1999 VOL. X(XVIII)

MATHEMATICS

ALIEV S.A.

ON SOME PROPERTIES OF THE MAIN CHARACTERISTICS OF JIRINA PROCESSES

Abstract

In the paper the dependence of the functions completely characterizing Jirina processes on the real number acting role of the parameter at passage to the limit is investigated.

First of all, remember that Jirina processes (branching random processes with continuous state space) whose definition we will give below are limited for many classes of branching processes such as Galton- Vatson processes, Bellman- Hariss processes and etc.

Let $\xi_m^{(n)}$, $m=0,1,2,\ldots$ for each natural n be Galton- Vatson branching process with generating function $F^{(n)}(S)$. Denote by A_n the mean of offspring of one particle in n-th process, that is $A_n = \left.\frac{dF^{(n)}(S)}{dS}\right|_{S=1}$ and suppose that $A_n \to 1$.

In [1] it was proved that under some conditions for the generating function, the normalized by some way branching Galton- Vatson processes

$$\mu_m^{(n)}(t)=\frac{\xi_{[mt]}^{(n)}}{b_m^{(n)}},\mu_m^{(n)}(0)=\frac{\left[xb_m^{(n)}\right]}{b_m^{(n)}},\quad n=1,2,3\ldots,\quad m=0,1,2,\ldots,\quad x>0,\ \ b_m^{(n)}\uparrow\infty$$

converge weakly to Jirina process.

So is called the homogeneous Markov process $\mu(t)$ with continuous time taking real non- negative values, if

$$M\left[e^{\lambda\mu(t)}\middle|\mu(0)=x\right]=e^{xK(t,\lambda)}.$$

Here $x\in[0,\infty)$, $\lambda\le 0$, and $K(t,\lambda)$ for each $t\ge 0$ there is a logarithm Laplace transform of some boundless divisible distribution in $[0,\infty]$ and consequently it has a view

$$K(t,\lambda)=a_t\lambda+\int_0^\infty\left(e^{\lambda x}-1\right)\Pi_t(dx), \tag{1}$$

where $a_t\ge 0$ and measure Π_t is such that integral converges. For any $t,\tau\ge 0$, $\lambda\le 0$ function $K(t,\lambda)$ satisfies the correlation

$$K(t+\tau;\lambda)=K(t;K(\tau;\lambda)).$$

To every Jirina process corresponds cumulant which is determined as

$$H(\lambda)=\lim_{\tau\to 0}\frac{K(\tau;\lambda)-\lambda}{\tau}.$$

The function $K(t,\lambda)$ satisfies the differential equation

$$\frac{\partial K(t;\lambda)}{\partial t} = H(K(t;\lambda)),\ K(0;\lambda) = \lambda . \qquad (2)$$

All these and other characteristics of Jirina processes can be found in [2].

In this work we investigate behavior of the main characteristics of Jirina processes.

Theorem. *Let* $n \to \infty, m \to \infty, m(A_n - 1) \to c$, *where* c *is any real number. Then, if*

$$mb_m^{(n)}\left[F^{(n)}\left(e^{\lambda/b_m^{(n)}}\right) - e^{\lambda/b_m^{(n)}}\right] \to H(\lambda),\quad \lambda \le 0 ,$$

then finitedimensional distributions of processes $\mu_m^{(n)}(t)$ *converge weakly to finitedimensional distribution of Jirina processes with cumulant* $H(\lambda)$ *and it takes place*

$$b_m^{(n)} P\left\{\frac{\xi_m^{(n)}}{b_m^{(n)}} \ge x\right\} \to \Pi(x)$$

in every point $x > 0$ *of continuity* Π, *where*

$$\int_0^\infty e^{\lambda x}\,\Pi(x)dx = \frac{K(\lambda)}{\lambda} - a,\quad a = \lim_{\lambda\to-\infty}\frac{K(\lambda)}{\lambda}, K(\lambda) = K(1;\lambda). \qquad (3)$$

Proof. By the results of the first part of statement of theorem, proved in [1] we can write

$$b_m^{(n)} \log F_{[mt]}^{(n)}\left(e^{\lambda/b_m^{(n)}}\right) \to K(t;\lambda)\quad \text{for}\quad n\to\infty,\ m\to\infty,\ m(A_n - 1) \to c ,$$

where $K(t,\lambda)$ satisfies differential equation (2) and has the view (1).

