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CONSTRUCTION OF HOMOGENEOUS SOLUTIONS FOR A TRANSVERSALLY-ISOTROPIC HOLLOW CYLINDER

Abstract

A general theory of transversally isotropic cylindrical shell is developed in the paper. The theory contains the methods for construction of non-homogeneous and homogeneous solutions, that admits to show characteristically peculiarities of stress-strain state of transversally-isotropic cylindrical shell. Behavior of solutions for three-dimensional boundary-value problems both in interior of the shell, and near the shell borders is studied on the premises of one of asymptotically method variants. The comparison of asymptotic solution with the solutions obtained by applied theries is carried out.

1. Consider an axiallysymmetric problem of elasticity theory for a transversally-isotropic hollow cylinder. Let the cylinder be of the volume

$$\Gamma=\{r\in[R_1,R_2],\quad \varphi\in[0,2\pi],\quad Z\in[-l,l]\}.$$

In displacements, balance equations have the form [1].

$$\begin{gathered} b_{11}\left(\Delta_0 U_\rho-\frac{U_\rho}{\rho}\right)+\frac{\partial^2 U_\rho}{\partial\xi^2}+(1+b_{13})\frac{\partial^2 U_\xi}{\partial\rho\partial\xi}=0 \\ (1+b_{13})\frac{\partial}{\partial\xi}\left(\frac{\partial U_\rho}{\partial\rho}+\frac{U_\rho}{\rho}\right)+\Delta_0 U_\xi+b_{33}\frac{\partial^2 U_\xi}{\partial\xi^2}=0 \end{gathered} \tag{1.1}$$

here $\rho=R_0^{-1}r, \xi=R_0^{-1}z, U_\rho=R_0^{-1}U_r, U_\xi=R^{-1}U_z$

$R_0=1/2(R_1+R_2)$ is the radius of a mean surface of the shell; $mb_{11}=2G_0(l-\nu_1\nu_2)$, $mb_{13}=2G_0\nu_1(l+\nu)$, $mb_{33}=2G_0E_0(1-\nu^2)$, $b_{12}=b_{11}-2G_0$, $E_0=E_1E^{-1}$, $G_0=GG_1^{-1}$ and $\nu_2=E_0^{-1}\nu_1$ are dimensionless values, E,G,ν are mass constants on the isotropy plane and E_1,G_1,ν_1 are mass constants on the plane that is perpendicular to isotropy plane.

Assume, that the shell is under the loading

$$\sigma_r\Big|_{r=R_k}=Q_k(z),\qquad \tau_{rz}\Big|_{r=R_k}=\tau_k(z) \tag{1.2}$$

from the lateral surfaces.

We shall not revise the character of boundary conditions at end-walls. We shall consider that the shell is in a balanced state.

2. A partial solution of balance equations (1.1) we call non-homogeneous solutions that satisfy non-homogeneous boundary conditions (1.2) at lateral surfaces. Methods of paper [2] may be used for the construction of non-homogeneous solutions. However, it is not the only method to unload the lateral surface of the cylinder.

One of the methods is the following.

External forces given at a lateral surface are decomposed in Fourier series, and it is necessary that external forces $Q_k(z),\tau_k(z)$ satisfy balance conditions.

For the simplicity, we shall consider them symmetric with regard to the plane $\xi=0$. Then $Q_k(z)$ is even, and $\tau_k(z)$ is odd with regard to the plane $(\xi=0)$. The skewsymmetric case is considered similarly. Then σ_z is even, and τ_{rz} is odd with regard to ξ and we present their boundary values by trigonometric series

$$Q_k(z)=\sum_{n=0}^{\infty}P_{kn}\cos\frac{p\pi\xi}{l},\ \tau_k(z)=\sum_{n=0}^{\infty}T_{kn}\sin\frac{p\pi\xi}{l} \tag{2.1}$$

$2l$ is the height of the cylinder.

It is natural to look for the components U_p, U_ξ of displacement vector in the form

$$U_p=\sum_{n=0}^{\infty}U_n(\rho)\cos\frac{n\pi\xi}{l},\ U_\xi=\sum_{n=0}^{\infty}W_n(\rho)\sin\frac{n\pi\xi}{l}. \tag{2.2}$$

By virtue of orthogonality of trigonometric functions, an input boundary value problem is led to one-dimensional boundary value problems with respect to the functions $U_n(\rho), W_n(\rho)$.

$$\left.\begin{aligned}&b_{11}\left(\frac{dU_n}{d\rho}+\frac{U_n}{\rho}\right)-n^2U_n+(1+b_{13})W_n'=0\\&\frac{d^2W_n}{d\rho^2}+\frac{dW_n}{d\rho}-b_{33}n^2W_n-(1+b_{13})n^2\left(U_n'+\frac{U_n}{\rho}\right)=0\end{aligned}\right\} \tag{2.3}$$

$$\left.\begin{aligned}&\left(b_{11}U_n'+b_{12}\frac{U_n}{\rho}+b_{13}W_n\right)\Bigg|_{\rho=\rho_k}=P_{kn}\\&\left(W_n'-n^2U_n\right)\Big|_{\rho=\rho_n}=T_{kn}\end{aligned}\right\} \tag{2.4}$$

We can use different methods for solution of obtained boundary-value problems, including numerical methods. The described method for the construction of non-homogeneous solutions is sufficiently universal and it doesn't depend on defferent parameters of the shell, including its thickness.

