}{\gamma}}+h_0,\quad(\gamma>1)$$

correspondingly. The system of equations (1) after introduction of denotations $P_{12}=h$, $P_{13}=-P$, $P_{23}=q$, $x^1=t$, $x^2=\xi$, will have a form

$$\begin{cases}\tilde{\tau}_1\equiv dq\wedge d\xi-dP\wedge dt=0\\ \tilde{\tau}_2\equiv d(h_p)\wedge d\xi-d(h_q)\wedge dt=0\end{cases}\tag{12}$$

where the arrangement of signs (plus, minus) is different from the arranggement of signs in the system of equation (1). Applying the É.Cartan criterion to forms in the subspace $\{\lambda_1\tilde{\tau}_1+\lambda_2\tilde{\tau}_2\}$ we again obtain equality (4), where $A=h_{pp}$, $B=h_{qq}$, $C=h_{pq}$. Making the same actions for the system of equations (12) as it was done for the system (1), we obtain for one-forms $\tilde{a}$ and $\tilde{b}$ the following expressions:

$$\tilde{a}_\pm = a_1^\pm dt + d\xi + a_3 dp + b_4^\pm a_3 dq\,, \qquad \tilde{b}_\pm = a_1^\pm b_2 dt + b_2 d\xi + dp + b_4^\pm dq\,,$$

where functions $a_1^\pm(p,q)$ and $b_4^\pm(p,q)$ are the solutions of system of equations

$$\begin{cases} a_1 + b_4 = \dfrac{2C}{A} \\ a_1 \cdot b_4 = \dfrac{B}{A} \end{cases} \tag{13}$$

and functions a_3 and b_2 are arbitrary functions, which hold the condition $a_3 b_2 \neq 1$. As it was done above, supposing $a_3 = b_2 \equiv 0$ and finding the integrant multiplier with the help of equations (8), we get for the model of isothermal flows of ideal gas the characteristic velocities and Riemann- Earnshaw invariants

$$a_1^\pm = \mp\left(\frac{m}{kT}\right)^{\frac{1}{2}} \frac{P}{\rho_0}, \quad R^\pm = \rho_0 \left(\frac{kT}{m}\right)^{\frac{1}{2}} \Phi_\pm\left(\left(\frac{m}{kT}\right)^{\frac{1}{2}} \frac{1}{\rho_0} q + \ln\left(\frac{P}{P_0}\right)\right)$$

and for the model of adiabatic flows of polytropic gases

$$a_1^\pm = \mp\left(\frac{\gamma}{B}\right)^{\frac{1}{2}} \left(\frac{P_0}{\rho_0}\right)^{\frac{1}{2}} \left(\frac{P}{P_0}\right)^{\frac{1+\gamma}{2\gamma}},$$

$$R^\pm = (\rho_0 P_0)^{\frac{1}{2}} \Phi_\pm\left(\frac{1}{(\rho_0 P_0)^{\frac{1}{2}}} q \pm \frac{2\gamma}{\gamma - 1}\left(\frac{B}{\gamma}\right)^{\frac{1}{2}} \left(\frac{P}{P_0}\right)^{\frac{\gamma-1}{2\gamma}}\right).$$

Remark. Obtained characteristic velocities are velocities with respect to Lagrange coordinates. For the finding the characteristic velocities with respect to Euler coordinates (frequently use in mathematical literature) it is necessary in one-form $\tilde{a}_\pm = a_1^\pm dt + d\xi$, taking into account $\xi = \xi(t,x)$, calculate the differential $d\xi$ and use the equalities $\dfrac{dx}{d\xi} = h_p$, $\mathrm{v} \equiv \dfrac{dx}{dt} = h_q$.

The characteristic velocities and Riemann-Earnshaw invariants were obtained above for the equations in Lagrange coordinates (t,ξ). To find these objects for equations in Euler coordinates (t,x) (for «pure» equtions in Euler coordinates [1]) it is necessary to use system of equations (6) of paper [1], from which it is easy to obtain the following system of equations for gas-dynamic models:

$$\begin{cases} dq \wedge dx - dh \wedge dt = 0 \\ d(P_h) \wedge dx + d(P_q) \wedge dt = 0 \end{cases} \tag{14}$$

where the function $P = P(q,h)$ is obtained from the equation of determinant surface of in it model, if the quantity P is expressed via q and h. The system of equations (14) coincides with the system of equations (12) with accuracy up to the interchange of denotations $P \underset{\leftarrow}{\overset{\rightarrow}{}} h$, therefore coefficients $a_1^\pm$ and $b_4^\pm$ are determined by the system of equations (13), where it is necessary to put

$$A=\frac{\partial^2 P}{\partial h^2},\quad B=\frac{\partial^2 P}{\partial q^2},\quad C=\frac{\partial^2 P}{\partial q\partial h}.$$

