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INEQUALITIES OF SOBOLEV AND POINCARE TYPE ON HEISENBERG GROUP

Abstract

In this paper we prove two-weighted Sobolev and Poincare type inequalities on the Heisenberg group.

The inequalities of Sobolev and Poincare have important significance for the investigation of regularity of solutions of some class of hypoelliptic equations and its nonlinear analogous. As it is known one of the methods obtaining these inequalities supposes usage of fractional integrals. In this paper we prove two-weighted theorems of Sobolev and Poincare type inequality on Heisenberg group, applying the results for fractional integrals from paper [8], (see also [9,10]).

Last years the theory of Sobolev spaces in geometry of vector fields satisfying the Hörmander condition of hypoellipticity is successfully developed and its applications to investigation of reqularity of solution of quasilinear subelliptic equations are adduced. The analytical difficulties in problems of such kind connected with presence nontrivial commutative correlation's, because of which in series cases important for applications the direct transfer technique, developed in Euclidean space for standard vector fields $\frac{\partial}{\partial x_i}$ becomes impossible. The basis of theory of hypoelliptic equations was founded after paper of A.Sanchez-Calle [1], where it was established the connection between the character of singularity of fundamental solution and geometry determined by vector fields X_i. The Poincare inequality for vector fields satisfying the Hormander condition was derived by D.Jerison in [2], and at papers G.Lu, B.Franchi, C.E.Gutierres, R.L.Wheeden and others (see [3-7] and other) was obtained weighted analogous of Sobolev and Poincare inequalities. These results allow to study the properties of solutions of subelliptic and its quasilinear generalizations. Therefore the obtaining weighted Poincare and Sobolev inequalities for «work» weights is actual. From the other side, the general approach for obtaining weighted Sobolev and Poincare inequalities for vector fields, satisfying the Hormander condition, requires investigation of weighted L_p-inequalities for the integral of potential type in spaces of homogeneous type.

Let H^n be a Heisenberg group, realized as a set of points $x=(x_0,x_1,\dots,x_{2n})= =(x_0,x')\in R^{2n+1}$ with multiplication

$$xy=\left(x_0+y_0+\frac{1}{2}\sum_{i=1}^{n}(x_i y_{n+i}-x_{n+i}y_i),x'+y'\right).$$

Corresponding Lee algebra generated by left-invariant vector fields

$$X_0=\frac{\partial}{\partial x_0},\quad X_i=\frac{\partial}{\partial x_0}+\frac{1}{2}x_{n+i}\frac{\partial}{\partial x_0},\quad X_{n+i}=\frac{\partial}{\partial x_{n+i}}-\frac{1}{2}x_i\frac{\partial}{\partial x_0},$$

satisfying the commutational correlations

$$[X_i, X_{n+1}] = \frac{1}{4}X_0,\ [X_0, X_i] = [X_0, X_{n+i}] = [X_i, X_j] = [X_{n+i}, X_{n+j}] =$$
$$= [X_i, X_{n+j}] = 0, \quad i, j = 1, \ldots, n, \quad i \neq j.$$

In H^n it is determined the dilatation $\delta_t : \delta_t x = (t^2 x_0, tx'), t > 0$. The Haar measure on this group coincides with Lebesque measure $dx = dx_0 dx_1 .. dx_{2n}$. The unit element in H^n is $e = 0 \in R^{2n+1}$, the inverse to x element x^{-1} is $(-x)$. The distance from e to x is

$$|x|_H = \left(x_0^2 + \left(\sum_{i=1}^{2n} x_i^2 \right)^2 \right)^{1/4},$$

$$|y^{-1}x|_H = \left(\left(x_0 - y_0 - \frac{1}{2}\sum_{i=1}^{n}(x_i y_{n+i} - x_{n+i} y_i) \right)^2 + \left(\sum_{i=1}^{2n} (x_i - y_i)^2 \right)^2 \right)^{1/4}$$

$|y^{-1}x|_H$ is distance between x and y.

More detale of analysis on Heisenberg group can be found, for example, in V.S.Guliev's monographs [11] (see also [12-15]).

Let B_R be a ball with radius $R > 0$ and center at the point e in H^n. For local summable positive function $\omega(|x|_H)$, $x \in H$, we suppose

$$\omega(B_R) = \int_{B_R} \omega(|x|_H) dx.$$

Theorem 1 (embedding theorem). *Let* $1 < p < Q = 2n + 2, 1/q = 1/p - 1/Q \leq \leq k \leq Q/(Q - p)$ *and* $\omega(t)$, $\omega_1(t)$ *are positive increasive functions on* $(0, R), \omega(0) = \omega_1(0) = 0$ *and*

$$\sup_{0<t<R} \left(\int_t^R \omega_1'(\tau) \tau^{-q/p'} d\tau \right)^{p/q} \left(\int_0^t \omega'(\tau)^{1-p'} \tau^{1-p'} d\tau \right)^{p-1} < \infty.$$