However, as in paper [3], if the thickness of the shell is sufficiently small, but the load given on lateral surfaces is sufficiently smooth, it is expedient to use the first iterational process of asymptotic method for the construction of non-homogeneous solutions. This method is less laborious and the final aim is achieved faster. However, we shall not stop on this case. In the given paper, it is constructed a system of homogeneous solutions admitting to unload the end-wall of the cylinder, and leaving the lateral surfaces of the cylinder free from stresses.

3. Now construct homogeneous solutions leaving the lateral surfaces of the cylinder free from stresses $Q_k(z)=0,\quad \tau_k(z)=0$

$$\sigma_r=0,\quad \tau_{rz}=0 \tag{3.1}$$

The solution (1.1), (3.1) will be looked for as

$$U_\rho=U(\rho)m(\xi),\qquad U_\xi=W(\rho)m(\xi), \tag{3.2}$$

where the function $m(\xi)$ is subjected to the condition

$$m''(\xi)-\mu^2m(\xi)=0. \tag{3.3}$$

By substituting (3.2) in (1.1), (3.1) we get the following boundary-value problem

$$\left.\begin{aligned} & b_{11}\left(U'+\frac{U}{\rho}\right)+\mu^2 U+(1+b_{13})W'=0 \\ & (1+b_{13})\mu^2\left(U'+\frac{U}{\rho}\right)+W''+\frac{1}{\rho}W'+b_{33}\mu^2 W=0 \end{aligned}\right\} \quad (3.4)$$

$$\left.\begin{aligned} & \left[b_{11}U'+b_{12}\frac{U}{\rho}+b_{13}W\right]_{\rho=\rho_{1,2}}=0 \\ & \left[\mu^2 U+W'\right]_{\rho=\rho_n}=0 \end{aligned}\right\} \quad (3.5)$$

Cite a general solution of the equation (3.4)

$$\left.\begin{aligned} & U(\rho)=(b_{33}\mu^2-\alpha_1^2)Z_1(\alpha_1\rho)+(b_{33}\mu^2-\alpha_2^2)Z_1(\alpha_2\rho) \\ & W(\rho)=-(1+b_{13})\mu^2[\alpha_1 Z_0(\alpha_1\rho)+\alpha_2 Z_0(\alpha_2\rho)] \end{aligned}\right\} \quad (3.6)$$

where $Z_k(\rho)=C_1 J_k(\rho)+C_2 Y_k(\rho)$, and the functions $J_k(\rho), Y_k(\rho)$ are linear-dependent solutions of the Bessel function; C_1, C_2 are constants;

$\alpha_n=\sqrt{t_n}, t_n$ are the roots of the quadratic equation

$$\left.\begin{aligned} & t^2-2q_1\mu^2 t+q_2\mu^4=0, \quad q_1=\frac{v_1}{v_2}(1-v_1v_2)^{-1}(1+v)(G_0-v_2) \\ & q_2=\frac{v_1}{v_2}(1-v_1v_2)^{-1}(1-v^2), \quad t_n=\mu^2 S_n \end{aligned}\right\} \quad (3.7)$$

$$S_n=\sqrt{q_1+(-1)^n\sqrt{q_1^2-q_2}}\,.$$

By satisfying the homogeneous boundary conditions (3.5) we get a chracteristic equation

$$\Delta(\mu,\rho_1,\rho_2)=8\pi^{-2}l_1l_2a_1a_2d_1d_2+(a_2b_1-a_1b_2)x$$
$$x\{a_1d_2[l_1L_{10}(\alpha_2)+l_2L_{01}(\alpha_2)]L_{11}(\alpha_1)-a_2d_1[l_1L_{10}(\alpha_1)+l_2L_{01}(\alpha_1)]L_{11}(\alpha_2)\}-$$
$$-(a_2b_1-a_1b_2)^2(\rho_1\rho_2)^{-1}L_{11}(\alpha_1)L_{11}(\alpha_2)+a_1a_2d_1d_2[L_{01}(\alpha_1)L_{10}(\alpha_2)+L_{01}(\alpha_2)L_{10}(\alpha_1)]-$$
$$a_2^2d_1^2L_{00}(\alpha_1)L_{11}(\alpha_2)-a_1^2d_2^2L_{00}(\alpha_2)L_{11}(\alpha_1)=0 \quad (3.8)$$
$$a_n=\mu^2(b_{33}\mu^2+b_{13}\alpha_n^2), \quad b_n=-2G_0(b_{33}\mu^2-\alpha_n^2),$$
$$d_n=\alpha_n(B_0\mu^2-b_{11}\alpha_n^2), \quad l_n=(\alpha_n\rho_n)^{-1},$$
$$B_0=b_{11}b_{33}-b_{13}^2-b_{13}$$
$$L_{ij}(x)=J_i(x\rho_1)Y_j(x\rho_2)-J_j(x\rho_2)Y_i(x\rho_1)$$
$$(i,j=0,1)$$

