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THE INVERSE SCATTERING PROBLEM FOR NON-STATIONARY DIRAC'S EQUATIONS SYSTEM

Abstract

In this paper the direct and inverse scattering problems for a non-stationary Dirac system on whole-axis are studied. The coefficients of the system are uniquely determined according to scattering operator. In this paper for solving of the inverse scattering problem also minimal information was introduced.

Consider non-stationary Dirac's equations system of the $2n$-th order

$$\frac{\partial}{\partial t}\psi(x,t)-\tau\frac{\partial}{\partial x}\psi(x,t)=U(x,t)\psi(x,t),$$

$$\psi(x,t)=\begin{pmatrix}\psi_1(x,t)\\ \psi_2(x,t)\end{pmatrix},\quad \tau=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix},\quad U(x,t)=\begin{pmatrix}0 & u_1(x,t)\\ u_2(x,t) & 0\end{pmatrix} \tag{1}$$

Here $-\infty<x,t<+\infty$; $\psi_i(x,t)\,(i=1,2)$ are vector-functions with n components; I is $n\times n$ unique matrix; $u_i(x,t)\,(i=1,2)$ is $n\times n$ matrix-functions with measurable complex-valued elements.

Assume that in (1) the euclidean norms of coefficients $u_i(x,t)$ $(i=1,2)$ satisfy the inequalities

$$\|u_i(x,t)\|\le C(1+|t|+|x|)^{-2-\varepsilon},\quad i=1,2, \tag{2}$$

where C and ε are positive numbers.

If $U(x,t)$ doesn't depend on t and $\psi(x,t)=\psi(x)e^{\lambda t}$; $\lambda=a+bi$, then (1) is reduced to Dirac's stationary equations system of variables and to solution of the inverse problems of spectral scattering theory [4,5].

In case when $n=1$ the inverse problem for Dirac's non-stationary system was considered in [1,3].

1. The scattering problem.

Let $U(x,t)\equiv 0$ in the equations system (1), then it is easy to check, that the vector-function

$$\psi(x,t)=\{\varphi_1(x-t),\varphi_2(x+t)\}\equiv T_t\varphi(x),$$

where $\varphi_i(x)$ $(i=1,2)$ are arbitrary locally integrated vector-functions with n components in generalized sense satisfy the unperturbed system (1).

We'll understand as solution of (1) any function $\psi(x,t)=(\psi_1(x,t),\psi_2(x,t'))\in L_\infty(R^2,C^{2n})$ satisfying (1) in a generalized sense.

Theorem 1. *For equations system (1) satisfying the condition (2) the following statement is valid.*

1) For any functions $a(x)\in L_\infty(R,C^{2n})$ there is only solution $\psi(x,t)\in L_\infty(R^2,C^{2n})$ of the system (1) such that

$$\operatorname{Vrai}\max_x|\psi(x,t)-T_ta(x)|\to 0,\ \text{for}\ t\to-\infty,$$

$$Vrai\max_{x}\left|\psi(x,t)-T_t b(x)\right|\to 0,\ \text{ for }\ t\to+\infty. \tag{3}$$

(modulus sign here is means the norm in C^{2n}).

2) For any solutions $\psi(x,t)\in L_\infty(R^2,C^{2n})$ of (1) there exists only solution $T_t a(x)$ and $T_t b(x)\in L_\infty(R^2,C^{2n})$ of the unperturbed system such that

$$Vrai\max_{x}\left|\psi(x,t)-T_t a(x)\right|\to 0,\ \text{ for }\ t\to-\infty,$$

$$Vrai\max_{x}\left|\psi(x,t)-T_t b(x)\right|\to 0,\ \text{ for }\ t\to+\infty.$$

Proof of the theorem is analogous to methods of [1,3].

