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THE SCHRÖDINGER INVERSE SINGULAR PROBLEM

Abstract

The whole solution of the inverse problem by two spectrums on the finite interval for one class of singular potentials with nonintegrable singularity at the beginning of interval was shown at this paper.

1. Consider boundary value problems generated on the interval $(0;\pi)$ of one-dimensional Schrödinger equation:

$$-y''(x)+q(x)y(x)=\lambda^2 y(x) \quad (\lambda^2=\mu) \tag{1}$$

with singular complex potential

$$q(x)=\sum_{i=1}^{m}\frac{A_i}{x^{p_i}}+q_0(x), \tag{2}$$

where $A_i, p_i \in (1;2), (i=\overline{1,m})$ are arbitrary real numbers and $q_0(x)\in L_2[0;\pi]$, and with separable boundary conditions of the form

$$y(0)=y(\pi)=0 \tag{3}$$

or,

$$y(0)=y'(\pi)=0\,. \tag{4}$$

By virtue of [1,2] for eigenvalues of the boundary value problems (1), (3) and (1), (4) the following asymptotic formulas are correct:

$$\mu_k=\lambda_k^2, \nu_k=s_k^2, (k=1,2,\ldots),$$

where

$$\lambda_k=k\frac{1}{\pi k}\int_0^{\pi}q(t)\sin^2 ktdt+O(r_k^2), \tag{5}$$

$$s_k k-\frac{1}{2}+\frac{1}{\pi\left(k-\frac{1}{2}\right)}\int_0^{\pi}q(t)\sin^2\left(k-\tfrac{1}{2}\right)dt+O(\tilde{r}_k^2), \tag{6}$$

where

$$r_k=\int_0^{1/_{k-1}}t|q(t)|dt+\int_{1/_{k+1}}^{\pi}|q(t)|dt, \tilde{r}_k=\frac{1}{k}+r_k\,.$$

Formulas (4) and (5) was obtained for the potentials that fulfills to the condition

$$\int_0^{\pi}x|q(x)|dx<\infty\,.$$

<u>Theorem 1.</u> *For the eigenvalues in case of potential (2) asymptotic formulas accurate in the form:*

$$\lambda_k k+\frac{1}{\pi}\sum_{i=1}^{m}\frac{A_iC_{p_i}}{k^{2-p_i}}-\frac{1}{2\pi k}\left[\sum_{i=1}^{m}\frac{A_i}{(p_i-1)\pi^{p_i-1}}+\int_0^\pi q_0(t)[1-\cos 2kt]dt\right]+O\left(\frac{1}{k^{4-2p}}\right) \quad (7)$$

$$s_k=k-\tfrac{1}{2}+\frac{1}{\pi}\sum_{i=1}^{m}\frac{A_iC_{p_i}}{\left(k-\tfrac{1}{2}\right)^{2-p_i}}-\frac{1}{2\pi\left(k-\tfrac{1}{2}\right)}\left[\sum_{i=1}^{m}\frac{A_i}{(p_i-1)\pi^{p_i-1}}+\right.$$
$$\left.+\int_0^\pi q_0(t)[1-\cos(2k-1)t]dt\right]+O\left(\frac{1}{k^{4-2p}}\right) \quad (8)$$

where $p=\max_i p_i, C_{p_i}=\int_0^\infty\frac{\sin^2\xi}{\xi^{p_i}}d\xi=\frac{2^{p_i-3}\pi}{(p_i-1)\Gamma(p_i-1)\sin\left[\frac{\pi(p_i-1)}{2}\right]}.$

Proof. Substitute potential into (4), we obtain

$$\lambda_k=k+\frac{1}{\pi k}\int_0^\pi\left[\sum_{k=1}^{m}\frac{A_i}{t^{p_i}}+q_0(t)\right]\sin^2 kt dt+O\left\{\left[\int_0^{1/k-1}\left[\sum_{k=1}^{m}\frac{A_i}{t^{p_i}}+q_0(t)\right]dt+\right.\right.$$
$$\left.\left.+\frac{1}{k}\int_{1/k-1}^{\pi}\left[\sum_{k=1}^{m}\frac{A_i}{t^{p_i}}+q_0(t)\right]dt\right]^2\right\}. \quad (9)$$

