MAMEDOV V.T.

STRESS DISTRIBUTION IN ELASTIC ELEMENTS WITH ECCENTRIC HOLES

Abstract

Stress distribution in elastic elements with eccentrically arranged holes is studied. Muskhelishvili's method is used in elastic element with tree eccentric holes. Obtained results admit to define stress distribution in a complicated configuration packing with holes.

An elastic element with eccentric holes is widely used as a packing of hermetically sealed nodes in oil field equipment. It is natural that the stress-strain state in these elements is of a complicated character [1].

Stress distribution of more simple form elastic element with symmetrically arranged holes has been studied in papers [2,3 etc.]. However, the stress-strain state of the packing with eccentric holes have not been studied.

In this paper we attempt to study the stress distribution in elastic elements with eccentric holes. Here we adopt that the elastic element (rubber packing) in packing process is compressed under the thrust washers and a part of the elastic material under the thrust washers undergoes a complicated strain. With some approximation for the part of an elastic element under the thrust washer a plane section is accepted, i.e. the elastic element is loaded by the uniform conditional initial stress σ_{ow} through a thrust washer.

Experimentally it has been obtained that by achieving the packing, initial tension (conditional stress) in the elastic element must be [1]:

$$\sigma_{ow} = \frac{\mu}{1-\mu}\frac{Q}{S_w}\ln\frac{R_w}{R_0}, \tag{1}$$

where μ is a Poisson coefficient for elastic material; Q is the force applied to the elastic element; R_0 is a mean internal radius of the elastic element, $R_0 = 0.5(r_1 + r_2)$; r_1, r_2 are different radiuses of eccentric holes; R_w and S_w are the radius and the square of the thrust washer.

To solve this problem we use Muskhelishvili's method [4]:

$$\sigma_x + \sigma_y = 4\operatorname{Re}[\Phi'(z)] = 2[\Phi'(z) + \overline{\Phi}'(z)] \tag{2}$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2[\bar{z}\Phi''(z) + X''(\bar{z})] \tag{3}$$

in polar coordinates:

$$\sigma_r + \sigma_\theta = \sigma_x + \sigma_y = 4\operatorname{Re}[\Phi'(z)] \tag{4}$$

$$\sigma_\theta - \sigma_r + 2i\tau_{i\theta} = 2[\bar{z}\Phi''(z) + X''(z)]e^{2i\theta} \tag{5}$$

here $\Phi'(z)$ and $X'(z)$ are analytic stress functions.

By using Muskhalishvili's method [4] we assume that at first (fig. 1a) the I hole is cut out, (fig. 1b) and we determine the distribution stresses around the hole and its influence. Then the II hole is cut out (fig 1c) and we determine its stress distribution.

The second hole will break the distribution of stresses of the first hole, if the requirement that the resulting stress equals to zero along the hole be fulfilled. This process will be debalanced successively [4].

Boundary conditions for stresses are accepted by Cauchy equations [4]:

$$\Phi_1(z)=-\frac{1}{2\pi i}\int\frac{f_1(t)dt}{t-z} \tag{6}$$

$$X'_1(z)=-\frac{1}{2\pi i}\int\frac{\bar{f}_1(\bar{t})}{t-z}+\frac{1}{2\pi i}\int\frac{\bar{f}_1(\bar{t})}{t}-\frac{r^2}{z}\Phi_1(z) \tag{7}$$

The resulting stresses way be written as [5]:

$$f(s')=i\int(X-iY)ds=\Phi(z)+z\overline{\Phi}'(\bar{z})+\overline{X}'(\bar{z}) \tag{8}$$

the adjoint function will be

$$\bar{f}(\bar{s})=-i\int(X-iY)ds=\overline{\Phi}(\bar{z})+\bar{z}\Phi'(z)+X'(\bar{z}). \tag{9}$$