By virtue of Theorem 1 Cauchy problem with data at infinity is solvable one-to-one in the class of significant bounded functions. It allows to determine the scattering operator as the operator transforming $T_t a(x)$ into $T_t b(x)$. More exactly identifying the solutions $T_t a(x)$ and $T_t b(x)$ with the initial data $a(x)$ and $b(x)$ for $t=0$ we determine the scattering operator S (accordingly to the general definition) as operator

$$S: a(x)\to b(x) \tag{4}$$

Operator S is matrix one $S=\left\|S_{ij}\right\|_{i,j=1}^{2}$, where $S_{ij}(i,j=1,2)$ are $n\times n$ matrix and linear bounded operator in the space $L_\infty(R,C^{2n})$. Further operator S will be studied in space $L_2(R,C^{2n})$, that is under S we will understand the closure of operator S narrowed on $L_\infty(R,C^{2n})\cap L_2(R,C^{2n})$.

The deeper properties of the scattering operator S will be studied below.

2. Transformation operators.

For solving the inverse scattering problem, that is the problem of restoration of the coefficients of the equation by known scattering operator the Volterra type integral representations of solutions have an important role. For the system (1) the following lemma is valid

Lemma 1. *Every bounded solution $\psi(x,t)=(\psi_1(x,t),\psi_2(x,t))$ of (1) with condition (2) admits the representation*

$$\psi(x,t)=T_t f^+(x)+\int_{-\infty}^{x}A^+(x,t,s)T_t f^+(s)ds, \tag{5}$$

$$\psi(x,t)=T_t f^-(x)+\int_{x}^{+\infty}A^-(x,t,s)T_t f^-(s)ds, \tag{6}$$

where

$$T_t f^\pm(x)=\left(f_1^\pm(x+t),f_2^\pm(x-t)\right),\quad f^\pm(x)\in L_\infty(R,C^{2n}),$$

$A^\pm(x,t,s)=\left\|A_{ij}^\pm(x,t,s)\right\|_{i,j=1}^{2}$, $A_{ij}^\pm(x,t,s)(i,j=1,2)$ *are $n\times n$ matrix-functions. Kernels $A_{ij}^\pm(x,t,s)(i,j=1,2)$ are determined one-to-one by the coefficients $u_i(x,t)(i=1,2)$ of (1) and for the fixed x they are summed with square by t and s, that is they are Hilbert-Shmidt kernels.*

The coefficients $u_i(x,t)(i=1,2)$ of (1) are expressed via the kernels of transform operators by formulas:

$$\begin{aligned} u_1(x,t) &= \mp 2A_{12}^{\pm}(x,t,x) \\ u_2(x,t) &= \pm 2A_{21}^{\pm}(x,t,x) \end{aligned} \tag{7}$$

Proof. Let prove the representation (5). The bounded solution of (1) with the given asymptotics $T_t f^+(x)$ for $x \to -\infty$ satisfies the system of integral equations

$$\begin{aligned} \psi_1(x,t) &= f_1^+(x+t) - \int_{-\infty}^{x} (u_1\psi_2)(s,t+x-s)ds \\ \psi_2(x,t) &= f_2^+(x-t) + \int_{-\infty}^{x} (u_2\psi_1)(s,t-x+s)ds \end{aligned} \tag{8}$$

If the solution of (8) is represented in form (5) for any $f_1^+(x), f_2^+(x) \in L_\infty(R,C^n)$, then substituting (5) into (8), we obtain the system of equations for kernels

$$\begin{aligned} A_{11}^+(x,t,\tau) &= -\int_{-\infty}^{x} u_1(s,x+t-s)A_{21}^+(s,x+t-s,\tau-x+s)ds, \\ A_{21}^+(x,t,\tau) &= \frac{1}{2}u_2\left(\frac{\tau+x}{2}, t+\frac{\tau-x}{2}\right) + \\ &+ \int_{\frac{\tau+x}{2}}^{x} u_2(s,t-x+s)A_{11}^+(s,t-x+s,\tau+x-s)ds, \\ A_{12}^+(x,t,\tau) &= -\frac{1}{2}u_1\left(\frac{\tau+x}{2}, t+\frac{\tau-x}{2}\right) - \\ &- \int_{\frac{\tau+x}{2}}^{x} u_1(s,x+t-s)A_{22}^+(s,t-x+s,\tau+x-s)ds, \\ A_{22}^+(x,t,\tau) &= \int_{-\infty}^{x} u_2(s,t-x+s)A_{12}^+(s,t-x+s,\tau-x+s)ds \end{aligned} \tag{9}$$