Therefore for the residue member we obtain

$$O\left\{\left[\int_0^{1/k-1}\frac{dt}{t^{p-1}}+\frac{1}{k}\int_{1/k+1}^{\pi}\frac{dt}{t^p}+\int_0^{1/k-1}t|q_0(t)|dt+\frac{1}{k}\int_{1/k+1}^{\pi}|q_0(t)|dt\right]^2\right\}=$$
$$=O\left\{\left[\frac{1}{k^{2-p}}+\sqrt{\int_0^{1/k-1}t^2dt}\sqrt{\int_0^{1/k-1}q_0^2(t)dt}\right]^2\right\}=O\left(\frac{1}{k^{4-2p}}\right) \quad (10)$$

where $p=\max p_i$.

$$\int_0^\pi\frac{\sin^2 kt}{t^{p_i}}dt\overset{nt=\xi}{=}n^{p_i-1}\left[\int_0^\infty\frac{\sin^2\xi}{\xi^{p_i}}d\xi-\int_{n\pi}^\infty\frac{1-\cos 2\xi}{2\xi^{p_i}}d\xi\right]=$$
$$=n^{p_i-1}C_{p_i}-\frac{1}{2(p_i-1)\pi^{p_i-1}}+O\left(\frac{1}{n}\right). \quad (11)$$

Substitute (10) and (11) into the formula (9), we obtain desired formula (7) of the theorem 1. The formula (8) can be proved similarly.

It must be noted, that from the classical theorems on oscillation follows alternative property of this eigenvalues, more exactly: $-\infty<\nu_1<\mu_1<\nu_2<\mu_2<\nu_3<\dots$.

2. Let us show the necessary for the solution of the inverse problem by two spectrums statements. For its solution we use method from [3], and we reduce it to the considered inverse problem of quantum theory of scattering. The following boundary value problems are considered:

$$-y''(x)+q(x)y(x)=\lambda^2 y(x) \qquad (0\le x<\infty) \quad (12)$$

$$y(0)=0 \quad (13)$$

which have properties:

1) $q(x)=\begin{cases} q_1(x)+q_0(x), & 0\le x\le \pi \\ 0, & x>\pi \end{cases}$ (14)

the discrete spectrum is absent.

<u>Theorem 2.</u> *For arbitrary function $S(\lambda)(-\infty<\lambda<\infty)$ to be the scattering function of some boundary value problem (12), (13) satisfying to the conditions 1), 2) it is necessary and sufficient the fulfillment of the following conditions:*

I.Function $S(\lambda)(-\infty<\lambda<\infty)$ is continuous on the whole axis,

$$S(\lambda)=\overline{S(-\lambda)}=[S(-\lambda)]^{-1}$$

and

$$1-S(\lambda)\overset{|\lambda|\to\infty}{\to} 0$$

and is the Fourier transformation of function $F_s(x)=\frac{1}{2}\int_{-\infty}^{\infty}[1-S(\lambda)]e^{i\lambda x}dx$, that can be represented in the form of the sum of two functions, one of which belong to space $L_1(-\infty,\infty)$ and other one is bounded and belong to $L_2(-\infty,\infty)$. On the positive half axis function $F_s(x)$ have a derivative, satisfying the condition $\int_0^{\infty}x|F_s'(x)|dx<\infty$.

I.For the argument of function $S(\lambda)$ the following equality is true

$$\frac{\ln s(+0)-\ln s(+\infty)}{\pi i}=\frac{1-s(0)}{2}.$$

II.Function $F_s(x)=0$ for $x>2\pi$ and is differentiable on interval $(0;2\pi)$ and $\left[F_s'(x)-q_1\left(\frac{x}{2}\right)\right]\in L_2[0;2\pi]$.

The proof of the theorem 2 was considered in the paper [4]. From [3] we need two lemmas.