The left-hand side of the equation (3.8), as an entire function of parameter μ, has a countable set of zeros with a concentration point at infinity. For effective study of its roots, we assume, that the shell is thin walled. Set

$$\rho_1=1-\varepsilon, \quad \rho_2=1+\varepsilon, \quad \varepsilon=(2R_0)^{-1}(R_2-R_1). \quad (3.9)$$

We assume that ε is a small parameter. By substituting (3.7) to (3.6), we get

$$D(\mu,\varepsilon)=\Delta(\mu,\rho_1,\rho_2)=0\,. \quad (3.10)$$

Concerning the zeros of the function $D(\mu,\varepsilon)$ we prove the statement: for $\varepsilon\to 0$ the function $D(\mu,\varepsilon)$ has three groups of zeros with following asymptotic properties:

a) the first group consists of double zero $\mu = 0$;

б) the second group consists of four zeros, and they have the order $0(\varepsilon^{-1/2})$

в) the third group contains a countable set of zeros, and they have the order $0(\varepsilon^{-1})$.

Mention the proof of this statement. Expand the function $D(\mu,\varepsilon)$ in a series of ε.

$$D(\mu,\varepsilon) = A\mu^2\varepsilon^2\left\{b_0 + \frac{1}{3}\left[E_0\mu^4 - 4G_0(1+\nu)(E_0G_0 - \nu_1)\mu^2 + 9b_0\right]\varepsilon^2 + \right.$$

$$\left. + \frac{1}{45}\left[-8(1+\nu)b_0^{-1}(E_0G_0 - \nu_1)\mu^6 + \ldots\right]\varepsilon^4 + \ldots\right\} = 0 \qquad (3.11)$$

$$b_0 = 1 - \nu_1\nu_2; \quad A = 128(1+\nu)^2 m^{-1}\pi^{-2}G_0^3E_0\left(\alpha_1^2 - \alpha_2^2\right)^2 b_{33}(b_{13}+1)^2$$

Hence it is seen that $\mu = 0$ is a double zero of the function $D(\mu,\varepsilon)$.

Similar to isotropic case, we can prove that all remaining zeros of the function $D(\mu,\varepsilon)$ are unrestrictedly increasing, for $\varepsilon \to 0$. We can divide them into two groups, depending on their behavior for $\varepsilon \to 0$;

1) $\varepsilon\mu_k \to 0$ for $\varepsilon \to 0$; 2) $\varepsilon\mu_k \to const$ for $\varepsilon \to 0$.

Determine such μ_k, for which $\varepsilon\mu_k \to 0$ for $\varepsilon \to 0$.

As in isotropic case, the zeros of the second group have the asymptotic expansions

$$\mu_k = \frac{\gamma_k}{\sqrt{\varepsilon}} \quad \gamma_k = \mu_{k0} + \varepsilon\mu_{k2} + \ldots \qquad (k = 1,2,3,4), \qquad (3.12)$$

where

$$\mu_{k0}^4 + \frac{3}{E_0}(1 - \nu_1\nu_2) = 0, \quad \mu_{k2} = \frac{G_0}{\mu_{k0}E}(1+\nu)(E_0G_0 - \nu_1)\left(1 - \frac{2}{5E_0}\right)$$

We look for $\mu_k(k = 5,6,\ldots)$ in the form of

$$\mu_k = \varepsilon^{-1}\delta_k + O(\varepsilon) \qquad (3.13)$$

to construct the asymptotics for the zeros of the third group.

However, as it is noted in [5,6] depending on the characteristics ν, ν_1, ν_2, G_0 of the material, the parameters q_1, q_2 adopt different values implies a different writing of solutions by a Bessel function. And this brings to various asymptotic representations of a Bessel function.

Consider the following possible cases:

1. $q_1 > 0, q_1 - q_2 \neq 0, \quad \alpha_{1,2} = \pm\mu s_1, \alpha_{3,4} = \pm\mu s_2$,

$$s_{1,2} = \sqrt{q_1 \pm \sqrt{q_1^2 - q_2}}, \quad q_1^2 > q_2$$

$$s_{1,2} = \chi + i\beta = \sqrt{q_1 \pm i\sqrt{q_2 - q_1^2}} \quad q_1^2 < q_2$$

2. The roots of a characteristic function are multiple

$$\alpha_{1,2} = \alpha_{3,4} = \mu\rho, \quad q_1 > 0, \quad q_1^2 - q_2 = 0$$

3. $q_1 < 0 \quad q_1^2 - q_2 \neq 0$

$$\alpha_{1,2} = \pm i\mu s_1, \quad \alpha_{3,4} = \pm i\mu s_2$$

$$s_{1,2} = \sqrt{|q_1| \pm \sqrt{q_1^2 - q_2}}, \quad q_1^2 > q_2$$

$$s_{1,2}=\sqrt{|q|_1 \pm i\sqrt{q_2-q_1^2}},\quad q_1^2<q_2$$

$$4.\ q_1<0,\quad q_1^2-q_2=0,\quad \alpha_{1,2}=\alpha_{3,4}=\pm i\mu\rho,\quad p=\sqrt{|q_1|}$$