On the contrary, the kernels $A_{ij}(x,t,\tau)$ satisfy (9), the (5) gives the bounded solution for any $f_1^+(x), f_2^+(x) \in L_\infty(R,C^n)$. Therefore, for the proof of (5) it is enough to prove that (9) has the solution. It follows from volterrity of these systems on the base of the theorem 4.1.1. [2]. Equality (7) follows immediately from (9) for $\tau = x$. The proof of the representation (6) is quite analogous to proof of the representation (5).

Formulas (5) and (6) can be written in the operator view:

$$\psi(x,t) = \left(I + A^+(t)\right)T_t f^+(t)\ , \tag{5'}$$

$$\psi(x,t) = \left(I + A^-(t)\right)T_t f^-(t), \tag{6'}$$

where the indices «+» and «–» mean polarity of volterra integral operator.

From lemma 1 it follows that there is one-to-one correspondence between the prototypes (between $f^+(x)$ and $f^-(x)$) of one and the same solution of equation system (2). Therefore, there exists the passage operator connecting the prototypes of one and same solution:

$$\widetilde{S} : f^+(x) \to \bar{f}(x).$$

This operator is matrix one in $L_2(R,C^{2n})$ and from (5') and (6') it is seen that the operator $T_t\widetilde{S}T_t^{-1}$ admits factorization on volterra matrix integral Hilbert-Shmidt, that is

$$T_t\widetilde{S}T_t^{-1}=\left(I+A^-(t)\right)^{-1}\left(I+A^+(t)\right). \tag{10}$$

Structure of operators S and $\widetilde{S}$ is investigated in the following

Lemma 2. *For operators S and $\widetilde{S}$ the following representations are valid:*

$$S=\begin{pmatrix} I & 0 \\ R_1 & I+R_{1+}\end{pmatrix}\begin{pmatrix} I+R_{2+} & R_2 \\ 0 & I\end{pmatrix}^{-1}=\begin{pmatrix} I+R_{2-} & R_3 \\ 0 & I\end{pmatrix}\begin{pmatrix} I & 0 \\ R_4 & I+R_{1-}\end{pmatrix}^{-1}, \tag{11}$$

$$\widetilde{S}=\begin{pmatrix} I & 0 \\ R_4 & I+R_{1-}\end{pmatrix}^{-1}\begin{pmatrix} I+R_{2+} & R_2 \\ 0 & I\end{pmatrix}=\begin{pmatrix} I+R_{2-} & R_3 \\ 0 & I\end{pmatrix}^{-1}\begin{pmatrix} I & 0 \\ R_4 & I+R_{1+}\end{pmatrix}^{-1}, \tag{12}$$

where $R_i\ \left(i=\overline{1,4}\right)$ are $n\times n$ Hilbert-Shmidt matrix integral operators, R_{i-} and $R_{i+}\ (i=1,2)$ are Hilbert-Shmidt matrix volterra operators of the corresponding polarity.

Proof. Substituting the integral representations (5) into the system of integral equations of the scattering problem satisfying the asymptotics $T_t a(x)=\left(a_1(x+t),a_2(x-t)\right)$ for $t\to-\infty$, that is in the system

$$\psi_1(x,t)=a_1(x+t)+\int_{-\infty}^{t}(u_1\psi_2)(x+t-s,s)ds,$$

$$\psi_2(x,t)=a_2(x+t)+\int_{-\infty}^{t}(u_2\psi_1)(x-t+s,s)ds$$

taking into account (8) and (9) for kernels $A_{ij}^+(x,t,\tau)$ it is obtained, that