Then for all $u \in C_0^\infty(B_R)$ *there exists positive constant* C *independent on* R *and* $u(x)$ *such that*

$$\left(\frac{1}{\omega_1(B_R)} \int_{B_R} |u(x)|^{kp} \omega_1(|x|_H) dx \right)^{\frac{1}{kp}} \leq$$

$$\leq C \frac{\omega(B_R)^{1/p}}{\omega_1(B_R)^{1/p}} \left(\frac{1}{\omega(B_R)} \int_{B_R} |\nabla_H u(x)|^p \omega(|x|_H) dx \right)^{\frac{1}{p}}.$$

Here $\nabla_H = (X_1, \ldots, X_{2n})$ *is a subgradient.*

Proof. Since $kp \leq q \leq Q/(Q - p)$, then by Hölder inequality

$$\left(\int_{B_R} |u(x)|^{kp} \omega_1(|x|_H) dx \right)^{\frac{1}{kp}} \leq \left(\int_{B_R} |u(x)|^q \omega_1(|x|_H) dx \right)^{\frac{1}{q}} \times$$

$$\times \left(\int_{B_R} \omega_1(|x|_H) dx \right)^{\frac{1}{kp} - \frac{1}{q}} = \omega_1(B_R)^{\frac{1}{kp} - \frac{1}{q}} \left(\int_{B_R} |u(x)|^q \omega_1(|x|_H) dx \right)^{\frac{1}{kp} - \frac{1}{q}}$$

or

$$\left(\frac{1}{\omega_1(B_R)}\int_{B_R}|u(x)|^{kp}\omega_1(|x|_H)dx\right)^{\frac{1}{kp}} \le \left(\frac{1}{\omega_1(B_R)}\int_{B_R}|u(x)|^{q}\omega_1(|x|_H)dx\right)^{\frac{1}{q}}. \qquad (1)$$

On the other hand for $\forall u \in C_0^\infty(B_R)$ [16]

$$u(x) = C_0 \int_{H^n} \frac{\Delta_H u(y)}{|xy^{-1}|_H^{Q-2}} dy =$$

$$= C_0 \int_{H^n} \nabla_H u(y) \nabla_H |xy^{-1}|_H^{2-Q} dy = C_0 \int_{H^n} \frac{\nabla_H u(y)}{|xy^{-1}|_H^{Q-1}} dy.$$

Hence , for $\forall u \in C_0^\infty(B_R)$

$$|u(x)| \le C \int_{H^n} \frac{|\nabla_H u(y)|}{|xy^{-1}|_H^{Q-1}} dy$$

with a constant independent on $u(x)$ and R.

Then by virtue of corollary 1.3.1 ([14], see also [15, 16]) we have

$$\left(\int_{B_R}|u(x)|^q \omega_1(|x|_H)dx\right)^{\frac{1}{q}} \le C\left[\int_{B_R}\left(\int_{B_R}\frac{|\nabla_H u(y)|}{|xy^{-1}|_H^{Q-1}}dy\right)\omega_1(|x|_H)dx\right]^{\frac{1}{q}} \le$$

$$\le C\left(\int_{B_R}|\nabla_H u(x)|^p \omega(|x|_H)dx\right)^{\frac{1}{p}}$$

with a constant independent on R and $u(x)$. Taking into account the last inequality in the right-hand side of (1), we obtain the required inequality.

Theorem is proved.

By the same way the next theorem is proved.

Theorem 2 (embedding theorem). *Let* $1 < p < Q = 2n+2, 1/q = 1/p - 1/Q$, $1 \le k \le Q/(Q-p)$ *and* $\omega(t), \omega_1(t)$ *are positive decreased functions on* $(0,R), \omega(R) = \omega_1(R) = 1$ *and*

$$\sup_{0<t<R}\left(\int_0^t(-\omega_1')(\tau)\tau^{-q/p'}d\tau\right)^{p/q}\left(\int_t^R(-\omega')(\tau)^{1-p'}\tau^{-p'/q-p'}d\tau\right)^{p-1} < \infty.$$

Then for all $u \in C_0^\infty(B_R)$ there exists a positive constant C independent on R and $u(x)$, such that

$$\left(\frac{1}{\omega_1(B_R)}\int_{B_R}|u(x)|^{kp}\omega_1(|x|_H)dx\right)^{\frac{1}{kp}} \le$$

$$\le C\frac{\omega(B_R)^{1/p}}{\omega_1(B_R)^{1/q}}\left(\frac{1}{\omega(B_R)}\int_{B_R}|\nabla_H u(x)|^p\omega(|x|_H)dx\right)^{\frac{1}{p}}.$$

Remark. *In case $B(x_0,R)$ is a ball with radius R and center at arbitrary point $x_0 \in H^n$ theorem 1 and 2 are proved by changing variables x to xx_0^{-1}.*

Remark. *Theorems 1 and 2 stay valid, if we change the ball B_R to any bounded domain containing the point e.*

Theorem 3. *Let $1<p<Q=2n+2$, $1/q=1/p-1/Q$, $1\le k\le Q/(Q-p)$, the pair of weighted functions (ω_1,ω_2) satisfies the conditions of either theorem 1, or theorem 2 and Ω is a bounded domain in H^n. Then for all $u\in C_0^\infty(\Omega)$ there exists positive constant C_Ω, dependent on diameter of domain Ω, such that*

$$\|u\|_{L_{kp}(\omega_1,\Omega)} \le C_\Omega \|\nabla_H u\|_{L_p(\omega,\Omega)}.$$

For prove of theorem 3 it is enough to put this domain in ball of sufficient big radius with center at the point e and apply theorems 1 and 2.