In cases 1 and 2, after substituting of (3.13*) in (3.8) and transformation it by means of asymptotic expansions $J_v(x)$, $Y_v(x)$ for δ_k we get

$$(S_2-S_1)\sin(S_1+S_2)\delta_n \pm (S_1+S_2)\sin(S_2-S_1)\,\delta_n=0,\quad q_1^2>q_2 \tag{3.14}$$

$$\chi\sin 2\beta\delta_n \pm \beta sh2\chi\delta_n=0,\quad q_1^2<q_2 \tag{3.15}$$

$$\sin 2\rho\delta_k \pm 2p\delta_k=0,\quad q_1>0,\ q_1^2-q_2=0 \tag{3.16}$$

The results for cases 3 and 4 are obtained from cases 1 and 2 with formal substitution of S_1,S_2 for iS_1,iS_2.

These equations coincide with equations that define the exponents of Saint-Venant's boundary effects in theory of transversally-isotropic thick plates [6]. The roots of these equations have been studied there.

4. Reduce an asymptotic construction of homogeneous solutions corresponding to different groups of characteristic equation by assuming that ε is a small parameter.

The displacements and stresses

$$u_r=-v_1C_0\rho,\quad u_z=C_0\xi$$

$$\sigma_z=\frac{2G_1v_1(1+v)}{v_2}C_0 \tag{4.1}$$

$$\sigma_r=\sigma_\varphi=\tau_{rz}=0$$

correspond to the double root $\mu_0=0$.

Here, C_0 is a constant. So, as the isotropic case, a pure extension along a symmetry axis corresponds to the root $\mu_0=0$ of the first group. This stresses state penetrates into the shell domain without damping.

Group (2). The function $m_k(\xi)$ is found from the equation

$$\frac{d^2m_k}{d\xi^2}-\frac{\gamma_k}{\sqrt{\varepsilon}}m_k(\xi)=0,\quad \gamma_k=\mu_{k0}+\varepsilon\mu_{k2}$$

where μ_{k0},μ_{k2} are given by the relation (3.12). hence

$$m_k(\xi)=E_x\exp\left(\frac{\gamma_k}{\sqrt{\varepsilon}}\xi\right)+N_k\exp\left(-\frac{\gamma_k}{\sqrt{\varepsilon}}\xi\right)$$

E_k,N_k are constants.

By assuming $\rho=1+\varepsilon\eta,\quad -1\le\eta\le1$ and decomposing the solution of the second group in small parameter ε, we find for them the asymptotic expression (the values of displacements are cited. Hook's generalized law may find the stresses)

$$u_\rho=\sqrt{\varepsilon}\sum_{k=1}^{4}C_k\left[-\mu_{k0}^2+O(\varepsilon)\right]\frac{dm_k}{d\xi}$$

$$u_\xi=-\sqrt{\varepsilon}\sum_{k=1}^{4}C_k\left[3b_0\frac{G_0}{E_0}\eta+2G_0b_{13}\left(b_{13}^2-b_{11}b_{33}\right)^{-1}+O(\varepsilon)\right]m_k(\xi) \tag{4.2}$$

C_k are constants.

Group (3). In this case, by using the first term of the asymptotics of Bessel's function, for displacements and stresses at the first approximation we get two classes of solutions. The first of them corresponds tot the zeros of the function

$$(S_2 - S_1)\sin(S_2 + S_1)\delta_k - (S_2 + S_1)\sin(S_2 - S_1)\delta_k = 0,$$

and the second one corresponds to the zeros of the function

$$(S_2 - S_1)\sin(S_2 + S_1)\delta_k + (S_2 + S_1)\sin(S_2 - S_1)\delta_k = 0.$$

They have the same structure, and may be represented by the expressions

$$u_{\rho n} = \varepsilon \frac{b_{33}}{b_{11}} \sum_{n=1,3,\ldots}^{\infty} B_n \left[A_1 \cos S_2\delta_n \cos S_1\delta_n\eta - A_2 \cos S_1\delta_n \cdot \cos S_2\delta_n\eta + O(\varepsilon)\right]\frac{dm_n}{d\xi}$$

$$u_{\xi n} = \sum_{n=1,3,\ldots}^{\infty} B_n\delta_n\left[S_1K_2 \cos S_2\delta_n \sin S_1\delta_n\eta - S_2K_1 \cos S_1\delta_n \cdot \sin S_2\delta_n\eta + O(\varepsilon)\right]m_n(\xi)$$

$$\sigma_z^{(n)} = G_1 \frac{b_{11}b_{33} - b_{13}^2}{b_{11}} \times$$

$$\times \sum_{n=1,3,\ldots}^{\infty} B_n\delta_n\left[S_1 \cos S_2\delta_n \sin S_1\delta_n\eta - S_2 \cos S_1\delta_n \cdot \sin S_2\delta_n\eta + O(\varepsilon)\right]\frac{dm_n}{d\xi} \tag{4.3}$$