<u>Lemma 1.</u> *For all λ from the closed upper halfplane, the equation (12) have solution in the form $e(\lambda,x)$, which represented in the form:*

$$e(\lambda,x)=e^{i\lambda x}+\int_x^{\infty}K(x,t)e^{i\lambda t}dt, \tag{15}$$

where the kernel K(x,t) satisfies to the inequality

$$|K(x,t)|\le\frac{1}{2}\sigma\left(\frac{x+t}{2}\right)\exp\left\{\sigma_1(x)-\sigma_1\left(\frac{x+t}{2}\right)\right\},$$

where

$$\sigma(x)=\int_x^{\infty}|q(t)|dt,\ \sigma_1(x)=\int_x^{\infty}\sigma(t)dt,\ K(x,x)=\frac{1}{2}\int_x^{\infty}q(t)dt.$$

2. <u>Lemma 2.</u> *For all functions $f(x)\in L_1(0;\pi)$ the following inequalities are true:*

$$\lim_{|\lambda|\to\infty}e^{-|\operatorname{Im}\lambda|\pi}\int_0^{\pi}f(x)\cos\lambda x dx=\lim_{|\lambda|\to\infty}e^{-|\operatorname{Im}\lambda|\pi}\int_0^{\pi}f(x)\sin\lambda x dx=0 \tag{16}$$

<u>Theorem 3.</u> *If in the equation (12) the potential of the type (14), then*

$$e(0,\lambda)=e^{i\lambda\pi}[s'(\pi,\lambda)-i\lambda s(\pi,\lambda)], \tag{17}$$

where $s(x,\lambda)$ is a solution of equation (12): $s(0,\lambda)=0,\ s'(0,\lambda)=1$.

Proof. So as functions $s(x,\lambda)$ and $c(x,\lambda)$, $(c(0,\lambda)=1)$ makes fundamental system of solutions of equation (12) with potential (14), then solution $e(x,\lambda)$ of this equation is equal to the linear combination:

$$e(x,\lambda)=C_1 s(x,\lambda)+C_2 c(x,\lambda).$$

By virtue of (15) potential is equal to zero for $x>\pi$, therefore $e(x,t)=e^{i\lambda x}$ for $x\geq\pi$ and at the point $x=\pi$ the following inequalities are fulfilled:

$$e^{i\lambda\pi}(x,\lambda)=C_1 s(x,\lambda)+C_2 c(x,\lambda),\ i\lambda e^{i\lambda\pi}=C_1 s'(\pi,\lambda)+C_2 c'(\pi,\lambda).$$

From these inequalities we determine C_1 and C_2, and obtain:

$$e(x,\lambda)=e^{i\lambda\pi}\{[i\lambda c(\pi,\lambda)-c'(\pi,\lambda)]s(x,\lambda)+[s'(\pi,\lambda)-i\lambda s(\pi,\lambda)]c(x,\lambda)\}.$$

And therefore formula (17) is obvious.

Theorem 4. *For the functions $u(z), v(z)$ to allow the representation*

$$u(z)=\sin\pi z+\int_0^\pi\frac{\sin z(\pi-t)}{z}q(t)\sin zt\,dt+A\pi\frac{4z\cos\pi z}{4z^2-1}+\frac{f(z)}{z} \tag{18}$$

$$v(z)=\cos\pi z+\int_0^\pi\frac{\sin z(\pi-t)}{z}q(t)\sin zt\,dt-B\pi\frac{\sin\pi z}{z}+\frac{g(z)}{z} \tag{19}$$

where $q(t)=\sum_{i=1}^{m}\frac{A_i}{t^{p_i}}+q_0(t)$, *$A, B, A_i$ are constant numbers,*

$$p_i\in\left(1;\tfrac{5}{4}\right),\ q_0(t)\in L_2[0;\pi]$$

$$f(z)=\int_0^\pi\tilde{f}(t)\cos zt\,dt,\ \tilde{f}(t)\in L_2[0;\pi],\ \int_0^\pi\tilde{f}(t)dt=0,$$