By contour of the holes (8) and (9) will have the form:

$$f_1=-f=-\overline{\Phi}(\bar{z})-z\overline{\Phi}'(\bar{z})-\overline{X}'(\bar{z}). \tag{10}$$

If the contour defined by the function f, becomes the boundary of the I hole at the contour, the function in adjoint function:

$$\bar{f}_1=-\bar{f}=-\overline{\Phi}(\bar{z})-\bar{z}\Phi'(z)-X'(z)\ . \tag{11}$$

By Muskhelishvili's method in the absence of the I hole, the stresses will be:

$$\Phi(z)=\Phi_1(z)+\Phi_0(z) \tag{12}$$

$$X'(Z)=X'_1(z)+X'_0(z) \tag{13}$$

where $\Phi_0(z)$ and $X'_0(z)$ are analytical stress functions without holes: $\Phi_1(z)$ and $X'_1(z)$ are the functions characterizing the influence of holes.

Considering (12) and (13) in (2) and (3) we get:

$$\Phi_0=\frac{\sigma_{ow}z}{2} \tag{14}$$

$$X''_0(z)=0 \tag{15}$$

$$X'_0(z)=C=0 \tag{16}$$

We can write from (10), (11), (14)-(16)

$$f_1(t)=-\sigma_{ow}z=-\sigma_{ow}t \tag{17}$$

$$\bar{f}_1(t)=-\sigma_{ow}z-\frac{\sigma_{ow}\bar{t}}{2}=-\frac{\sigma_{ow}r_1^2}{t} \tag{18}$$

where $t=r_1e^{i\theta}$ and $\bar{t}=\frac{r_1^2}{t}$.

Considering (17) and (18) in (6) and (7) we get:

$$\Phi(z)=\frac{\sigma_{ow}z}{2} \tag{19}$$

$$X'(z)=-\frac{\sigma_{ow}r_1^2}{z} \tag{20}$$

By substituting the equations (19) and (20) into the equation (8) we find what coincides to the boundary condition according to which the stress around the border of the hole equals to zero.

The stress corresponding to (19) and (20) we get from (4) and (5) (table 1)

$$\sigma_r = \sigma_{ou}\left(1-\frac{r_1^2}{r^2}\right) \tag{21}$$

$$\sigma_\theta = \sigma_{ou}\left(1+\frac{r_1^2}{r^2}\right) \tag{22}$$

$$\tau_{r\theta} = 0 \tag{23}$$

Now we assume that the II hole is cut out (fig. 1b) and the coordinate origin is led to the point (O,C):

$$z = c + z_1 \tag{24}$$

Similar to above mentioned we can write:

$$\Phi(z_1) = \frac{\sigma_{ou}(z_1+c)}{2} = \frac{\sigma_{ou}r_1}{2} \tag{25}$$

$$X''(z_1)_2 = c\Phi''(z)_1 + X''(z)_1 \tag{26}$$

or

$$X'(z_1)_2 = c\Phi'(z)_1 + X'(z)_1 + c_1 \tag{27}$$

We get from (6), (7) and (27)

$$X'(z_1)_2 = \frac{c\sigma_{ou}}{2} - \frac{\sigma_{ou}r_1^2}{c+z_1} \tag{28}$$

$$\Phi(z_1)_2 = -\frac{1}{2\pi i}\int\left[-\sigma_{ou}t + \frac{\sigma_{ou}r_1^2}{\left(c+\frac{r_2^2}{t}\right)}\right]\frac{dt}{t-z} \tag{29}$$