$$\sigma_\varphi^{(n)} = G_1 \frac{b_{33}}{b_{11}} \sum_{n=1,3,\ldots}^{\infty} B_n\delta_n \begin{bmatrix} S_1(b_{11}K_2 - b_{12}A_1)\cos S_2\delta_n \sin S_1\delta_n\eta - S_2(b_{11}K_1 - b_{12}A_2)x \\ \cos S_1\delta_n \sin S_2\delta_n\eta + O(\varepsilon) \end{bmatrix}\frac{dm_n}{d\xi}$$

$$\sigma_r^{(n)} = -G_1S_1S_2\left(b_{11}b_{33} - b_{13}^2\right)\times$$

$$\times \sum_{n=1,3,\ldots}^{\infty} B_n\delta_n\left[S_2 \cos S_2\delta_n \sin S_1\delta_n\eta - S_1 \cos S_1\delta_n \sin S_2\delta_n\eta + O(\varepsilon)\right]\frac{dm_n}{d\xi}$$

$$\tau_{rz}^{(n)} = \frac{G_1}{\varepsilon}\frac{b_{11}b_{33} - b_{13}^2}{b_{11}} \times$$

$$\times \sum_{n=1,3,\ldots}^{\infty} B_n\delta_n\left[\cos S_2\delta_n \cos S_1\delta_n\eta - \cos S_1\delta_n \cos S_2\delta_n\eta + O(\varepsilon)\right]m_n(\xi) \quad (S_1 \neq S_2)$$

$$K_i = b_{33} + b_{13}S_i^2; \quad A_i = b_{11}b_{33} - b_{13}^2 - b_{13} - b_{11}S_i^2 \qquad S_1 \neq S_2$$

Similarly, in case $\chi \sin 2\beta\delta_n + \beta sh 2\chi\delta_n = 0$ we have

$$u_{\rho n} = \varepsilon \sum_{n=1,3,\ldots}^{\infty} D_n\left[\left(b_{33} + \chi^2 - \beta^2\right)ch\chi\delta_n \cos\beta\delta_n\eta\Delta_1 + 2\chi\beta sh\chi\delta_n\eta \sin\beta\delta_n\eta\Delta_2 + O(\varepsilon)\right]\frac{dm_n}{d\xi}$$

$$u_{\xi n} = -(b_{13} + 1)\sum_{n=1,3,\ldots}^{\infty} \delta_n D_n\left[\alpha sh\chi\delta_n\eta \cos\beta\delta_n\eta\Delta_1 + \beta ch\chi\delta_n\eta \sin\beta\delta_n\eta\Delta_2 + O(\varepsilon)\right]m_n(\xi)$$

$$\sigma_z^{(n)} = G_1 \sum_{n=1,3,\ldots}^{\infty} \delta_n D_n \times$$

$$\times \left\{ \begin{matrix} \begin{bmatrix} \alpha\left(b_{13}\chi^2 - b_{13}\beta^2 - b_{33}\right)sh\chi\delta_n\eta \cos\beta\delta_n\eta - \beta b_{13}\left(b_{33} + \chi^2 - \beta^2\right)x \\ ch\chi\delta_n\eta \sin\beta\delta_n\eta \end{bmatrix} \times \Delta_1 + \\ \beta\left[\left(b_{13}\chi^2 - b_{13}b_{33} - b_{33}\right)ch\chi\delta_n\eta \sin\beta\delta_n\eta + 2\chi\beta b_{13} sh\chi\delta_n\eta \cos\beta\delta_n\eta\right]\times \\ \times \Delta_2 + O(\varepsilon) \end{matrix} \right\}\frac{dm_n}{d\xi}$$

$$\sigma_\varphi^{(n)} = G_1 \sum_{n=1,3,\ldots}^{\infty} \delta_n D_n \times$$

$$\times\left\{\begin{aligned}&\left[\begin{aligned}&\chi\left(b_{12}b_{33}-b_{12}b_{13}-b_{12}+b_{12}\chi^2-b_{12}\beta^2\right)sh\chi\delta_n\eta\cos\beta\delta_n\eta-\\&\beta b_{12}\left(b_{33}+\chi^2-\beta^2\right)ch\chi\delta_n\eta\sin\beta\delta_n\eta\end{aligned}\right]\Delta_1+\\&+\beta\left[\begin{aligned}&\left(2b_{12}\chi^2-b_{12}b_{13}-b_{12}\right)ch\chi\delta_n\eta\sin\beta\delta_n\eta+\\&2\chi\beta b_{12}sh\chi\delta_n\eta\cos\beta\delta_n\eta\end{aligned}\right]\Delta_2+O(\xi)\end{aligned}\right\}\frac{dm_n}{d\xi} \quad (4.4)$$

$$\sigma_r^{(n)}=G_1\sum_{n=1,3,\dots}^{\infty}\delta_nD_n\left\{\begin{aligned}&\left[\begin{aligned}&\chi\left(b_{11}b_{33}-b_{13}^2-b_{13}+b_{11}\chi^2-b_{11}\beta^2\right)sh\chi\delta_n\eta\cos\beta\delta_n\eta-\\&-\beta b_{11}\left(b_{33}+\chi^2-\beta^2\right)ch\chi\delta_n\eta\sin\beta\delta_n\eta\end{aligned}\right]\Delta_1+\\&+\left[\begin{aligned}&\beta\left(b_{13}^2+b_{13}-2b_{11}\chi^2\right)ch\chi\delta_n\eta\sin\beta\delta_n\eta-\\&-2\chi\beta^2b_{11}sh\chi\delta_n\eta\cos\beta\delta_n\eta\end{aligned}\right]\Delta_2+O(\xi)\end{aligned}\right\}\frac{dm_n}{d\xi}$$