$$\begin{pmatrix} a_1(x+t) \\ a_1(x-t)\end{pmatrix}=\begin{pmatrix} I+R_{2+}(t) & R_2(t) \\ 0 & I\end{pmatrix}\begin{pmatrix} f_1^+(x+t) \\ f_2^+(x-t)\end{pmatrix}, \tag{13}$$

where kernels of operators $R_{2+}(t)$ and $R_2(t)$ have a view:

$$R_{2+}(x,t,\tau)=\int_{-\infty}^{+\infty}u_1(s,x+t-s)A_{21}^+(s,x+t-s,\tau-x+s)ds,$$

$$R_2(x,t,\tau)=\frac{1}{2}u_1\left(\frac{\tau+x}{2},t+\frac{\tau-x}{2}\right)+$$

$$+\int_{\frac{\tau+x}{2}}^{+\infty}u_1(s,x+t-s)A_{22}^+(s,x+t-s,\tau+x-s)ds$$

Analogously the following equalities are obtained:

$$\begin{pmatrix} b_1(x+t) \\ b_2(x-t)\end{pmatrix}=\begin{pmatrix} I & 0 \\ R_1(t) & I+R_{1+}(t)\end{pmatrix}\begin{pmatrix} f_1^+(x+t) \\ f_1^+(x-t)\end{pmatrix} \tag{14}$$

$$\begin{pmatrix} b_1(x+t) \\ b_2(x-t) \end{pmatrix} = \begin{pmatrix} I & 0 \\ R_4(t) & I+R_{1-}(t) \end{pmatrix} \begin{pmatrix} f_1^-(x+t) \\ f_1^-(x-t) \end{pmatrix} \quad (15)$$

$$\begin{pmatrix} a_1(x+t) \\ a_2(x-t) \end{pmatrix} = \begin{pmatrix} I+R_{2-}(t) & R_3(t) \\ 0 & I \end{pmatrix} \begin{pmatrix} \bar{f}_1(x+t) \\ \bar{f}_2(x-t) \end{pmatrix} \quad (16)$$

From equalities (13)-(16) validity of (11) and (12) follows

Corollary. *Operators S and $\widetilde{S}$ are connected one-to-one among themselves.*

The proof follows from (11) and (12).

3. The inverse scattering problem. The inverse scattering problem for the system of equations (1) is confined in finding the matrix potential $U(x,t)$ by known scattering operator S.

Theorem 2. *Let S be the scattering operator for the hyperbolic system of the differential equations (1) with potential $U(x,t)$ satisfying the estimation (2). Then potential $U(x,t)$ is determined one-to-one by the known scattering operator S.*

Proof. Let the scattering operator S_1 correspond to the potential $U_1(x,t)$ and the operator S_2 correspond to the potential $U_2(x,t)$. Show that if $S_1 = S_2$, then $U_1(x,t) = U_2(x,t)$. According to Lemma 2 each scattering operator S_1 and S_2 admit the structure (11), that is

$$S_k = \begin{pmatrix} I & 0 \\ R_1^k & I+R_{1+}^k \end{pmatrix} \begin{pmatrix} I+R_{2+}^k & R_2^k \\ 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} I+R_{2-}^k & R_3^k \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ R_4 & I+R_{1-}^k \end{pmatrix}^{-1}, k=1,2.$$

As these representations are unique and $S_1 = S_2$, then $R_i^1 = R_i^2 \left(i=\overline{1,4}\right)$, $R_{i+}^1 = R_{i+}^2$, $R_{i-}^1 = R_{i-}^2$ $(i=1,2)$. So the operator $\widetilde{S}_k, k=1,2$ also coincides according to (12).

By virtue of uniqueness of factorization into volterra multipliers the factorization multipliers coincide in the factorizations

$$T_t \widetilde{S}^k T_t^{-1} = \left(I + A_k^-(t)\right)\left(I + A_k^+(t)\right), k=1,2.$$

Hence $U_k(x,t), k=1,2$ also coincide obtained by these factorization multipliers by (7).