Then

$$\Phi(z_1)_3 = -\frac{\sigma_{ou}r_1^2 r_2^2}{c\left(cz_1 + r_2^2\right)} \tag{30}$$

$$X'(z_1)_3 = -\frac{1}{2\pi i}\int\left(-\frac{\sigma_{ou}r_2^2}{t} - \frac{\sigma_{ou}r_1^2}{c+t}\right)\frac{dt}{t-z} + \frac{1}{2\pi i}\int\left(-\frac{\sigma_{ou}r_2^2}{t} + \frac{\sigma_{ou}r_1^2}{t(c+t)}\right)dt - \frac{\rho^2\sigma_{ou}r_1^2 r_2^2}{z_1\left(cz_1 + r_2^2\right)^2}$$

or

$$X'(z_1)_3 = -\frac{\sigma_{ou}r_2^2}{z_1} + \frac{\sigma_{ou}r_1^2}{c} - \frac{\rho^2\sigma_{ou}r_1^2 r_2^2}{z_1\left(cz_1 + r_2^2\right)^2}. \tag{31}$$

Stress along the line that connects the centers of holes:

$$\psi = \pi; \quad z_1 = \rho e^{i\theta} = -\rho$$

$$\sigma_\rho + \sigma_\psi = 2\sigma_{ou} + \frac{4\sigma_{ou}r_1^2 r_2^2}{\left(-c\rho + r_2^2\right)^2}. \tag{32}$$

Similar to (5) we can write:

$$\sigma_\psi - \sigma_\rho = 2\sigma_{ou}\left[\frac{r_2^2}{\rho^3} + \frac{r_1^2}{(c-\rho)^2} + \frac{r_1^2 r_2^2}{\left(r_2^2 - c\rho\right)^2}\right]. \tag{33}$$

We get from (32) and (33)

$$\sigma_\psi = \sigma_{ou}\left[1+\frac{r_2^2}{\rho^2}+\frac{r_1^2}{(c-\rho)^2}+\frac{3r_1^2r_2^2}{(r_2^2-c\rho)^2}\right];$$
$$\sigma_\rho = \sigma_{ou}\left[1-\frac{r_2^2}{\rho^2}-\frac{r_1^2}{(c-\rho)^2}+\frac{3r_1^2r_2^2}{(r_2^2-c\rho)^2}\right] \quad (34)$$

Consequently, stresses around the two eccentric holes (table 1-2)

$$\sigma_r = \sigma_{ou}\left[1-\frac{r_1^2}{r^2}-\frac{r_2^2}{(c-r)^2}+\frac{r_1^2r_2^2}{(cr-r_1^2)^2}\right]; \quad (35)$$

$$\sigma_\theta = \sigma_{ou}\left[1+\frac{r_1^2}{r^2}+\frac{r_2^2}{(c-r)^2}+\frac{3r_1^2r_2^2}{(cr-r_1^2)^2}\right] \quad (36)$$

Now define the stress distribution with regard to the influence of interaction of holes of radiuses r_1 and r_2:

a) stresses for the contour of the hole of r_1 radius and the influence of the hole of r_2 radius on it:

$$\Phi(z)_{r_1} = -\frac{\sigma_{ou}r_1^2r_2^2}{c(cr-l^2)} \quad (37)$$

$$\Phi(z_1)_{r_1} = -\frac{\sigma_{ou}r_1^2r_2^2}{c(cr+d^2)} \quad (38)$$

$$X'(z)_{r_1} = -\frac{r^2\sigma_{ou}r_1^2r_2^2}{z(cz-l^2)^2} \quad (39)$$

$$X'(z_1)_{r_1} = -\frac{\rho^2\sigma_{ou}r_1^2r_2^2}{z_1(cz_1+d^2)^2} \quad (40)$$

$$\Phi(z)_{r_1,r_2} = -\frac{\sigma_{ou}r_1^2r_2^2}{c(cz-l^2)} \quad (41)$$

$$X'(z)_{r_1,r_2} = -\frac{r^2\sigma_{ou}r_1^2r_2^2}{z(cz-l^2)^2}+\frac{\sigma_{ou}r_1^2r_2^2}{c(cr_1^2-l^2z)} \quad (42)$$