Assume $t = 1$. Then we obtain

$$b_m^{(n)} \log F_m^{(n)}\left(e^{\lambda/b_m^{(n)}}\right) \to K(\lambda) = K(1;\lambda)\ \text{ for }\ n\to\infty,\ m\to\infty,\ m(A_n - 1)\to c$$

or

$$b_m^{(n)}\left(F_m^{(n)}\left(e^{\lambda/b_m^{(n)}}\right) - 1\right) \to K(\lambda)\quad \text{for}\quad n\to\infty,\ m\to\infty,\ m(A_n-1)\to c ,$$

$$K(\lambda) = a\lambda + \int_0^\infty \left(e^{\lambda x} - 1\right)\Pi(dx).$$

Hence it follows that Laplace transform of function $\Pi(x)$ can be calculated by formulas (3).

Denote

$$\sigma(t;c) = \lim_{\substack{n\to\infty \\ m\to\infty \\ m(A_n-1)\to c}} \frac{b_{[mt]}^{(n)}}{b_m^{(n)}}$$

and write the obvious identity

$$\frac{b_{[mts]}^{(n)}}{b_m^{(n)}} = \frac{b_{[mts]}^{(n)}}{b_{[mt]}^{(n)}} \cdot \frac{b_{[mt]}^{(n)}}{b_m^{(n)}} . \qquad (4)$$

If t is a rational number, then it is possible to choice such subsequence $m_k \to m$, that number $m_k t$, $k = 1,2,\ldots$ was entire.

Then if in (4) we turn to the limit for $n \to \infty$, $m \to \infty$, $m(A_n - 1) \to c$, then we will obtain the following:

$$\sigma(ts,c) = \sigma(s,tc)(\sigma(t,c)). \tag{5}$$

If $c > 0$, then (5) must be fulfilled for all non- negative c. Assume $c = 1$. Then

$$\sigma(ts,1) = \sigma(s,t)\sigma(t,1) \quad \text{or}$$

$$\sigma(s,t) = \frac{\sigma(ts,1)}{\sigma(t,1)}.$$

Introduce the denotation $\sigma_+(t) = \sigma(t,1)$. In this denotation we have

$$\sigma(t,c) = \frac{\sigma_+(tc)}{\sigma_+(c)}.$$

And if $c < 0$, then by analogy we have

$$\sigma(t,c) = \frac{\sigma_-(tc)}{\sigma_-(c)},$$

where $\sigma_-(t) = \sigma(t,-1)$.

Note that $\sigma_+(1) = \sigma_-(-1) = 1$.

Further, from (5) $c = 0$ we obtain

$$\sigma(st,0) = \sigma(s,0)\sigma(t,0).$$

Hence it follows, that $\sigma(t,0) = t^{\beta}$, $\beta \geq 0$.

Therefore, finally we obtain, that if $n \to \infty$, $m \to \infty$, $m(A_n - 1) \to c$ and $t > 0$ is any real number, then the following is valid:

$$\sigma(t,c) = \begin{cases} \dfrac{\sigma_+(tc)}{\sigma_+(c)}, & for \quad c > 0, \\ \dfrac{\sigma_-(tc)}{\sigma_-(c)}, & for \quad c < 0, \\ t^{\beta}; & for \quad c = 0. \end{cases}$$

Using it we find the dependence of functions $H(\lambda)$ and $K(t,\lambda)$ on c.

We denote the dependence of functions $H(\lambda)$, $K(t,\lambda)$ and $\Pi(x)$ on c, correspondingly, by $H_c(\lambda)$, $K_c(t,\lambda)$ and $\Pi^{(c)}(x)$.

Three variants are possible: $c > 0$, $c < 0$, $c = 0$.