Thus, by the matrix kernel of the operator $F = S - I$ containing $4n^2$ of given functions $F_{ij}(x,t)\left(i,j=\overline{1,2n}\right)$, the potential $U(x,t)$ can be found in the system (1) containing $2n^2$ of the desired functions.

Under the scattering data we will understand the minimal information by which the inverse scattering problem can be solved. For the hyperbolic system of $n \geq 3$ equations $(\xi_1 > \xi_2 > \ldots > \xi_n)$ such minimal information was introduced in [6] and for the system (1) for $n=2$ when $S = I - \left\|F_{ij}\right\|_{i,j=1}^2, S^{-1} = I + \left\|G_{ij}\right\|_{i,j=1}^2$ in [1] the scattering data as a pair of functions $\{F_{12}(x,y), G_{21}(x,y)$ or $\{F_{21}(x,y), G_{12}(x,y)$ which are the kernels of the integral operators $\{F_{12}, G_{21}\}, \{F_{21}, G_{12}$ had been introduced long ago.

Definition. *The scattering data for the system of differential equations (1) will be called some pair of the matrix functions $\{R_1(x,y), R_3(x,y)\}$ or $\{R_2(x,y), R_4(x,y)\}$ which are the kernels of the matrix integral operators $\{R_1, R_3\}$, $\{R_2, R_4$ in the representations (11), (12).*

It can be shown that by these scattering data, for example by $\{R_1, R_3$ the operators S and $\widetilde{S}$ are found one-to-one.

From the matrix equality (12) the following equalities are obtained:

$$(I + R_{2-})(I + R_{2+}) = I - R_3 R_1, \tag{17}$$

$$(I + R_{1-})^{-1}(I + R_{1+})^{-1} = I + R_1(I - R_3 R_1)^{-1} R_3, \tag{18}$$

$$R_4 = -(I + R_{1-}) R_1 (I + R_{2+})^{-1}, \tag{19}$$

$$R_2 = -(I + R_{2-})^{-1} R_3 (I + R_{1+}). \tag{20}$$

Really, if we know the pair of integral operators $\{R_1, R_3$, then from the factorization equalities (17) and (18) the factorization multipliers are found, that is, the matrix operators $R_{2\pm}$ and $R_{1\pm}$. Finding of these multipliers is the well-known problem and it is reduced to solution of the system of Gelfand-Levitan-Marchenko type integral equations (see [2]). Equalities (19) and (20) give us possibility to determine operators R_2 and R_4.

Therefore, by the pair of operators $\{R_1, R_3$ all other operators in representations (11) and (12) of operators S and $\widetilde{S}$ are determined.

Solution of the inverse scattering problem for the system of differential equations (1) with the potential satisfying the estimation (2) is unique. Algorithm of solving of this problem is in that the constructed by the scattering data operator $\widetilde{S}$ to (17)-(20) admits factorization (10) by whose factorization multipliers the potential is found according to (7).

References

[1]. Нижник Л.П. *Обратная нестационарная задача рассеяния.* Киев, Наукова думка, 1973, 182 с.

[2]. Нижник Л.П. *Обратная задача рассеяния для гиперболических уравнений.* Киев, Наукова думка, 1991, 232 с.

[3]. Шабат А.Б. *Одномерная теория рассеяния.* Диф. ур. ,т.8, №1, 1972, с.164-178.

[4]. Гасымов М.Г. *Обратная задача теории рассеяния для систем уравнений Дирака порядка* $2n$. Труды Моск-го матем. об-ва, т.19, 1968, с.41-112.

[5]. Гусейнов И.М. *Обратная задача теории рассеяния для системы уравнений Дирака* $2n$ *-го порядка.* ДАН СССР, 1977, т.232, №5, с.993-996.

[6]. Тарасов В.Г. *Данные рассеяния в обратной задаче рассеяния для гиперболической системы дифференциальных уравнений первого порядка.* Донов Нац. АН. Укр., 1997, №7, с.60-64.

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