$$g(z)=\int_0^\pi\tilde{g}(t)\sin zt\,dt,\ \tilde{g}(t)\in L_2[0;\pi].$$

It is necessary and sufficient that,

$$u(z)=\pi z\prod_{k=1}^{\infty}\frac{u_k^2-z^2}{k^2},\ v(z)=\prod_{k=1}^{\infty}\frac{v_k^2-z^2}{\left(k-\tfrac{1}{2}\right)^2}, \tag{20}$$

where

$$u_k=k+\frac{1}{\pi}\sum_{i=1}^{m}\frac{A_iC_{p_i}}{k^{2-p_i}}-\frac{1}{k}\left[A+\sum_{i=1}^{m}\frac{A_i}{2(p_i-1)\pi^{p_i}}\right]+\frac{a_k}{k},$$

$$v_k=k-\frac{1}{2}+\frac{1}{\pi}\sum_{i=1}^{m}\frac{A_iC_{p_i}}{\left(k-\tfrac{1}{2}\right)^{2-p_i}}-\frac{1}{k-\tfrac{1}{2}}\left[B+\sum_{i=1}^{m}\frac{A_i}{2(p_i-1)\pi^{p_i}}\right]+\frac{b_k}{k-\tfrac{1}{2}}$$

and moreover, $\sum_{k=1}^{\infty}|a_k|^2+|b_k|^2<\infty$.

The proof of this theorem is similar to analogous at paper [5]. In the theorem 4 functions *u(z)* and *v(z)* are taken as analogies of functions $zs(\pi,z)$ and $s'(\pi,z)$ correspondingly.

3. Theorem 5. *For two sequences of real numbers $\{\mu_k\}, \{\nu_k\}, (k=1,2,\ldots)$ to be spectrums of boundary value problems, generated by the same Schrödinger equation*

$$-y''(x)+q(x)y(x)=\mu y(x),\ (0\leq x\leq\pi)$$

and with real potential

$$q(x)=\sum_{i=1}^{m}\frac{A_i}{x^{p_i}}+q_0(x),$$

A_i are real constants, $p_i\in\left(1;\frac{5}{4}\right)$, $q_0(x)\in L_2[0;\pi]$, it is necessary and sufficient for these functions to alternate and satisfies to asymptotic formulas:

$$\mu_k=k^2+\frac{2}{\pi}\sum_{i=1}^{m}A_iC_{p_i}k^{p_i-1}-2A+a_k, \tag{21}$$

$$\nu_k=\left(k-\frac{1}{2}\right)^2+\frac{2}{\pi}\sum_{i=1}^{m}A_iC_{p_i}\left(k-\frac{1}{2}\right)^{p_i-1}-2A+b_k \tag{22}$$

where $C_{p_i}=\int_0^{\pi}\frac{\sin^2\xi}{\xi^{p_i}}d\xi$, A is arbitrary real number and $\sum_{k=1}^{\infty}\left|a_k\right|^2+\left|b_k\right|^2<\infty$.

Proof. The necessity of these conditions was state at the previous points. Without loose of generality, we can suppose that all numbers μ_k,ν_k are positive. Asymptotic formulas (21), (22) are equivalent to the formulas:

$$\sqrt{\mu_k}=\left[k^2+\frac{2}{\pi}\sum_{k=1}^{m}A_iC_{p_i}k^{p_i-1}-2A+a_k\right]^{\frac{1}{2}}=$$

$$=k+\frac{1}{\pi}\sum_{k=1}^{m}\frac{A_iC_{p_i}}{k^{2-p_i}}-\frac{A}{k}+\frac{a_k}{2k}+O\left[\frac{1}{k^{5-2p}}\right]= \tag{23}$$

$$=k+\frac{1}{\pi}\sum_{k=1}^{m}\frac{A_iC_{p_i}}{k^{2-p_i}}-\frac{A}{k}+\frac{\widetilde{a}_k}{2k}$$

and by analogous calculations:

$$\sqrt{\nu_k}=k-\frac{1}{2}+\frac{1}{\pi}\sum_{i=1}^{m}\frac{A_iC_{p_i}}{\left(k-\frac{1}{2}\right)^{2-p_i}}-\frac{A}{k-\frac{1}{2}}+\frac{\widetilde{b}_k}{k-\frac{1}{2}}, \tag{24}$$

where $\sum_{k=1}^{\infty}\left|\widetilde{a}_k\right|^2+\left|\widetilde{b}_k\right|^2<\infty$.