Let first $c > 0$. We have

$$mb_m^{(n)}\left[F^{(n)}\left(e^{\lambda/b_m^{[n]}}\right) - e^{\lambda/b_m^{(n)}}\right] \xrightarrow[\substack{n\to\infty \\ m\to\infty \\ m(A_n-1)\to c}]{} H_c(\lambda).$$

Now substitute m by $[mt]$. Then it is obvious, that $[mt](A_n - 1) \to tc$ and

$$[mt]b_{[mt]}^{(n)}\left[F\left(e^{\lambda/b_{[mt]}^{(n)}}\right) - e^{\lambda/b_{[mt]}^{(n)}}\right] \to H_{tc}(\lambda). \tag{6}$$

Hence

$$H_{tc}(\lambda) = t\frac{\sigma_+(tc)}{\sigma_+(c)} H_c\left(\lambda \frac{\sigma_+(c)}{\sigma_+(tc)}\right).$$

If assume here $c=1$ then we obtain

$$H_t(\lambda)=t\sigma_+(t)H_1(\lambda/\sigma_+(t))$$

or

$$H_c(\lambda)=c\sigma_+(c)H_+(\lambda/\sigma_+(c)),$$

where $H_+(\lambda)=H_1(\lambda)=\lim\limits_{\substack{n\to\infty\\ m\to\infty\\ m(A_n-1)\to 1}} mb_m^{(n)}\left[F^{(n)}\left(e^{\lambda/b_m^{[n]}}\right)-e^{\lambda/b_m^{[n]}}\right].$

Now let $c<0$. Then from (6) we have

$$H_{tc}(\lambda)=t\frac{\sigma_-(tc)}{\sigma_-(c)}H_c\left(\lambda\frac{\sigma_-(c)}{\sigma_-(tc)}\right).$$

Assume $c=-1$.

$$H_{-t}(\lambda)=t\sigma_-(-t)H_{-1}(\lambda/\sigma_-(-t))$$

or

$$H_c(\lambda)=-c\sigma_-(c)H_-(\lambda/\sigma_-(c)),$$

$$H_-(\lambda)=H_{-1}(\lambda)=\lim_{\substack{n\to\infty\\ m\to\infty\\ m(A_n-1)\to -1}} mb_m^{(n)}\left[F^{(n)}\left(e^{\lambda/b_m^{(n)}}\right)-e^{\lambda/b_m^{(n)}}\right].$$

If $c=0$, then from (6) we obtain

$$H_0(\lambda)=t^{1+\beta}H_0(\lambda t^{-\beta}).$$

Two cases are possible: $\beta>0, \beta=0$.

Let first $\beta>0$. Assume $\lambda=-1$

$$H_0(-1)=t^{1+\beta}H_0(-t^{-\beta}).$$

Denote $h=H_0(-1)$, $\lambda=-t^{-\beta}$. Then $H_0(\lambda)h|\lambda|^{1+\alpha}$, where $\alpha=\dfrac{1}{\beta}$. From $\alpha=\dfrac{1}{\beta}$ and from the common form of cumulant $H(\lambda)$ it follows that $0<\alpha\le 1$. If $\beta=0$, then we obtain $H_0(\lambda)\equiv 0$.

Therefore, finally

$$H_c(\lambda)=\begin{cases} c\sigma_+(c)H_+(\lambda/\sigma_+(c)), & c>0,\\ -c\sigma_-(c)H_-(\lambda/\sigma_-(c)), & c<0,\\ h|\lambda|^{1+\alpha}, & c=0\end{cases}$$

By analogy the dependence of function $K(t;\lambda)$ on c can be found. We have

$$b_m^{(n)}\left[F_{[mt]}^{(n)}\left(e^{\lambda/b_m^{[n]}}\right)-1\right]\xrightarrow[\substack{n\to\infty\\ m\to\infty\\ m(A_n-1)\to c}]{} K_c(t;\lambda),$$

where $\dfrac{\partial K_c(t;\lambda)}{\partial t}=H_c(K_c(t;\lambda)),\ K_c(0;\lambda)=\lambda$.