$$\tau_{rz}^{(n)}=-G_1\frac{\chi\beta}{\varepsilon}\left[\left(b_{13}-1\right)\left(b_{33}-b_{13}^2\chi^2-\beta^2\right)+\beta^2\left(b_{13}+1\right)\right]\times$$

$$\times\sum_{n=1,3,\dots}^{\infty}\delta_n^2D_n\left[sh\chi\delta_n\sin\beta\delta_nch\chi\delta_n\eta\cos\beta\delta_n\eta-ch\chi\delta_n\cos\beta\delta_nsh\chi\delta_n\eta\sin\beta\delta_n\eta+O(\varepsilon)\right]m_n(\xi)$$

$$\Delta_1=-\chi\beta\left(b_{13}-1\right)sh\chi\delta_n\sin\beta\delta_n+\beta^2\left(b_{13}+1\right)ch\chi\delta_n\cos\beta\delta_n$$

$$\Delta_2=\left(b_{33}-b_{13}\chi^2-\beta^2\right)ch\chi\delta_n\cos\beta\delta_n+\chi\beta\left(b_{13}+1\right)sh\chi\delta_n\sin\beta\delta_n$$

In case when $q_i>0$ and the roots of the quadratic equation (3.7) are multiple, the solutions have the form:

$$s_1=s_2=p=\frac{v_1}{v_2}\frac{(1+v)(G_0-v_2)}{v_2(1-v_1v_2)}$$

$$u_{\rho n}=\varepsilon p\sum_{n=1,3,\dots}^{\infty}C_n\left[\begin{aligned}&\left(p\delta_n\sin p\delta_n-\frac{b_{13}+2}{b_{13}+1}\cos p\delta_n\right)\cos p\delta_n\eta-\\&-\eta p\delta_n\cos p\delta_n\sin p\delta_n\eta+O(\varepsilon)\end{aligned}\right]\frac{dm_n}{d\xi}$$

$$u_{\xi n}=\sum_{n=1,3,\dots}^{\infty}C_n\delta_n\left[\begin{aligned}&\left(p\delta_n\sin p\delta_n-\frac{1}{b_{13}+1}\cos p\delta_n\right)\sin p\delta_n\eta+\\&+\eta p\delta_n\cos p\delta_n\cos p\delta_n\eta+O(\varepsilon)\end{aligned}\right]m_n(\xi)$$

$$\sigma_r^{(n)}=2G_1\sum_{n=1,3,\dots}^{\infty}C_n\delta_v\times$$

$$\times\left[(\cos p\delta_n-p\delta_n\sin p\delta_n)\sin p\delta_n\eta-\eta p\delta_n\cos p\delta_n\cos p\delta_n\eta+O(\varepsilon)\right]\frac{dm_n}{d\xi} \quad (4.5)$$

$$\sigma_z^{(n)}=2G_1\frac{b_{13}+2}{b_{11}}\sum_{n=1,3,\dots}^{\infty}C_n\delta_n\left[\begin{aligned}&(\cos p\delta_n+p\delta_n\sin p\delta_n)\sin p\delta_n\eta+\\&+p\delta_n\eta\cos p\delta_n\cos p\delta_n\eta+O(\varepsilon)\end{aligned}\right]\frac{dm_n}{d\xi}$$

$$\sigma_\varphi^{(n)}=G_1\sum_{n=1,3,\dots}^{\infty}C_n\delta_n\left\{\begin{aligned}&\left[\frac{b_{12}p^2+b_{13}}{b_{13}+1}\cos p\delta_n+\left(b_{13}-b_{12}p^2\right)p\delta_n\sin p\delta_n\right]x\\&\sin p\delta_n\eta+p\delta_n\left(b_{13}-b_{12}p^2\right)\eta\cos p\delta_n\cos p\delta_n\eta+O(\varepsilon)\end{aligned}\right\}\frac{dm_n}{d\xi}$$

$$\tau_{rz}^{(n)}=\frac{2G}{\varepsilon}p^2\sum_{n=1,3,\dots}^{\infty}C_n\delta_n^3\left[\sin p\delta_n\cos p\delta_n\eta-\eta\cos p\delta_n\sin p\delta_n\eta+O(\varepsilon)\right]m_n(\xi)$$

where B_n, D_n, C_n are constants. Expressions for $n=2,4,6\dots$ are obtained from (4.3), (4.4), (4.5) with $\cos x$ replaced by $\sin x$, and $\sin x$ by $\cos x$, chx by shx, and shx by $-chx$.

In formulae (4.3), (4.4), (4.5) with the solution of is_1, is_2, ip for s_1, s_2, p we get the solutions of cases 3 and 4.

In papers [5,6] the roots of equations (3.14), (3.15) and (3.16) are studied and their calculation methods are developed. The character of these roots influences essentially to a general picture of a stress-strain state of the shell.