The increment of stress under the influence of the II hole on the I hole

$$\Delta\sigma_r = \frac{\sigma_{ou}r_1^2r_2^2}{(cr-l^2)^2}-\frac{\sigma_{ou}r_1^2r_2^2}{\left(cr-l^2\frac{r}{r_1}\right)^2} \quad (43)$$

$$\Delta\sigma_\theta = \frac{3\sigma_{ou}r_1^2r_2^2}{(cr-l^2)^2}+\frac{\sigma_{ou}r_1^2r_2^2}{\left(cr_1-l^2\frac{r}{r_1}\right)^2} \quad (44)$$

By summing (35), (36) to (43) and (44) we get

$$\sigma_r^* = \sigma_r + \Delta\sigma_r = \sigma_{ow}\left[1 - \frac{r_1^2}{r^2} - \frac{r_2^2}{(c-r)^2} + \frac{r_1^2 r_2^2}{\left(cr - r_1^2\right)^2} + \frac{r_1^2 r_2^2}{\left(cr - l^2\right)^2} - \frac{r_1^2 r_2^2}{\left(cr_1 - l^2\frac{r}{r_1}\right)^2}\right] \quad (45)$$

$$\sigma_\theta^* = \sigma_\theta + \Delta\sigma_\theta = \sigma_{ow}\left[1 + \frac{r_1^2}{r^2} + \frac{r_2^2}{(c-r)^2} + \frac{r_1^2 r_2^2}{\left(cr - r_1^2\right)^2} + \frac{r_1^2 r_2^2}{\left(cr - l^2\right)^2} - \frac{r_1^2 r_2^2}{\left(cr_1 - l^2\frac{r}{r_1}\right)^2}\right] \quad (46)$$

b) stresses for the contour of the II hole and the influence of the I hole on it:

$$\Phi(z_1)_{r_2,r_1} = -\frac{\sigma_{ow} r_1^2 r_2^2}{c\left(cz_1 + d^2\right)} \quad (47)$$

$$X'(z_1)_{r_2,r_1} = -\frac{\rho^2 \sigma_{ow} r_1^2 r_2^2}{z_1\left(cz_1 - d^2\right)^2} + \frac{\sigma_{ow} r_1^2 r_2^2 z_1}{c\left(cr_2^2 - d^2 z_1\right)} \quad (48)$$

Analogously, the stress increment

$$\Delta\sigma_\rho = \frac{\sigma_{ow} r_1^2 r_2^2}{\left(c\rho - d^2\right)^2} - \frac{\sigma_{ow} r_1^2 r_2^2}{\left(cr_2^2 - d^2\frac{\rho}{r_2}\right)^2} \quad (49)$$

$$\Delta\sigma_\psi = \frac{3\sigma_{ow} r_1^2 r_2^2}{\left(c\rho - d^2\right)^2} + \frac{\sigma_{ow} r_1^2 r_2^2}{\left(cr_2^2 - d^2\frac{\rho}{r_2}\right)^2} \quad (50)$$

If we sum (34) with (49) and (50) we get (table 3)

$$\sigma_\rho^* = \sigma_\rho + \Delta\sigma_\rho =$$

$$= \sigma_{ow}\left[1 - \frac{r_2^2}{\rho^2} - \frac{r_1^2}{(c-\rho)^2} + \frac{r_1^2 r_2^2}{\left(r_2^2 - c\rho\right)^2} + \frac{r_1^2 r_2^2}{\left(c\rho - d^2\right)^2} - \frac{r_1^2 r_2^2}{\left(cr_2 - d^2\frac{\rho}{r_2}\right)^2}\right] \quad (51)$$

$$\sigma_\theta^* = \sigma_\psi + \Delta\sigma_\psi =$$

$$= \sigma_{ow}\left[1 + \frac{r_2^2}{\rho^2} + \frac{r_1^2}{(c-\rho)^2} + \frac{3r_1^2 r_2^2}{\left(r_2^2 - c\rho\right)^2} + \frac{3r_1^2 r_2^2}{\left(c\rho - d^2\right)^2} + \frac{r_1^2 r_2^2}{\left(cr_2 - d^2\frac{\rho}{r_2}\right)^2}\right] \quad (52)$$

Consequently, equations (45) and (46) define the stress distribution in contour of eccentric holes, and (51) and (52) - mutual influence of these holes (table 4).