If we substitute m by $[ms], s>0$, then we will obtain $[ms](A_n-1)\to sc$ and

$$b_{[ms]}^{(n)}\left[F_{[mst]}^{(n)}\left(e^{\lambda/b_{[ms]}^{[n]}}\right)-1\right]\longrightarrow K_{sc}(t;\lambda).$$

If $c>0$, then hence it follows

$$K_{sc}(t;\lambda)=\frac{\sigma_+(sc)}{\sigma_+(c)}K_c\left(st;\lambda\frac{\sigma_+(c)}{\sigma_+(sc)}\right).$$

Assume $c=1$

$$K_s(t;\lambda)=\sigma_+(s)K_1\big(st;\lambda/\sigma_+(s)\big)$$

or

$$Kc(t;\lambda)=\sigma_+(c)K_+\big(tc;\lambda/\sigma_+(c)\big),$$

where function K_+ corresponds to H_+ and

$$K_+(t;\lambda)=\lim_{\substack{n\to\infty\\ m\to\infty\\ m(A_n-1)\to 1}} b_m^{(n)}\left[F_{[mt]}^{(n)}\left(e^{\lambda/b_m^{(n)}}\right)-1\right].$$

Similarly, the dependence of function $K(t;\lambda)$ on c is found when $c<0$.

If $c=0, \beta>0$, then solving the differential equation

$$\frac{\partial K_0(t;\lambda)}{\partial t}=h|K_0(t;\lambda)|^{1+\alpha},\quad K_0(0;\lambda)=\lambda,$$

we find

$$K_0(t;\lambda)=-\left\{h\alpha t+|\lambda|^{-\alpha}\right\}^{-\frac{1}{\alpha}}.$$

Note, that if $c=0, \beta=0$, then we will obtain $K_0(t;\lambda)=\lambda$, which means, that the number of particles in the process doesn't change.

Thus,

$$K_c(t;\lambda)=\begin{cases}\sigma_+(c)K_+\big(tc;\lambda/\sigma_+(c)\big), c>0,\\ \sigma_-(c)K_-\big(-tc;\lambda/\sigma_-(c)\big), c<0,\\ -\left\{h\alpha t+|\lambda|^{-\alpha}\right\}^{-\frac{1}{\alpha}}, \quad c=0.\end{cases}$$

In order to find the dependence of function $\prod$ on c assumes here $t=1$.

$$K_c(1;\lambda)=K_c(\lambda)=\begin{cases}\sigma_+(c)K_+\big(c;\lambda/\sigma_+(c)\big), c>0,\\ \sigma_-(c)K_-\big(-c;\lambda/\sigma_-(c)\big), c<0,\\ -\left\{h\alpha+|\lambda|^{-\alpha}\right\}^{-\frac{1}{\alpha}}, \quad c=0.\end{cases}$$

Hence we will obtain

$$\Pi_1^{(c)}(x)=\Pi^{(c)}(x)=\begin{cases}\sigma_+(c)\Pi_c^+\big(x\sigma_+(c)\big),\ c>0,\\ \sigma_-(c)\Pi_{-c}^-\big(x\sigma_-(c)\big),\ c<0,\\ \Pi^0(x), \quad c=0,\end{cases}$$

where $\Pi_t^{\pm}(x)$ is determined from expansion

$$K_\pm(t;\lambda)=a_t^\pm\lambda+\int_0^\infty\left(e^{\lambda x}-1\right)\Pi_t^\pm(dx).$$

References.

[1]. Алиев С.А., Шуренко В.М. *Переходные явления и сходимость процессов Гальтона-Ватсона к процессам Иржины.* Теория вероятности и ее применения, 1982, 27, №3, с. 443-445.

[2]. Jirina M. *Stochastic branching processes with continuous state space.* -Czechoslovak. Mathem. Jour., v.8, №2, 1958, p. 292-313.

Aliev S.A.
Institute of Matematics and Mechanics of AS Azerbaijan.
9, F. Agayev str., 370141, Baku, Azerbaijan.
Tel.: 39-47-20.

Received April 13, 1999; Revised June 11, 1999.
Translated by Soltanova S.M.