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CARTAN TYPE CRITERION FOR SOLVABILITY OF SUBALGEBRAS OF ASSOCIATIVE BANACH ALGEBRA

Abstract

For finite-dimensional subalgebras of associative Banach algebra, an analogue of Cartan type criterion for solvability of matrix subalgebras is proved.

We remind the classical result on solvability of Lie algebras of matrixes known as Cartan criterion.

Theorem 1. ([1,P.1, Ch. 5, §7]) *Let k be a filed of zero characteristic, V be a finite-dimensional vector space over k and $\wp$ be a Lie subalgebra of all linear transformations $B(V)$ on the space V. The following conditions are equivalent:*

(i) $\wp$ is solvable Lie algebra;

(ii) $tr_V(x \cdot y)=0$ for all $x \in \wp$ and $y \in [\wp,\wp]$, where tr_V means the trace of a matrix.

In this note we shall consider finite-dimensional Lie subalgebras of a complex (associative) Banach algebra A (in particular, algebra of all bounded linear operators acting on a complex Banach space) and prove the analogue of the theorem 1 for these Lie subalgebras only with replacement of the condition $tr_V(x \cdot y)=0$ in (ii) by $\rho_A(x \cdot y)=0$, where ρ_A is the spectral radius with respect to A. Moreover, the theorem 1 for $k=\mathbf{C}$ (the field of complex numbers) is easily obtained from the last one.

Theorem 2. *Let A be a complex Banach algebra and $\wp$ be a finite-dimentional Lie subalgebra in A. The following conditions are equivalent:*

(i) $\wp$ is solvable;

(ii) $\rho_A(x \cdot y)=0$ for all $x \in \wp$, $y \in [\wp,\wp]$.

Proof. Let $\wp$ be a solvable Lie algebra and L be its closed associative envelope in A. By [3] we get that L is commutative modulo Jacobson radial $\mathrm{Rad}\, L$. In particular, $[\wp,\wp] \subset \mathrm{Rad}\, L$. Then $x \cdot y \in \mathrm{Rad}\, L$, where $x \in \wp$, $y \in [\wp,\wp]$.

Back, let we have the condition (ii). Then for any $y \in [\wp,\wp]$ we have $\rho_A(y^2)=0$, i.e. y is quasinilpotent element of A. The operator of adjoined representation $\mathrm{ad}(y) \in B(A)$ is equal to the difference $L_y - R_y$ of commuting quasinilpotent multiplication operators and consequently is quasinilpotent. Then by finite dimensionality of $\wp$ the operator $\mathrm{ad}(y) \in B(\wp)$ is nilpotent for any $y \in [\wp,\wp]$. By Engel theorem we get that $[\wp,\wp]$ is nilpotent Lie algebra and consequently $\wp$ is solvable. The theorem is proved.

Corollary 3. Let $\Im$ be a finite-dimensional solvable Lie subalgebra of a associative Banach algebra B and A its closed associative envelope in B. Then any finite-dimensional Lie subalgebra $\wp \subseteq A$ is also solvable.

Proof. By [3] the algebra A is commutative modulo the radial $\mathrm{Rad}\, A$. Then $[\wp,\wp]\subseteq \mathrm{Rad}\, A$. Therefore $x \cdot y$ is quasinilpotent for all $x \in \wp$, $y \in [\wp,\wp]$. By the theorem 2 $\wp$ is solvable Lie algebra. The corollary is proved.

Now we are going to show that how the particular case of the theorem 1 (for $k = \mathbf{C}$) is obtained from the theorem 2 by using the following proposition 4 given below. Firstly let us remind some concepts from the theory of replicas [2, Chapter 4]. Let V be a finite-dimensional space over the field k, V^* be the dual space to V, $V_{r,s}$ be the tensor product $\underbrace{V\otimes\ldots\otimes V}_{r}\otimes\underbrace{V^*\otimes\ldots\otimes V^*}_{s}$. The spaces $V_{r,s}$ are considered to be $B(V)$- module by means of the natural way of extending of the representation (see [2, Ch.3, section 5]). The image of the operator $T \in B(V)$ at this representation we shall denote by $T_{r,s}$. One remind [2,Ch.4, item 2], that $T' \in B(V)$ is a replica of the operator T, if $\ker(T_{r,s}) \subseteq \ker(T'_{r,s})$ for all r,s. The Lie subalgebra $\wp \subseteq B(V)$ is said to be algebraic, if together with any operator $T \in \wp$ it contains every of its replicas T'. An algebraic envelope $\overline{\wp}$ of the Lie subalgebra $\wp \subseteq B(V)$ is the intersection of all algebraic Lie subalgebras containing $\wp$. One easily checks that $[\wp,\wp]=[\overline{\wp},\overline{\wp}]$.

Proposition 4. Let $\wp$ be a Lie subalgebra in $B(V)$. The following conditions are equivalent:

(i) $X \cdot T$ *is nilpotent for all* $X \in \wp$, $T \in [\wp,\wp]$;

(ii) $tr_V(X \cdot T) = 0$ *for all* $X \in \wp$, $T \in [\wp,\wp]$.

Proof. It is obvious that the trace of the nilpotent operator is equal to zero. Now let us suppose that the condition (ii) holds. Consider the algebraic envelope $\overline{\wp}$ of the Lie subalgebra $\wp$ in $B(V)$ and let $X, Y \in \wp$ and $Z \in \overline{\wp}$. One easily checks that $tr_V([X,Y]\cdot Z) = tr_V(X \cdot [Y,Z])$. As we have been noted $[\wp,\wp]=[\overline{\wp},\overline{\wp}]$. It follows that $[Y,Z] \in [\wp,\wp]$. Therefore $tr_V([X,Y]\cdot Z)=0$. Thus, $tr_V(T \cdot Z)=0$ for all $T \in [\wp,\wp]$ and $Z \in \overline{\wp}$. In particular, $tr(T \cdot T')=0$ for any replica T' of the operator $T \in [\wp,\wp]$. By using [2, Ch.4, theorem 1] we get that T is nilpotent operator. Thus, $[\wp,\wp]$ consists of the nilpotent operators, i.e. $\wp$ is solvable Lie algebra. Further by the Lie theorem there exists a flag $\{V_i : 1 \le i \le n\}$ of invariant subspaces with respect to $\wp$. It is obvious, that the element $T \in [\wp,\wp]$ annuls the gaps V_i / V_{i+1}. Therefore the operator $X \cdot T$ has zero spectrum for any $X \in \wp$. The proposition is proved.

It follows that for $k = \mathbf{C}$ the theorem 1 is deduced from the theorem 2.

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