Consider the stress distribution in an elastic element with eccentrically arranged hole of radius r_3 on an example

$$\sigma^{*}_{\theta 3} = \sigma_{ош}\left[1 + \frac{r_3^2}{r^2} + \frac{r_1^2}{(c_2 - r)^2} + \frac{3r_1^2 r_3^2}{(c_2 r - r_3^2)^2} + \frac{3r_1^2 r_2^2}{(cr - l^2)^2} + \frac{r_1^2 r_3^2}{\left(c_2 r_3 - l_3^2 \frac{r}{r_3}\right)^2}\right]$$

$$\sigma^{*}_{r3} = \sigma_{ош}\left[1 - \frac{r_3^2}{r^2} - \frac{r_1^2}{(c_2 - r)^2} + \frac{r_1^2 r_3^2}{(c_2 r - r_3^2)^2} + \frac{r_1^2 r_3^2}{(c_2 r - l_3^2)^2} - \frac{r_1^2 r_3^2}{\left(c_2 r_3 - l_3^2 \frac{r}{r_3}\right)^2}\right]$$

$$\sigma^{*}_{\psi 3} = \sigma_{ош}\left[1 + \frac{r_3^2}{\rho^2} + \frac{r_3^2}{(c_2 - \rho)^2} + \frac{3r_3^2 d^2}{(r_2^2 - c_2\rho)^2} + \frac{3r_3^2 r_2^2}{(c_2\rho - d^2)^2} + \frac{r_3^2 r_2^2}{\left(c_2 r_2 - d^2 \frac{\rho}{r_2}\right)^2}\right]$$

$$\sigma^{*}_{\rho 3} = \sigma_{ош}\left[1 - \frac{r_2^2}{\rho^2} - \frac{r_3^2}{(c_2 - \rho)^2} + \frac{r_3^2 r_2^2}{(r_2^2 - c_2\rho)^2} + \frac{r_3^2 r_2^2}{(c_2\rho - d^2)^2} - \frac{r_3^2 r_2^2}{\left(c_2 r_2 - d^2 \frac{\rho}{r_2}\right)^2}\right]$$

The first approximation for the stresses around the hole:

Table 1.

$r/r_1 \rightarrow$	1,0 3,0	2,0 2,5	3,0 2,0	4,0 1,5	5,0 1,0	ρ/r_2
$\sigma_\theta/\sigma_{ош}$	2,44	1,48	1,39	1,52	2,05	
$\sigma_\theta/\sigma_{ош}$	0,09	0,61	0,65	0,51	-0,03	

The second approximation

Table 2.

r/r_1	1,0	2,0	3,0	4,0	5,0
σ_θ	$2,444\sigma_{ош}$	$1,482\sigma_{ош}$	$1,39\sigma_{ош}$	$1,522\sigma_{ош}$	$2,055\sigma_{ош}$
σ_r	0	$0,614\sigma_{ош}$	$0,649\sigma_{ош}$	$0,498\sigma_{ош}$	$-0,035\sigma_{ош}$
$\Delta\sigma_\theta$	$0,012\sigma_{ош}$	$0,015\sigma_{ош}$	$0,024\sigma_{ош}$	$0,039\sigma_{ош}$	$0,12\sigma_{ош}$
$\Delta\sigma_r$	0	$0,004\sigma_{ош}$	$0,008\sigma_{ош}$	$0,013\sigma_{ош}$	$0,04\sigma_{ош}$
σ_θ	$2,456\sigma_{ош}$	$1,497\sigma_{ош}$	$1,415\sigma_{ош}$	$1,561\sigma_{ош}$	$2,17\sigma_{ош}$
σ_r	0	$0,618\sigma_{ош}$	$0,657\sigma_{ош}$	$0,511\sigma_{ош}$	$0,005\sigma_{ош}$