As it is seen from paper [7], in case of essential anisotropy, holding for sufficiently great values of G_0, Saint-Venant's boundary layer is damping very weakly, and solutions (4.3), (4.4), (4.5) should be attached to penetrating solutions. Therefore, in this case, a stress-strain state of a transversally-isotropic shell and an isotropic shell strongly differ. Consider the connection of homogeneous solutions with a principal vector of stresses P, that acts on the section $\xi = const$. We have

$$P = \int_0^{2\pi} \int_{R_1}^{R_2} (\sigma_z + \tau_{rz}) r dr d\varphi . \tag{4.6}$$

Represent the stresses σ_z and τ_{rz} in the form

$$\sigma_z = \sigma_z^0 + \sum_{n=1}^{\infty} \sigma_{zn}(r) \frac{dm_n}{d\xi} , \tag{4.7}$$

$$\tau_{2z} = \tau_{2z}^0 + \sum_{n=1}^{\infty} T_n(2) m_n(\xi) .$$

The summands σ_z^0, τ_{rz}^0 correspond to eigenvalues $\mu = 0$. The second summand contains the stresses, definable by the second and the third groups of solutions.

By substituting (4.7) in (4.6) with regard to solutions (4.1)-(4.5) we get

$$P = P_0 + \sum_{n=1}^{\infty} P_n$$

$$P_n = 2\pi \int_{R_1}^{R_2} \left[\sigma_{zn}(r) \frac{dm_n}{d\xi} + T_n(r) m_n(\xi) \right] r dr \tag{4.8}$$

By a solvability condition of the elasticity theory problem, P_n must not depend from the variable ξ. However, the right hand side of relation (4.8) by virtue of a linear independence of $m_n(\xi), \frac{dm}{d\xi}$ depends on ξ. Hence, it follows that $P_n = 0$ for any $n = (n = 1, 2, \ldots)$. So, we get

$$P = P_0 = 2\pi G_1 (1 + \nu) \frac{\nu_1}{\nu_2} C_0 \left(R_2^2 - R_1^2 \right) \tag{4.9}$$

for the principal vector P.

The stress-state corresponding to the zeros of the second and third groups is self-balanced at each segment of $\xi = const$.

The solutions (4.1) and (4.2) determine the internal stress-strain state of the shell. The first terms of their asymptotic expansions in thin-shellness parameter ε define the momentless stress state. At the first term of asymptotics, we may consider them as a solution on applied theory of shells.

Stress state corresponding to solutions (4.3). (4.4) and (4.5) is of a boundary layer character. The first terms of its asymptotic expansion are absolutely equivalent to Saint-venant's boundary effect of transversally-isotropic plate.

5. Now reduce an analysis of characteristic equations obtained by Kirghoff-Liav and S.A. Ambartsumyan theories for comparison. We are to note, that a transversally-isotropic shell by Kirghoff-Liev and S.A.Ambartsumyan theories in a coordinate system (z,φ,r)(in our coordinate system (r,φ,z)) behaves as an orthotropic shell.

At the first case, balance equations in displacements have the form of [8].

$$C_{11}\frac{d^2u}{ds^2}+C_{12}\frac{dw}{ds}=0$$
$$C_{12}\frac{du}{ds}+\frac{D_{11}}{R_0^2}\frac{d^4w}{ds^4}+C_{22}w=0 \qquad (5.1)$$

Here $u=u(s), w=w(s)$ are the corresponding components of a displacement vector along and on the thickness of the generator

$$C_{ij}=hB_{ij},\quad D_{ij}=\frac{h^3}{12}B_{ij},\quad B_{11}=\frac{E_1}{1-\nu_1\nu_2},\quad B_{22}=\frac{E}{1-\nu_1\nu_2},\quad B_{12}=\frac{\nu_2 E}{1-\nu_1\nu_2}.$$

We shall look for the solution of the system (5.1) in the form of

$$u=Ae^{\mu\xi},\quad w=Be^{\mu\xi}.$$

We obtain the characteristic equation

$$\mu^2\left[\varepsilon^2\mu^2+3(1-\nu_1\nu_2)\frac{E}{E_1}\right]=0 \qquad (5.2)$$

from the existence condition of non-trivial solutions.

The following groups of roots are obtained from (5.2)

1. $\mu=0$ is double
2. $\mu_k=\frac{\beta_k}{\sqrt{\varepsilon}},\quad \beta_k=\beta_{k0}+\varepsilon\beta_{k2}+\dots$

$$\beta_{k0}^4+3(1-\nu_1\nu_2)\frac{E}{E_1}=0,\quad \beta_{k2}=0 \qquad (5.3)$$

So, the given applied theory admits to find only the first term of expansion of a boundary effect exponent. But it is impossible to define the next terms by this theory.

S.A.Ambartsumyan's theory [8]. In this case, a characteristic equation has the form of [8] (p.309, in our notations)

$$\mu^2\left[3(1-\nu_1\nu_2)\frac{E}{E_1}+\varepsilon^2\left(\mu^4-\frac{6}{5}G_1^{-1}E\mu^2\right)\right]=0 \qquad (5.4)$$

The equation (5.4) has the following groups of zeros.