It is known the following theorem (see [2]).

Theorem. *If $u,\nabla_H u\in L_p(H^n)$, $1<p<\infty$, then*

$$|u(x)-u_{B_R}|\le C\int_{B_{2R}}\frac{|\nabla_H u(y)|}{|xy^{-1}|_H^{Q-1}}dy,$$

for almost all points $x\in B$, where constant C is independent on R and u, where $u_{B_R}=\frac{1}{|B_R|}\int_{B_R}u(x)dx$.

Theorem 4. *(Poincare inequality). Let $1<p<Q=2n+2$, $1/q=1/p-1/Q$, $1\le k\le Q/(Q-p)$ and pair of weighted functions (ω,ω_1) satisfy the conditions of either theorem 1, or theorem 2. Then for all $u\in C^\infty(\overline{B_{2R}})$ there exist positive constant C, independent on R and $u(x)$ such that*

$$\left(\frac{1}{\omega_1(B_R)}\int_{B_R}|u(x)-u_{B_R}|^{kp}\,\omega_1(|x|_H)dx\right)^{\frac{1}{kp}}\le$$

$$\le C\frac{(\omega(B_R))^{1/p}}{(\omega_1(B_R))^{1/p}}\left(\frac{1}{\omega(B_R)}\int_{B_{2R}}|\nabla_H u(x)|^p\,\omega(|x|_H)dx\right)^{\frac{1}{2}},$$

where $u_{B_R}=\frac{1}{|B_R|}\int_{B_R}u(x)dx$ or $u_{B_R}=\frac{1}{\omega_1(B_R)}\int_{B_R}u(x)\omega_1(|x|_H)dx$.

Proof. a) Let $u_{B_R}=\frac{1}{|B_R|}\int_{B_R}u(x)dx$.

According to Hölder inequality with exponent $\frac{q}{kp}\ge 1$ and further, by virtue of inequality of previous theorem

$$\left(\frac{1}{\omega_1(B_R)}\int_{B_R}|u(x)-u_{B_R}|^{kp}\,\omega_1(|x|_H)dx\right)^{\frac{1}{kp}}\le$$

$$\leq C\left(\frac{1}{\omega_1(B_R)}\int_{B_R}\left(\int_{B_{2R}}\frac{|\nabla_H u(y)|}{|xy^{-1}|_H^{Q-1}}dy\right)^q \omega_1(|x|_H)dx\right)^{\frac{1}{q}}.$$

Now applying theorem 1 and 2 to the right-hand side of last inequality, we finish the proof in case a).

b) Let now $u_{B_R}=\frac{1}{\omega_1(B_R)}\int_{B_R}u(x)\omega_1(|x|_H)dx$.

It is obvious

$$|u(x)-u_{B_R}|\leq\left|u(x)-\frac{1}{|B_R|}\int_{B_R}u(y)dy\right|+\left|u_{B_R}-\frac{1}{|B_R|}\int_{B_R}u(y)dy\right|. \tag{2}$$

Further,

$$\left|u_{B_R}-\frac{1}{|B_R|}\int_{B_R}u(y)dy\right|=\left|\frac{1}{\omega_1|B_R|}\int_{B_R}\left(u(x)-\frac{1}{|B_R|}\int_{B_R}u(y)dy\right)\omega_1(|x|_H)dx\right|\leq$$

$$\leq\frac{1}{\omega_1(B_R)}\int_{B_R}\left|u(x)-\frac{1}{|B_R|}\int_{B_R}u(y)dy\right|\omega_1(|x|_H)dx.$$

According to Hölder inequality with exponent $kp>1$ the right-hand side of last inequality doesn't exceed

$$\left(\frac{1}{\omega_1(B_R)}\int_{B_R}\left|u(x)-\frac{1}{|B_R|}\int_{B_R}u(y)dy\right|^{kp}\omega_1(|x|_H)dx\right)^{\frac{1}{kp}}.$$

Now, according to (2)

$$\left(\frac{1}{\omega_1(B_R)}\int_{B_R}|u(x)-u_{B_R}|^{kp}\omega_1(|x|_H)dx\right)^{\frac{1}{kp}}\leq$$

$$\leq 2\left(\frac{1}{\omega_1(B_R)}\int_{B_R}\left|u(x)-\frac{1}{|B_R|}\int_{B_R}u(y)dy\right|^{kp}\omega_1(|x|_H)dx\right)^{\frac{1}{kp}}$$

and then we apply case a).

Theorem is proved.

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Received May 26, 1999; Revised July 22, 1999.
Translated by Panarina V.K.