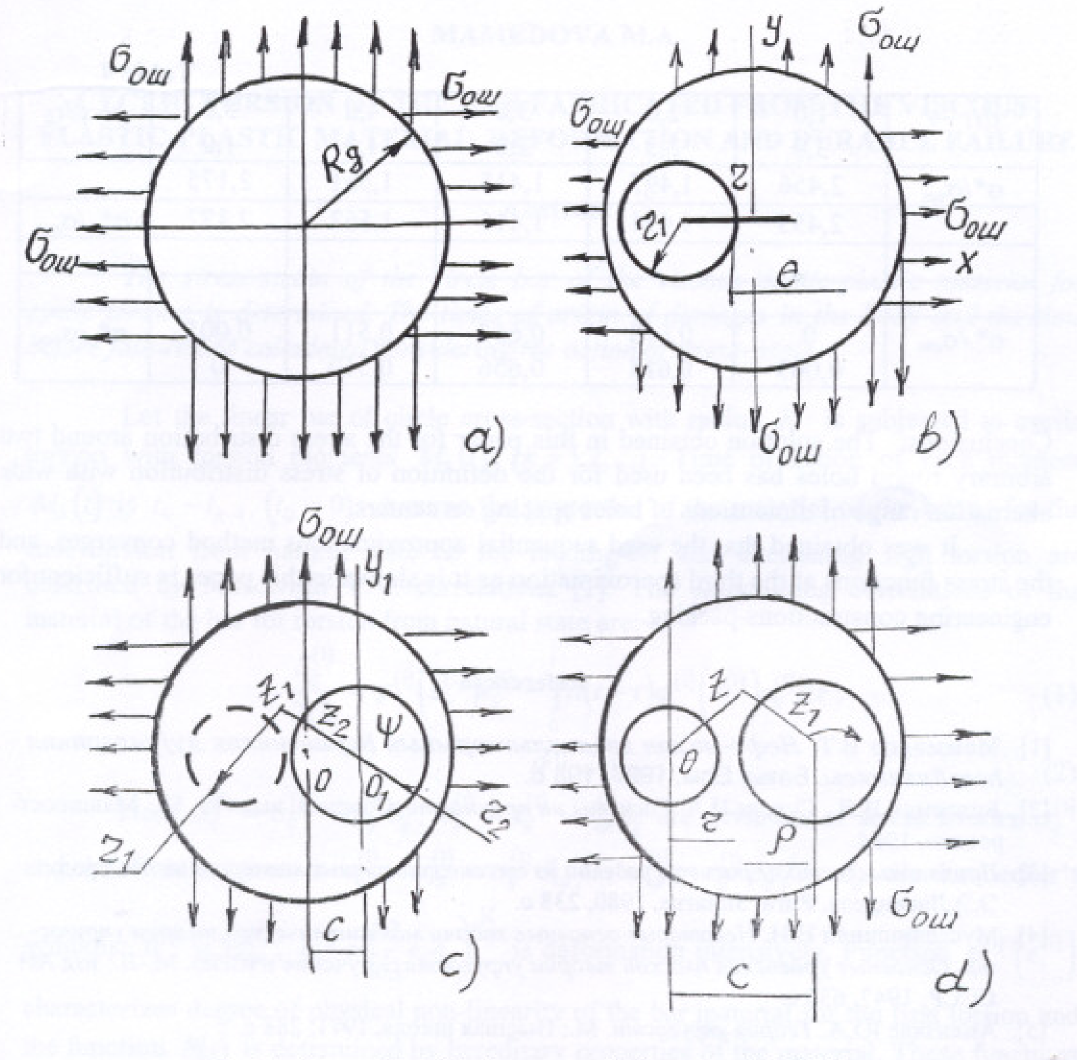

Fig.1. Stress distribution around the eccentric holes

a) without holes;
b) with a hole;
c) with two holes;
d) around the holes and under their mutual influence.