1. $\mu=0$ is a double zero
2. $\mu_k=\frac{\omega_k}{\sqrt{\varepsilon}},\quad \omega_k=\omega_{k0}+\varepsilon\omega_{k2}+\dots$

$$\omega_{k0}^4+3(1-\nu_1\nu_2)\frac{E}{E_1}=0;\quad \omega_{k2}=\frac{3}{10\omega_{k0}}\frac{E}{G_1} \qquad (5.5)$$

By comparing (5.5) with precise expansion (3.12) we get that the first terms coincide, but the second terms essentially differ.

For $G_1\to\infty$ the expansions of (5.5) coincide with the expansion of (5.4).

The study of existing applied theories shows that, all of them at the first term of the asymptotics exactly approximate the solutions that correspond to zeros, and definable by the formulae (3.12). At the same time they can't pretend to some revision at consequent approximations, since in these theories the second term of approximation doesn't coincide with its exact value given by the formula (3.12). The zeros of a characteristic equation definable by formulas (3.14), (3.15) and (3.16), can't be determined by Kirghoff-Liav and S.A.Ambartsumyan theories, and quality distinction of anisotropic shells theory from isotropic one just appears in these zeros.

The results of [4] and above-mentioned analysis show that it is impossible to get a refined theory by means of introducing artificially correcting terms to the classic shell theory equation.

Only the exact analysis of corresponding three-dimensional problems can show the way for construction of refined theories, that admit in a concrete case to take into account various phenomena arising in thin shells.

6. By summing in all roots of a characteristic equation, we can represent the homogeneous solutions as

$$
\begin{aligned}
u_\rho &= R_0\sum_{k=1}^{\infty} C_k u_k(\rho)\frac{dm_k}{d\xi} \\
u_\xi &= R_0\sum_{k=1}^{\infty} C_k W_k(\rho) m_k(\xi) \\
\sigma_r &= G_1\sum_{k=1}^{\infty} C_k Q_{rk}(\rho)\frac{dm_k}{d\xi} \\
\sigma_\varphi &= G_1\sum_{k=1}^{\infty} C_k Q_{\varphi k}(\rho)\frac{dm_k}{d\xi} \\
\sigma_z &= G_1\sum_{k=1}^{\infty} C_k Q_{zk}(\rho)\frac{dm_k}{d\xi} \\
\tau_{rz} &= G_1\sum_{k=1}^{\infty} C_k T_k(\rho) m_k(\xi)
\end{aligned}
\qquad (6.1)
$$

Here C_k are constants. As in the isotropic case [4] we can prove that a system of homogeneous solutions satisfy the generalized conditions of orthogonality, that allows to solve exactly elasticity theory problems under mixed boundary conditions at the end-walls of the cylinder

$$\int_{\rho_1}^{\rho_2}\left[T_p(\rho)u_k(\rho)-Q_{zk}(\rho)W_p(\rho)\right]\rho d\rho \quad k\neq p. \qquad (6.2)$$

At all other cases we need to use different approximate methods to satisfy boundary conditions at the end-walls of the cylinder. Therefore, consider a problem on satisfaction of boundary conditions at the end-walls of the cylinder by means of a class of homogeneous solutions. Let systems of stresses $\sigma_z^i, \tau_{rz}^i (i=1,2)$ be given for $\xi=\pm l_0 \quad \left(l_0=R_0^{-1}\cdot l\right)$. Here, as we noted above, it is sufficient to consider the cases when the load is symmetric with respect to the plane $\xi=0$. A skewsymmetric case is considered similarly (we may put $m_k=ch\mu_k\xi$ in a symmetric case, but in a skewsymmetric case we take $m_k=sh\mu_k\xi$). So, let the following conditions

$$\sigma_z=Q(\rho), \quad \tau_{rz}=\tau(\rho) \text{ for } \xi=\pm l_0 \qquad (6.3)$$

be given.

We shall search the solution in the form of (6.1). Use Langranges's variational principle for the definition of constants C_k $(k=1,2,\ldots)$, whose variations will be considered as independent.

Since the homogeneous solutions satisfy the balance equation and boundary conditions on a cylindrical surface, the variational principle adopts the form

$$\int_{\rho_1}^{\rho_2}[(\sigma_z - Q)\delta W + (\tau_{rz} - \tau)\delta u]\rho d\rho = 0. \qquad (6.4)$$

We get an infinite system of algebraic equations from (6.4)

$$\sum_{k=1}^{\infty} M_{kp} C_k = N_p \qquad (p=1,2,\ldots)$$

$$M_{kp} = \int_{\rho_1}^{\rho_2}(Q_{zk}W_p + T_k u_p)\rho d\rho \qquad (6.5)$$

$$N_p = \int_{\rho_1}^{\rho_2}(QW_p + \tau u_p)\rho d\rho$$

By using the smallness of a parameter of the thin wall property of the shell ε, we can construct an asymptotic solution of the system (6.5). This approach is well-known [4] and we shall not stop on it in detail.

At the endnote that we get N.A.Bazarenko and I.I.Vorovich's known results in the isotropic case [9] for $G_0 = 1$.

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