Suppose that $A=\widetilde{A}+\sum_{i=1}^{m}\frac{A_i}{2(p_i-1)\pi^{p_i}}$ and construct by given sequences $\{\mu_k\},\{\nu_k\}$ the functions

$$zs(z)=\pi z\prod_{k=1}^{\infty}\frac{\mu_k-z^2}{k^2},\ s_1(z)=\prod_{k=1}^{\infty}\frac{\nu_k-z^2}{\left(k-\frac{1}{2}\right)^2} \tag{25}$$

and suppose also

$$e(z)=e^{iz\pi}\left[s_1(z)-izs(s)\right] \tag{26}$$

Using for asymptotic formulas (23), (24) the theorem 4, from (26) we obtain:

$$e(z)=1-\frac{i\widetilde{A}\pi}{z}+\frac{1}{2z}\sum_{i=1}^{m}A_i\int_0^{\pi}\frac{\sin 2zt}{t^{p_i}}dt+\frac{i}{z}\sum_{i=1}^{m}A_i\int_0^{\pi}\frac{\sin^2 zt}{t^{p_i}}dt-$$
$$-\frac{e^{iz\pi}}{z}\left[\frac{i\widetilde{A}}{4z^2-1}\cos\pi z-g(z)+if(z)\right]. \tag{27}$$

By virtue of Lemma 2 from the last formula for $|z|\to\infty$ we have:

$$e(z)=1+O\left\{\frac{1}{|z|}\left[1+\int_0^\pi \frac{\sin 2zt}{t^{\max p_l}}dt+\int_0^\pi \frac{\sin^2 zt}{t^{\max p_l}}dt\right]\right\}.$$

It is easy to show

$$e(z)=1+O\left[\frac{e^{2|\mathrm{Im}|\pi}}{|z|^{2-p}}\right]. \tag{28}$$

Further, from alternative property of the roots of functions $s_1(x), xs(x)$ and formula (25) it follows that on the segments $\left[\sqrt{\mu_{k-1}},\sqrt{\mu_k}\right]$ the argument of function $s_1(x)-ixs(x)$ obtains the increment that equal to $-\pi$. Therefore, when x takes the values from $\left[-\sqrt{\mu_k},\sqrt{\mu_k}\right]$, the argument of this function obtain the increment, that equal to $2k\pi$. And the increment of argument of function $e^{iz\pi}$ on the same segment $\left[-\sqrt{\mu_k},\sqrt{\mu_k}\right.$ is equal to $2\sqrt{\mu_k}\pi$.

Thus, on the segment $\left[-\sqrt{\mu_k},\sqrt{\mu_k}\right.$ the increment of argument of the function $e(x)$ is equal to $2\pi\left[\sqrt{\mu_k}-k\right.$ and tem to zero for $k\to\infty$. Therefore, taking into account that $\lim\limits_{x\to\pm\infty} e(x)=1$ we obtain:

$$\arg e(+\infty)-\arg e(-\infty)=0. \tag{29}$$

By virtue of argument principle from this equality and asymptotic equality (28) follows that function *e(z)* have not roots in closed upper halfplane.

Now, we will show that function

$$S(\lambda)=\frac{e(-\lambda)}{e(\lambda)} \tag{30}$$

satisfies to the conditions of theorem 2.