The system of equations in Riemann-Earnshaw invariants has the form

$$\begin{cases}\left(\bar{a}_1^+dt+dx\right)\wedge dR^+(q,h)=0\\ \left(\bar{a}_1^-dt+dx\right)\wedge dR^-(q,h)=0\end{cases}$$

where $\bar{a}_1^\pm=\dfrac{C\mp\sqrt{C^2-AB}}{A}$, and $R^\pm(q,h)$ are founded with the help of the integranting multiplier, which is the solution of equation (8). For example, for the model of isothermal flows of ideal gas we have

$$P=P_0\exp\left[\frac{m}{kT\rho_0}\left(h-\frac{1}{2\rho_0}q^2\right)\right].$$

Calculating the second partial derivatives of this function by q and h, solving of equations (13) and equation (8), we get

$$\bar{a}_1^\pm=-\left(\frac{1}{\rho_0}q\pm\left(\frac{kT}{m}\right)^{\frac{1}{2}}\right),$$

$$R^\pm(q,h)=\frac{kT\rho_0}{m}\Phi_\pm\left(\frac{m}{kT\rho_0}h-\frac{m}{kT\rho_0^2}q^2\pm\left(\frac{m}{kT}\right)^{\frac{1}{2}}\frac{1}{\rho_0}q\right),$$

where $\Phi_\pm(\cdot)$ are arbitrary twice differentiable functions.

Note, that Riemann- Earnshaw invariants in systems of equations with respect to Lagrange and Euler coordinates are «symmetric» to each other with respect to equation of determinant surface of a model. The numerical values of these invariants coincide (for identically chosen functions $\Phi_\pm(\cdot)$), but for the equtions in Lagrange coordinates these invariants are expressed via the density of impulse and pressure; and for the equations in Euler coordinates («pure» equations of Euler coordinates) these invariants for the isothermal flows of ideal gas are expressed via the density of impulse and density of termodynamic potential of Gibbs, and for adiabatic flows of polytropic gases via the density of impulse and density of enthalpy. In the literature on gas-dynamics as a system of equations in Euler coordinates it is considered the system of equations (in partial derivatives) of Euler. This system of equations deduced from the system of equations (in differentials) (1) (see [1]), therefore working with the system of Euler equations we obtain Riemann-Earnshaw invariants expressed via the density of impulse (or via velocity) and pressure. From here one usually makes not exactly conclusion, that when we pass from description of flows in Lagrange coordinates to description of flows in Euler coordinate, Riemann-Earnshaw invariants do not change not only numerically, but by form of expression via gas-dynamic quantities.

Now consider a question on the correspondence of characteristic velocities for the equations in Euler and Lagrange variables to real velocity of propogation of perturbations (non-shoke) in medium with respect to medium itself. In gas-dynamics for equations in Euler coordinates it we obtain characteristic velocities of the type $\mathrm{v}\pm a(\rho)$ (or $\mathrm{v}\pm\bar{a}(\rho)$), where v is velocity of gas particle with respect to a viewer's system of coordinates, and the quantity $a(\rho)$ is called velocity of propagation of perturbations in gas with respect to gas it-self (see, for example, [3], [4]). On the other side, considering the same flows in Lagrange coordinates, we obtain characteristic velocity $a_1(\rho)$ (or $\overline{a}_1(P)$), which is called some times acoustic impedance, although the quantity $a_1(\rho)$ also we can call velocity of propagation of perturbations in gas with respect to gas itself (as in dynamics of an elastic pivot). The quantities $a(\rho)$ and $a_1(\rho)$ are different from each other, therefore the following question arises: which of the quantities $a(\rho)$ and $a_1(\rho)$ should be called velocity of propagation of perturbations in gas with respect to gas itself? Consider, for example, characteristic velocities obtained for equations in Lagrange and Euler coordinates for the model of isothermal flows of ideal gas. Characteristic velocity obtained for equations in Lagrange coordinates is equal to $a_1(\rho)=\overline{a}_1(P)=\left(\frac{m}{kT}\right)^{\frac{1}{2}}\frac{P}{\rho_0}$, and for equations in Euler coordinates is equal to $\mathrm{v}\pm\overline{a}(P)\equiv\mathrm{v}\pm\left(\frac{kT}{m}\right)^{\frac{1}{2}}$, hence we can see that $\overline{a}_1(P)$ coincides with $\overline{a}(P)$ only if pressure P is equal to background pressure $P_0=\frac{kT}{m}\rho_0$. The perturbations propagating with velocity $\left(\frac{kT}{m}\right)^{\frac{1}{2}}$ is called in acoustics the perturbations with infinitely small amplitude and these perturbations described by a system of linear equations obtained after linearization of exact equations. The author thinks, that characteristic velocity obtained for equations in Lagrange coordinates, which in the considered model depends not only on temperature, but on density of gas at present moment of time as well, should be called the velocity of propagation of perturbations in gas with respect to gas it-self. Note that in dynamics of a pivot the characteristic velocity obtained for equations in Lagrange coordinates is called velocity of propagation of perturbations along the pivot.

For those manifolds (or parts of manifolds), which uniquely projected to the plane $\left(y^3,y^4\right)$ from the system of equations (11) it is easy to obtain the system of linear equations in partial derivatives of the first order for functions $y^1=y^1\left(R^+,R^-\right)$, $y^2=y^2\left(R^+,R^-\right)$

$$\begin{cases} a_1^+\left(R^+,R^-\right)\dfrac{\partial y^1}{\partial R^-}+\dfrac{\partial y^2}{\partial R^-}=0 \\ a_1^-\left(R^+,R^-\right)\dfrac{\partial y^1}{\partial R^+}+\dfrac{\partial y^2}{\partial R^+}=0 \end{cases}$$

which in gas-dynamics describe solutions which are not simple waves.

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