The third approximation

Table 3.

ρ/r_2	3,0	2,5	2,0	1,5	1,0
σ_ψ	$2{,}120\sigma_{\text{ош}}$	$1{,}422\sigma_{\text{ош}}$	$1{,}385\sigma_{\text{ош}}$	$1{,}54\sigma_{\text{ош}}$	$2{,}546\sigma_{\text{ош}}$
σ_ρ	$-1{,}08\sigma_{\text{ош}}$	$0{,}594\sigma_{\text{ош}}$	$0{,}647\sigma_{\text{ош}}$	$0{,}506\sigma_{\text{ош}}$	0
$\Delta\sigma_\psi$	$0{,}333\sigma_{\text{ош}}$	$0{,}072\sigma_{\text{ош}}$	$0{,}031\sigma_{\text{ош}}$	$0{,}016\sigma_{\text{ош}}$	$0{,}012\sigma_{\text{ош}}$
$\Delta\sigma_\rho$	$0{,}111\sigma_{\text{ош}}$	$0{,}024\sigma_{\text{ош}}$	$0{,}009\sigma_{\text{ош}}$	$0{,}004\sigma_{\text{ош}}$	0
σ_ψ	$0{,}453\sigma_{\text{ош}}$	$1{,}494\sigma_{\text{ош}}$	$1{,}416\sigma_{\text{ош}}$	$1{,}562\sigma_{\text{ош}}$	$2{,}172\sigma_{\text{ош}}$
σ_ρ	$0{,}003\sigma_{\text{ош}}$	$0{,}618\sigma_{\text{ош}}$	$0{,}656\sigma_{\text{ош}}$	$0{,}510\sigma_{\text{ош}}$	0

The results of comparison

Table 4.

$r/r_1 \rightarrow$	1,0 3,0	2,0 2,5	3,0 2,0	4,0 1,5	5,0 1,0	$\leftarrow \rho/r_2$
$\sigma^*/\sigma_{\text{ош}}$	2,456	1,497	1,415	1,561	2,175	
	2,453	1,494	1,416	1,562	2,172	$\sigma^*_\psi/\sigma_{\text{ош}}$
$\sigma^*_r/\sigma_{\text{ош}}$	0 0,003	0,618 0,618	0,657 0,656	0,511 0,510	0,005 0	$\sigma^*_\rho/\sigma_{\text{ош}}$

Conclusions: The solution obtained in this paper for the stress distribution around two arbitrary round holes has been used for the definition of stress distribution with wide alternation range of dimensions of holes spacing on centers.

It was obtained that the used sequential approximations method converges, and the stress functions at the third approximation as it is shown in this paper is sufficient for engineering constructions-packing.

References

[1]. Мәммәдов В.Т. *Нефт-мә'дән аваданлыгларынын һерметиклик дүјүнләринин һесабланмасы.* Бакы: Елм, 1997, 198 с.

[2]. Бидерман В.Л., Сухова Н.А. *Расчеты на прочность.* Сборник, вып. 13, М.: Машиностроение, 1964.

[3]. *Прикладные методы расчета изделий из высокоэластичных материалов.* Под ред. Э.Э.Лавендела. Рига: Зинатне, 1980, 238 с.

[4]. Мусхелишвили Н.И. *Некоторые основные задачи математической теории упругости. Основные уравнения плоской теории упругости.* Кручение и изгиб. М.-Л.: изд.АН СССР, 1947, 635 с.

[5]. Амензаде Ю.А. *Теория упругости.* М.: Высшая школа, 1971, 288 с.

Mamedov V.T.
Azerbaijan State Oil Academy.
20, Azadlyg av., 370601, Baku, Azerbaijan.
Tel.: 98-49-37.

Received June 3, 1999; Revised September 31, 1999.
Translated by Aliyeva E.T.