This function is continuous, $S(\lambda)=\overline{S(-\lambda)}=[S(\lambda)]^{-1}$, $S(0)=1$. From formulas (27) and (28), by virtue of the fact that function $f(\lambda)$ is even and function $g(\lambda)$ is odd for $|\lambda|\to\infty$ we have:

$$s(\lambda)=1+\frac{\dfrac{2i\widetilde{A}\pi}{\lambda}-\dfrac{2i}{\lambda}\sum_{i=1}^{m}A_i\int_0^\pi\dfrac{\sin^2\lambda t}{t^{p_i}}dt-\dfrac{2i}{\lambda}\operatorname{Im}e^{i\lambda\pi}[g(\lambda)-if(\lambda)]+O\left(\dfrac{1}{|\lambda|^2}\right)}{1+O\left(\dfrac{1}{\lambda^{2-p}}\right)}=$$

$$=1+\frac{2i\widetilde{A}\pi}{\lambda}-\frac{2i}{\lambda}\sum_{i=1}^{m}A_i\int_0^\pi\frac{\sin^2\lambda t}{t^{p_i}}dt-\frac{2i}{\lambda}\operatorname{Im}e^{i\lambda\pi}[g(\lambda)-if(\lambda)]+O\left(\frac{1}{|\lambda|^{4-2p}}\right) \tag{31}$$

So as functions $f(\lambda)$ and $g(\lambda)$ are bounded and belong to the space $L_2(-\infty;\infty)$, then function $F_s(x)=\dfrac{1}{2\pi}\int_{-\infty}^{\infty}[1-S(\lambda)]e^{i\lambda x}d\lambda$ is bounded and belong to the space $L_2(-\infty;\infty)$. Moreover, from formula (31) follows that:

$$1-S(\lambda)=-\frac{2i\widetilde{A}\pi}{\lambda+i}+\frac{2i}{\lambda}\sum_{i=1}^{m}A_i\int_0^\pi\frac{\sin^2\lambda t}{t^{p_i}}dt+\frac{\psi(\lambda)}{1+|\lambda|},$$

where $\psi(\lambda)\in L_2(-\infty,\infty)$. Therefore for positive x:

$$F_s(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{2i\widetilde{A}\pi}{\lambda+i}+\frac{2i}{\lambda}\sum_{i=1}^{m}A_i\int_0^{\pi}\frac{\sin^2\lambda t}{t^{p_i}}dt+\frac{\psi(\lambda)}{1+|\lambda|}\right\}e^{i\lambda x}d\lambda=$$
$$=\frac{i}{\pi}\sum_{i=1}^{m}A_i\int_{-\infty}^{\infty}\frac{e^{i\lambda x}}{\lambda}\left[\int_0^{\pi}\frac{\sin^2\lambda t}{t^{p_i}}dt\right]d\lambda+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\psi(\lambda)}{1+|\lambda|}e^{i\lambda x}d\lambda. \tag{32}$$

Further,

$$\frac{i}{\pi}\sum_{i=1}^{m}A_i\int_{-\infty}^{\infty}\frac{e^{i\lambda x}}{\lambda}\left[\int_0^{\pi}\frac{\sin^2\lambda t}{t^{p_i}}dt\right]d\lambda=-\frac{2}{\pi}\sum_{i=1}^{m}A_i\int_0^{\infty}\frac{\sin\lambda t}{\lambda}\left[\int_0^{\pi}\frac{\sin^2\lambda t}{t^{p_i}}dt\right]d\lambda\overset{\lambda t=\xi}{=}$$
$$\overset{\lambda t=\xi}{=}-\frac{2}{\pi}\sum_{i=1}^{m}A_i\int_0^{\infty}\frac{\sin\lambda x}{\lambda^{2-p_i}}\left[\int_0^{2\pi}\frac{\sin^2\xi}{\xi^{p_i}}d\xi\right]d\lambda=-\frac{2}{\pi}\sum_{i=1}^{m}A_iC_{p_i}\int_0^{\infty}\frac{\sin\lambda x}{\lambda^{2-p_i}}\left[1+O\left(\frac{1}{|\lambda|^{p_i}}\right)\right]d\lambda=$$
$$=-\frac{2}{\pi}\sum_{i=1}^{m}A_iC_{p_i}\int_0^{\infty}\frac{\sin\lambda x}{\lambda^{2-p_i}}d\lambda+O\left\{\int_0^{\infty}\frac{\sin\lambda x}{\lambda^2}d\lambda\right\}$$

Substituting the last equality into (30), we obtain

$$F_s(x)+\frac{2}{\pi}\sum_{i=1}^{m}A_iC_{p_i}\int_0^{\infty}\frac{\sin\lambda x}{\lambda^{2-p_i}}d\lambda=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\widetilde{\psi}(\lambda)}{1+|\lambda|}d\lambda \tag{33}$$

and here according to the table [6, p.434] we can find:

$$\frac{2}{\pi}\sum_{i=1}^{m}A_iC_{p_i}\int_0^{\infty}\frac{\sin\lambda x}{\lambda^{2-p_i}}d\lambda=\frac{2}{\pi}\sum_{i=1}^{m}A_i\frac{2^{p_i-3}\pi}{(p_i-1)\Gamma(p_i-1)\sin\left[\frac{\pi(p_i-1)}{2}\right]}\times$$

$$\times\frac{\Gamma(p_i-1)\sin\left[\frac{\pi(p_i-1)}{2}\right]}{x^{p_i-1}}=\sum_{i=1}^{m}\frac{2^{p_i-2}A_i}{(p_i-1)x^{p_i-1}}.$$

From this and formula (33), we obtain:

$$\left[F_s'(x)-\sum_{i=1}^{m}\frac{2^{p_i-2}A_i}{x^{p_i}}\right]\in L_2[0;2\pi].$$

Finally, so as *e(-z)* and *e(z)* are entire functions and function *e(z)* have not roots in the closed upper halfplane, then at this halfplane $1-S(z)$ is holomorphic, moreover, according to (27) and (30) and lemma 2

$$1-S(z)\le\frac{C}{|z|^{2-p}}e^{2\pi\,\mathrm{Im}\,z},\ (\mathrm{Im}\,z\ge 0),$$

Therefore, by virtue of Jordan lemma follows that $F_s(x)=0$ for $x>2\pi$. Thus, function $S(\lambda)$ satisfies to the conditions I and III of theorem 2. Condition II also fulfills because from (29) and (30) we have

$$\arg s(0)-\arg s(+\infty)=\arg e(0)-\arg e(-\infty)-\arg e(0)+\arg e(+\infty)=$$
$$=\arg e(+\infty)-\arg e(-\infty)=0$$

Following, there exist real potential (14) and function $S(\lambda)$ is scattering function of bounded value problem (12), (13).

Comparing definition of scattering function and formula (30), we obtain:

$$\frac{e(0,-\lambda)}{e(0,\lambda)}=\frac{e(-\lambda)}{e(\lambda)},\ \text{or}\ \frac{e(0,-\lambda)}{e(-\lambda)}=\frac{e(0,\lambda)}{e(\lambda)}.$$

In a right (left) side of this equality there is a function, holomorphic in the closed upper (low) half plane and is uniform tending in it to unit for $|\lambda| \to \infty$. Therefore, $e(\lambda) \equiv e(0,\lambda)$ and from this, according to (26) and theorem 4, follows that

$$s_1(z) - izs(z) = s'(\pi, z) - izs(,z)$$

that means

$$s(z) = s(\pi, z),\ s_1(z) = s'(\pi, z)$$

These equalities shows that sequences $\{\mu_k\}, \{\nu_k$ was formed by the squares of roots of function $s(\pi, z)$ and $s'(\pi, z)$, i.e. is spectrums of bounded problems generated by equation

$$-y''(x) + q(x)y(x) = \mu y(x),\ (0 \le x \le \pi)$$

and by bounded conditions

$$y(0) = y(\pi) = 0 \text{, and } y(0) = y'(\pi) = 0$$

The theorem is proved.

Present proof contains also reconstruction method for potential $q(x)$ by spectrums $\{\mu_k\}, \{\nu_k\}$. Really, when we define function $s(\lambda)$, we could solve the main equation of Helfand-Levitan-Marchenco

$$F_s(x+y) + K(x,y) + \int_x^\infty K(x,t)F_s(t+y)dt = 0,\ (x \le y < \infty),$$

which have unique solution for each $x \ge 0$ under the conditions of theorem 2. We can find function *K(x,y)* after solving that equation, and after that we could find potential $q(x)$ by formula

$$q(x) = -\frac{1}{2}\frac{d}{dx}K(x,x).$$

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